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HIGHER ORDER CLOSE-TO-CONVEX FUNCTIONS ASSOCIATED WITH RUSCHEWEYH DERIVATIVE OPERATOR

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Abstract. The purpose of this paper is to introduce and study certain subclasses of analytic functions by using Ruscheweyh derivative operator. We discuss various of interesting properties such as, necessary condition, arc length problem and growth rate of coefficient of newly defined class. Also rate of growth of Hankel determinant will be estimated.

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1. Introduction

Let $\mathbf{A}$ be the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

in the open unit disk $E = \{z : |z| < 1\}$. Also, let S, S^*, C and K denote the subclasses of $\mathbf{A}$ consisting of functions that are univalent, starlike, convex and close-to-convex in E respectively.

The convolution or Hadamard product of two functions $f, g \in \mathbf{A}$ is denoted by $f * g$ and is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in E. \tag{2}$$

A function $f \in \mathbf{A}$ is subordinate to $g \in \mathbf{A}$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwartz function w in E such that $f(z) = g(w(z))$.

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In [5], Janowski introduced the class $P[A,B]$. For $-1 \leq B < A \leq 1$, a function p analytic in E with $p(0)=1$ belongs to the class $P[A,B]$, if $p(z)$ is subordinate to $\frac{1+Az}{1+Bz}$.

Noor [11] extended the concept of Janowski functions in bounded rotation and defined certain subclasses of analytic functions as follows:

Let $p \in \mathbf{A}$ with $p(0)=1$. Then, for $m \geq 2$, $p \in P_m[A,B]$ if and only if

$$p(z) = \left(\frac{m}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{m}{4}-\frac{1}{2}\right)p_2(z), \text{ for } p_1, p_2 \in P[A,B].$$

$$R_m[A,B] = \left\{ f \in \mathbf{A} : \frac{zf'}{f} \in P_m[A,B] \right\}$$

and

$$V_m[A,B] = \left\{ f \in \mathbf{A} : zf' \in R_m[A,B] \right\}.$$

For $k \geqslant 0$, the conic domains Ω_k, defined as;

$$\Omega_k = \left\{ u+iv : u > k\sqrt{(u-1)^2+v^2} \right\}.$$

The domains Ω_k $(k=0)$ represents right half plane, Ω_k $(0<k<1)$ represents hyperbola, Ω_k $(k=1)$ represents a parabola and Ω_k $(k>1)$ represents an ellipse. The extremal functions for these conic regions are given as

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, & k=0 \\ 1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k=1 \\ 1+\frac{2}{1-k^2}\left[\left(\frac{2}{\pi}\arccos k\right)\arctan h\sqrt{z}\right], & 0<k<1 \\ 1+\frac{1}{k^2-1}\sin\left(\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{\sqrt{t}}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right)+\frac{1}{k^2-1}, & k>1, \end{cases} \tag{3}$$

where $u(z) = \frac{z-\sqrt{t}}{z-\sqrt{tz}}$, $t \in (0,1)$, $z \in E$ and z is chosen such that $k = \cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$, $R(t)$ is Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$. See [6, 7] for more information.

Let $P(p_k)$ denote the class of all those functions $p(z)$ which are analytic in E with $p(0)=1$ and satisfies $p(z) \prec p_k(z)$, $z \in E$.

Clearly $P(p_k) \subset P(\frac{k}{1+k}) \subset P$, where P is the well known class of Caratheodory functions.

Let $f \in \mathbf{A}$ and $D^\delta : \mathbf{A} \to \mathbf{A}$ be the operator defined by

$$D^\delta f(z) = \begin{cases} \frac{z}{(1-z)^{\delta+1}} * f(z); & \delta > -1 \\ \frac{z\left(z^{\delta-1}f(z)\right)^\delta}{\delta!} & \delta \in \mathbb{N}_0 = \{0,1,2,...\} \end{cases}.$$

Note that $D^0 f(z) = f(z)$ and $D^1 f(z) = zf'(z)$. We can easily verify the following identity, see [19].

$$z(D^\delta f)' = (\delta+1)D^{\delta+1}f - \delta D^\delta f. \tag{4}$$

Using Ruscheweyh derivative operator, we define:

$$R_m^\delta[A,B] = \left\{ f \in \mathbf{A} : D^\delta f \in R_m\,[A,B] \right\},$$
$$V_m^\delta[A,B] = \left\{ f \in \mathbf{A} : zf' \in R_m^\delta[A,B] \right\}$$

and

$$k - UT_m^\delta[A,B] = \left\{ f \in \mathbf{A} : \frac{\left(D^\delta f\right)'}{\left(D^\delta g\right)'} \in P(p_k), \text{ for } g \in V_m^\delta[A,B] \right\}.$$

We note that for special values of k, δ, m, A and B we obtain several known classes of analytic functions, see [3, 5, 10, 11].

2. Main Results

2.1. Necessary Condition.

Theorem 2.1. *Let $f \in k - UT_m^\delta[A,B]$ and $F(z) = D^\delta f(z)$. Then, for $\theta_1 < \theta_2$, $z = re^{\iota\theta}$*

$$\int_{\theta_1}^{\theta_2} Re \left\{ \frac{(zF'(z))'}{F'(z)} \right\} d\theta > - \left[\frac{(A-B)\,(m-2)}{2\,(1-B)} + \sigma \right] \pi,$$

where $\sigma = \frac{2}{\pi}\arctan(\frac{1}{k})$.

Proof. Let $f \in k - UT_m^\delta[A,B]$. Then there exists $g \in V_m^\delta[A,B]$ such that

$$\frac{F'(z)}{G'(z)} \in P\left(p_k(z)\right), \text{ where } G = D^\delta g$$

Equivalently

$$F'(z) = G'(z)p(z), \text{ where } p(z) \in P\left(p_k(z)\right). \tag{5}$$
$$F'(z) = G'(z)h^\sigma(z), \tag{6}$$

where $h \in P$ and $\sigma = \frac{2}{\pi}\arctan(\frac{1}{k})$.

Since $g \in V_m^\delta[A,B]$, so

$$\left(D^\delta g\right)(z) = G(z) \in V_m[A,B] \subset V_m(\rho),$$

where $\rho = \frac{1-A}{1-B}$, we have

$$G'(z) = \left(G_1'(z)\right)^{1-\rho},\ G_1 \in V_m, \quad \text{(see [16])}. \tag{7}$$

From (6) and (7), we get

$$\begin{aligned} F'(z) &= \left(G_1'(z)\right)^{1-\rho} h^\sigma(z) \\ zF'(z) &= \left(zG_1'(z)\right)^{1-\rho} z^\rho h^\sigma(z). \end{aligned} \tag{8}$$

Logarithmic differentiation of (8) yields

$$\begin{aligned} \frac{(zF')'(z)}{zF'(z)} &= (1-\rho)\frac{(zG_1'(z))'}{zG_1'(z)} + \frac{\rho}{z} + \sigma\frac{h'(z)}{h(z)} \\ \frac{(zF')'(z)}{F'(z)} &= (1-\rho)\frac{(zG_1'(z))'}{G_1'(z)} + \rho + \sigma\frac{zh'(z)}{h(z)}. \end{aligned}$$

Integrating from θ_1 to θ_2, where $\theta_1 < \theta_2$, for $z = re^{i\theta}$ we have

$$\int_{\theta_1}^{\theta_2} Re\left\{\frac{(zF')'(z)}{F'(z)}\right\} d\theta = (1-\rho)\int_{\theta_1}^{\theta_2} Re\left\{\frac{(zG_1'(z))'}{G_1'(z)}\right\} d\theta + \rho\,(\theta_2-\theta_1) + \sigma\int_{\theta_1}^{\theta_2} Re\left\{\frac{zh'(z)}{h(z)}\right\} d\theta. \quad (9)$$

We observe that, for $h \in P$

$$\begin{aligned}\frac{\partial}{\partial\theta}\arg h(re^{i\theta}) &= \frac{\partial}{\partial\theta} Re\left\{-i\ln h(re^{i\theta})\right\} \\ &= Re\left\{\frac{re^{i\theta}h'(re^{i\theta})}{h(re^{i\theta})}\right\}.\end{aligned}$$

This implies

$$\int_{\theta_1}^{\theta_2} Re\left\{\frac{re^{i\theta}h'(re^{i\theta})}{h(re^{i\theta})}\right\} d\theta = \arg h(re^{i\theta_2}) - \arg h(re^{i\theta_1}),$$

and

$$\int_{\theta_1}^{\theta_2} Re\left\{\frac{(zG_1'(z))'}{G_1'(z)}\right\} d\theta > -\left(\frac{m}{2}-1\right)\pi. \quad (10)$$

From $(8-10)$, we get for $\theta_1 < \theta_2$, $z = re^{i\theta}$

$$\begin{aligned}\int_{\theta_1}^{\theta_2} Re\left\{\frac{(zF')'(z)}{F'(z)}\right\} d\theta &> -(1-\rho)\left(\frac{m}{2}-1\right)\pi - \sigma\pi - 2\sigma\cos^{-1}\left(\frac{2r}{1+r^2}\right) \\ &> -\left[\frac{(A-B)(m-2)}{2(1-B)} + \sigma\right]\pi,\ (r \to 1).\end{aligned}$$

□

Remark 2.1. For $f \in k - UT_m^\delta[A,B]$, it follows that $D^\delta f$ is univalent for $2 \le m \le 4 - \frac{2\sigma}{1-\rho}$, where $\rho = \frac{1-A}{1-B}$, $\sigma = \frac{2}{\pi}\arctan(\frac{1}{k})$ and we restrict $\sigma \neq 1-\rho$.

Remark 2.2. Due to [3], Goodman introduced the class $K(\varsigma)$ of analytic functions which are close-to-convex of order $\varsigma \ge 0$. Let f be analytic and $f'(z) \neq 0$. Then for $\theta_1 < \theta_2$, $z = re^{i\theta}$

$$\int_{\theta_1}^{\theta_2} Re\left\{\frac{(zf'(z))'}{f'(z)}\right\} d\theta > -\varsigma\pi.$$

If $\varsigma = 1$, then $f \in K(1) = K$ is close-to-convex and hence univalent. We note, from Theorem 2.1, that

$$D^\delta f \in K(\varsigma), \text{ where } \varsigma = \left[\frac{(A-B)(m-2)}{2(1-B)} + \sigma\right]. \quad (11)$$

When $\delta = k = 0$, $A = 1$ and $B = -1$ we get well known reusult proved by Noor [10].

Corollary 2.2. *Let $f \in T_m$. Then for $z = re^{i\theta}$ and $\theta_1 < \theta_2$*

$$\int_{\theta_1}^{\theta_2} Re\left\{\frac{(zf'(z))'}{f'(z)}\right\} d\theta > -\frac{m}{2}\pi.$$

2.2. Arc Length Problem.

Theorem 2.3. *Let $f \in k-UT_m^{\delta}[A,B]$ and $F(z) = D^{\delta}f = z + \sum_{n=2}^{\infty} A_n z^n$. Then, for $m > \left\{\frac{(2-\sigma)}{1-\rho} - 2\right\}$, $\delta \in \mathbb{N}_0$ and $n \geq 2$ the arc length $L_r(F)$ of image of the circle $|z| = r$ under F is given by*

$$L_r(F) \leq c(m,\rho,k)n^{\alpha-1},$$

where $c(m,\rho,k)$ is constant depending on m, ρ and k and $\alpha = (1-\rho)\left(\frac{m+2}{2}\right) + \sigma$.

Proof. Let $f \in k-UT_m^{\delta}[A,B]$. Then there exists $g \in V_m^{\delta}[A,B]$ such that

$$\frac{F'(z)}{G'(z)} \in P\left(p_k(z)\right), \text{ where } F(z) = D^{\delta}f(z) \text{ and } G(z) = D^{\delta}g(z).$$

Equivalently

$$F'(z) = G'(z)p(z), \tag{12}$$

where $p(z) \in P\left(p_k(z)\right)$.

$$F'(z) = G'(z)h^{\sigma}(z), \tag{13}$$

where $h \in P$ and $\sigma = \frac{2}{\pi}\arctan(\frac{1}{k})$. Now for $z = re^{i\theta}$, we have

$$\begin{aligned} L_r(F) &= \int_0^{2\pi} |zF'(z)|\, d\theta \\ &= \int_0^{2\pi} |zG'(z)h^{\sigma}(z)|\, d\theta. \end{aligned} \tag{14}$$

Since $g \in V_m^{\delta}[A,B]$, so

$$G(z) = D^{\delta}g(z) \in V_m[A,B] \subset V_m(\rho),$$

where $\rho = \frac{1-A}{1-B}$, we have

$$G'(z) = \left(G_1'(z)\right)^{1-\rho}, \; G_1 \in V_m, \quad (\text{see [16]}). \tag{15}$$

For $G_1 \in V_m$, due to Brannan [1]

$$G_1'(z) = \frac{\left(\frac{s_1(z)}{z}\right)^{\frac{m+2}{4}}}{\left(\frac{s_2(z)}{z}\right)^{\frac{m-2}{4}}}, \quad s_1, s_2 \in S^*. \tag{16}$$

From $(14-16)$, we have

$$L_r(F) = \int_0^{2\pi} \left| z \left[\frac{\left(\frac{s_1(z)}{z}\right)^{\frac{m+2}{4}}}{\left(\frac{s_2(z)}{z}\right)^{\frac{m-2}{4}}}\right] h^{\sigma}(z)\right| d\theta, \qquad s_1, s_2 \in S^*.$$

$$\leq \quad r^{\rho}\left(\frac{4}{r}\right)^{(1-\rho)\left(\frac{m+2}{4}\right)}\int_{0}^{2\pi}|s_1(z)|^{(1-\rho)\left(\frac{m+2}{4}\right)}|h(z)|^{\sigma}\,d\theta. \qquad (17)$$

We have used distortion result for starlike function $s_2(z)$. Now by Holder's inequality together with subordination of starlike functions (17) implies

$$L_r(F) \leq 2\pi r^{\rho}\left(\frac{4}{r}\right)^{(1-\rho)\left(\frac{m+2}{4}\right)}\left[\frac{1}{2\pi}\int_{0}^{2\pi}\left\{\left(\frac{r}{|1-re^{i\theta}|}\right)^{(1-\rho)\left(\frac{m+2}{2}\right)}\right\}^{\frac{2}{2-\sigma}}d\theta\right]^{\frac{2-\sigma}{2}}$$

$$\times\left[\frac{1}{2\pi}\int_{0}^{2\pi}|h(z)|^{2}\,d\theta\right]^{\frac{\sigma}{2}}. \qquad (18)$$

Since $h(z)\in P$, so we have

$$\frac{1}{2\pi}\int_{0}^{2\pi}|h(z)|^{2}\,d\theta \leq \frac{1+3r^{2}}{1-r^{2}}. \qquad (\text{see [17]}). \qquad (19)$$

From (18) and (19), we obtain for $m > \left\{\frac{(2-\sigma)}{1-\rho}-2\right\}$

$$L_r(F) \leq c(m,\rho,k)\left(\frac{1}{1-r}\right)^{\alpha-1},$$

where $c(m,\rho,k)$ is constant depending on m, ρ and k and $\alpha=(1-\rho)\left(\frac{m+2}{2}\right)+\sigma$. Taking $r=1-\frac{1}{n}$, then we have

$$L_r(F) \leq c(m,\rho,k)n^{\alpha-1}, \qquad (n\to\infty).$$

□

2.3. Growth Rate of Coefficient.

Theorem 2.4. *Let $f\in k-UT_m^{\delta}[A,B]$ and $F(z)=D^{\delta}f(z)$. Then, for $m > \left\{\frac{2-\sigma}{1-\rho}-2\right\}$ and $\delta\in\mathbb{N}_0$*

$$|a_n| = O(1)n^{\alpha-(2+\delta)}, \qquad (20)$$

where $O(1)$ is constant depending on m, ρ and k and $\alpha=(1-\rho)\left(\frac{m+2}{2}\right)+\sigma$.

Proof. Making use of Cauchy's theorem, for $z=re^{\iota\theta}$

$$\begin{aligned} n\,|A_n| &= \frac{1}{2\pi r^{n}}\left|\int_{0}^{2\pi} zF'(z)e^{-\iota n\theta}d\theta\right| \\ &\leq \frac{1}{2\pi r^{n}}\int_{0}^{2\pi}|zF'(z)|\,d\theta \\ &= \frac{1}{2\pi r^{n}}L_r(F). \end{aligned}$$

From Theorem 2.3, we obtain

$$|A_n| \leq c_1(m,\rho,k)n^{\alpha-2}, \qquad (n\to\infty),$$

where $c_1(m, \rho, k)$ is constant depending on m, ρ and k and

$$\alpha = (1-\rho)\left(\frac{m+2}{2}\right) + \sigma.$$

Since $A_n = \left[\frac{(n+\delta-1)!}{\delta!(n-1)!}\right] a_n$, so we can easily write

$$|a_n| = O(1) n^{\alpha-(2+\delta)}, \qquad (n \to \infty),$$

where $O(1)$ is constant depending on m, ρ, δ and k with

$$\alpha = (1-\rho)\left(\frac{m+2}{2}\right) + \sigma.$$

□

2.4. The Hankel Determinant. Let $f \in \mathbf{A}$ and be given by (1). Then the *qth* Hankel determinant of $f(z)$ is given for $q \geq 1$, $n \geq 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{n+q-1} & .. & .. & a_{n+2q-2} \end{vmatrix} \tag{21}$$

The problem of determining the rate of growth of $H_q(n)$ as $n \to \infty$ for functions belonging to certain subclasses of analytic functions is well-known, see [4, 8, 10, 12, 13, 14, 15, 18].

Noonan and Thomas [8] have shown that, for a really mean p-valent functions,

$$H_q(n) = O(1)\begin{cases} n^{2p-1}; & q=1, \quad p > \frac{1}{4} \\ n^{2pq-q^2}; & q \geq 2,\ p \geq 2(q-1), \end{cases}$$

where $O(1)$ depends upon p, q and f and the exponent $\left(2pq - q^2\right)$ is best possible. For $p = 1$, Hayman [4] has shown that $H_2(n) = O(1).n^{\frac{1}{2}}$ as $n \to \infty$ and this is best possible. In [9], it was shown that if $f \in V_m$, then

$$H_q(n) = O(1)\begin{cases} n^{\frac{m}{2}-1}; & q=1, \\ n^{\frac{mq}{2}-q^2}; & q \geq 2,\ m \geq 8q-10. \end{cases}$$

The exponent $\left(\frac{mq}{2} - q^2\right)$ is best possible in some sense. Here we estimate the rate of growth of $f \in T_m(\varphi, \frac{1+Az}{1+Bz}, p_k(z))$, we need following known Lemmas, due to Noonan and Thomas [8].

Lemma 2.5. *Let $f \in A$ and be given by* (1). *Let qth Hankel determinant of f for $q \geq 1$, $n \geq 1$, be defined by* (21). *Then writing $\Delta_j(n) = \Delta_j(n, z_1, f)$, we have*

$$H_q(n) = \begin{vmatrix} \Delta_{2q-2}(n) & \Delta_{2q-3}(n+1) & \ldots & \Delta_{q-1}(n+q-1) \\ \Delta_{2q-3}(n+1) & \Delta_{2q-4}(n+2) & \ldots & \Delta_{q-2}(n+q) \\ . & . & . & . \\ . & . & . & . \\ \Delta_{q-1}(n+q-1) & .. & .. & \Delta_0(n+2q-2) \end{vmatrix},$$

where, with $\Delta_0(n, z_1, f) = a_n$*, we define for* $j \geq 1$*,*

$$\Delta_j(n, z_1, f) = \Delta_{j-1}(n, z_1, f) - z_1\Delta_{j-1}(n+1, z_1, f)$$

Lemma 2.6. *With* $x = \left(\frac{n}{n+1}y\right)$ *and* $v \geq 0$ *be any integer*

$$\Delta_j(n+v, x, zf') = \sum_{i=0}^{j} \left(\begin{array}{c} j \\ i \end{array} \right) \frac{y^i(v-(i-1)n)}{(n+1)^i}\Delta_{j-i}(n+v+i, y, f)$$

Theorem 2.7. *Let* $f \in k - UT_m^\delta[A, B]$ *and let the qth Hankel determinant of* $f(z)$ *for* $q \geq 1$*,* $n \geq 1$*, be defined by* (21)*. Then, for* $m \geq \frac{4(q-1)}{1-\rho} - 2$

$$H_q(n) = O(1).n^{\left\{(1-\rho)\left(\frac{m}{2}+1\right)+\sigma-1\right\}q-q^2},$$

where $O(1)$ *is constant depending upon* m*,* ρ *and* j *and* $\rho = \frac{1-A}{1-B}$*.*

Proof. Let $f \in k - UT_m^\delta[A, B]$. Then we can write

$$\frac{F'(z)}{G'(z)} \in P\left(p_k(z)\right), \text{ where } F(z) = D^\delta f(z) \text{ and } G(z) = D^\delta g(z).$$

Equivalently

$$F'(z) = G'(z)p(z), \text{ where } p(z) \in P\left(p_k(z)\right).$$

$$F'(z) = G'(z)h^\sigma(z), \tag{22}$$

where $h \in P$ and $\sigma = \frac{2}{\pi}\arctan(\frac{1}{k})$. Since $g \in V_m^\delta[A, B]$, so

$$G = D^\delta g \in V_m[A, B] \subset V_m(\rho),$$

where $\rho = \frac{1-A}{1-B}$, we have

$$G'(z) = \left(G_1'(z)\right)^{1-\rho}, \ G_1 \in V_m, \qquad \text{(see [16])}. \tag{23}$$

For $G_1 \in V_m$, due to Brannan [1]

$$G_1'(z) = \frac{\left(\frac{s_1(z)}{z}\right)^{\frac{m+2}{4}}}{\left(\frac{s_2(z)}{z}\right)^{\frac{m-2}{4}}}, \qquad s_1, s_2 \in S^*. \tag{24}$$

From $(22-24)$, we get

$$F'(z) = \left[\frac{\left(\frac{s_1(z)}{z}\right)^{\frac{m+2}{4}}}{\left(\frac{s_2(z)}{z}\right)^{\frac{m-2}{4}}}\right]^{(1-\rho)} .h^\sigma(z), \ \ s_1, s_2 \in S^*. \tag{25}$$

We can choose a $z_1 = z_1(r)$ with $|z| = r$ such that for any univalent function $s(z)$

$$\max_{|z|=r}|(z-z_1)s(z)| \leq \frac{2r^2}{1-r^2}; \qquad \text{(see [2])}. \tag{26}$$

Now for $j \geq 1$, z_1 be any non-zero complex number, consider

$$|\Delta_j(n, z_1, zF')| = \frac{1}{2\pi r^{n+j}} \left| \int_0^{2\pi} (z - z_1)^j \, zF'(z) e^{-i(n+j)\theta} d\theta \right|. \tag{27}$$

Putting (25) in (27), we get

$$|\Delta_j(n, z_1, zF')| \leq \frac{1}{2\pi r^{n+j}} \int_0^{2\pi} \left| (z - z_1)^j \, \frac{z \left(\frac{s_1(z)}{z} \right)^{(1-\rho)\left(\frac{m+2}{4}\right)}}{\left(\frac{s_2(z)}{z} \right)^{(1-\rho)\left(\frac{m-2}{4}\right)}} . h^{\sigma}(z) d\theta \right|. \tag{28}$$

From (26) and (28), we have for $m \geq \frac{4j}{1-\rho} - 2$

$$|\Delta_j(n, z_1, zF')| \leq \frac{1}{2\pi r^{n+j+\rho-1}} \left(\frac{2r^2}{1 - r^2} \right)^j \int_0^{2\pi} \frac{|s_1(z)|^{(1-\rho)\left(\frac{m+2}{4}\right)-j}}{|s_2(z)|^{(1-\rho)\left(\frac{m-2}{4}\right)}} |h^{\sigma}(z)| \, d\theta. \tag{29}$$

Using Holder's inequality along with employing distortion result for starlike function $s_1(z)$ and subordination for starlike function $s_2(z)$, on simplification, we obtain from (29)

$$|\Delta_j(n, z_1, zF')| \leq c(m, \rho, j) \left(\frac{1}{1 - r} \right)^j \left[\frac{1}{2\pi} \int_0^{2\pi} |s_1(z)|^{\left\{(1-\rho)\left(\frac{m+2}{2}\right)-2j\right\}\frac{2}{2-\sigma}} d\theta \right]^{\frac{2-\sigma}{2}} \times \left[\frac{1}{2\pi} \int_0^{2\pi} |h(z)|^2 \, d\theta \right]^{\frac{\sigma}{2}}.$$

$$\begin{aligned} |\Delta_j(n, z_1, zF')| &\leq c(m, \rho, j) \left(\frac{1}{1 - r} \right)^{\frac{\sigma}{2}+j} \left[\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1 - re^{i\theta}|^{\frac{(1-\rho)\left(\frac{m+2}{2}\right)-4j}{2-\sigma}}} \right]^{\frac{2-\sigma}{2}} \\ &\leq c(m, \rho, j) \left(\frac{1}{1 - r} \right)^{(1-\rho)\left(\frac{m+2}{4}\right)+\sigma-j-1}, \end{aligned}$$

where $c(m, \rho, j)$ is constant depending upon m, ρ and j. Choosing $r = 1 - \frac{1}{n}$, we have for $m \geq \frac{4j}{1-\rho} - 2$

$$|\Delta_j(n, z_1, zF')| = O(1).n^{(1-\rho)\left(\frac{m+2}{2}\right)+\sigma-j-1},$$

where $O(1)$ is constant depending upon m, ρ and j. Now applying Lemma 2.6 and putting $z_1 = \left(\frac{n}{n+1} e^{\iota\theta_n} \right)$ $(n \to \infty)$, we have for $m \geq \frac{4j}{1-\rho} - 2$

$$\left| \Delta_j(n, e^{\iota\theta_n}, F) \right| = O(1).n^{(1-\rho)\left(\frac{m+2}{4}\right)+\sigma-j-2}.$$

We use Lemma 2.5 and follow the similar arguments given in [8], we get for $m \geq \frac{4(q-1)}{1-\rho} - 2$

$$H_q(n) = O(1).n^{\left\{(1-\rho)\left(\frac{m+2}{4}\right)+\sigma-1\right\}q-q^2}.$$

□

3. Conclusion

The main aim of this paper is to define a new subclass of analytic functions by applying Ruscheweyh derivative operator. These classes are generalization of many of the well-known classes. We have discussed necessary condition, arc length problem, growth rate of coefficient and the Hankel determinant problem for the newly defined class. In these investigations concepts of Janowski functions and conic domains were used.

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