J. Appl. Math. & Informatics Vol. **39***(2021), No. 1 - 2, pp. 125 - 131*
https://doi.org/10.14317/jami.2021.125

THE RELATION BETWEEN THE NUMERICAL RANGE $W(\mathbf{A}^n)$ AND $W(\mathbf{A})$ FOR THE 2×2 COMPLEX MATRIX

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ABSTRACT. In the paper, we investigate the representation of the numerical range $W(\mathbf{A}^n)$ for the 2×2 complex matrix $\mathbf{A}$, in terms of the numerical range $W(\mathbf{A})$ of the matrix $\mathbf{A}$, and the elements of $\mathbf{A}$ or the eigenvalue of $\mathbf{A}$.

AMS Mathematics Subject Classification : 49M37, 65L07, 65L60, 90C30.
Key words and phrases : Numerical range, square of matrix, Schur decomposition, Unitary similar, eigenvalue.

1. Introduction

Let us consider the square complex matrices $\mathbf{A} \in \mathbb{C}^{n\times n}$ given by the following form:

$$\mathbf{A} := \begin{pmatrix} a_{11} & a_{12} & \cdots a_{1n} \\ a_{21} & a_{22} & \cdots a_{2n} \\ \vdots & \vdots & \ddots \vdots \\ a_{n1} & a_{n2} & \cdots a_{nn} \end{pmatrix} \tag{1.1}$$

where all the elements are complex number. Denote the numerical range of the matrix $\mathbf{A}$ by

$$W(\mathbf{A}) := \{\mathbf{x}^*\mathbf{A}\mathbf{x} \mid \mathbf{x} \in \mathbb{C}^n \text{ with } \mathbf{x}^*\mathbf{x} = 1\} \subset \mathbb{C}. \tag{1.2}$$

Here the notation $*$ means the conjugate transpose. The numerical range of a matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ has been studied for over 100 years. In 1918, Toeplitz and Hausdorff have proved that the numerical ranges are convex [12, 6]. The fact that $W(\mathbf{A})$ is an ellipse in case $n = 2$ is often used to prove convexity of $W(\mathbf{A})$ for arbitrary n.

We consider the 2×2 complex matrix $\mathbf{A}$ and we study an explicit expression of the ellipse which is formed from the numerical range $W(\mathbf{A})$ in terms of the

Received July 23, 2020. Revised September 14, 2020. Accepted October 13, 2020.
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four elements of a matrix $\mathbf{A}$. Therefore the center, the direction and the length of the half-axes of the ellipse is described in terms of the elements of a matrix $\mathbf{A}$. It is well known fact that the numerical range $W(\mathbf{A})$ of the 2×2 complex matrix $\mathbf{A}$ is generally ellipse with foci as two eigenvalues of $\mathbf{A}$.

Nevertheless, there is little known about the properties of the numerical range for the operation of matrix. For example, there are only these things such as $W(\alpha\mathbf{I} + \beta\mathbf{A}) = \alpha + \beta W(\mathbf{A})$, $W(\mathbf{U}^*\mathbf{A}\mathbf{U}) = W(\mathbf{A})$ for unitary matrix $\mathbf{U}$, $W(\mathbf{A}+\mathbf{B}) \subset W(\mathbf{A})+W(\mathbf{B})$ etc. So, in this paper, we investigate the numerical range $W(\mathbf{A}^n)$ for the 2×2 complex matrix $\mathbf{A}$.

2. Preliminaries

In this section, we introduce some notations, definitions and basic properties related to the numerical range.

The numerical range $W(\mathbf{A})$ of a matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ is the set of complex numbers. It is well known (see [7]) that $W(\mathbf{A})$ is a convex compact subset of $\mathbb{C}$, which contains all the eigenvalues of $\mathbf{A}$. The following basic properties of the numerical range $W(\mathbf{A})$ can be easily proved. [4, 7, 13]

Proposition 2.1. *Let $\mathbf{A}$ and $\mathbf{B}$ be an $n \times n$ complex matrix. Then we have the following:*

(a) $W(\alpha\mathbf{I} + \beta\mathbf{A}) = \alpha + \beta W(\mathbf{A})$, *for any* $\alpha, \beta \in \mathbb{C}$.
(b) $W(\mathbf{A}^*) = \{\overline{\lambda} \mid \lambda \in W(\mathbf{A})\} = \overline{W(\mathbf{A})}$.
(c) $W(\mathbf{U}^*\mathbf{A}\mathbf{U}) = W(\mathbf{A})$, *for any unitary matrix* $\mathbf{U}$.

In the sequel, we deal with 2×2 complex matrices $\mathbf{A} \in \mathbb{C}^{2\times 2}$ whose four elements are complex number $a, b, c, d \in \mathbb{C}$ as following:

$$\mathbf{A} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.1}$$

Definition 2.1. The numerical range $W(\mathbf{A})$ of a 2×2 complex matrix $\mathbf{A}$ is defined by

$$W(\mathbf{A}) = \{\mathbf{x}^*\mathbf{A}\mathbf{x} \in \mathbb{C} \mid \mathbf{x} \in \mathbb{C}^2, \ \mathbf{x}^*\mathbf{x} = 1\}. \tag{2.2}$$

Since $\mathbf{A}$ is a 2×2 complex matrix and $\mathbf{x} = (x, y)^T \in \mathbb{C}^2$, then the composite form $\mathbf{x}^*\mathbf{A}\mathbf{x} = a|x|^2 + b\bar{x}y + cx\bar{y} + d|y|^2$ assumes the complex values. Hence the numerical range $W(\mathbf{A})$ is the subset of complex numbers and induce a region in the complex plane which is covered by these values under the hypothesis that the number $\mathbf{x}^*\mathbf{x} = |x|^2 + |y|^2$ of $\mathbf{x}$ has the value unity.

For 2×2 complex matrices $\mathbf{A}$, a complete description of the numerical range $W(\mathbf{A})$ is well known. Namely, $W(\mathbf{A})$ is an ellipse with foci at the eigenvalues λ_1, λ_2 of $\mathbf{A}$ and a minor axis of the length $s = (\mathrm{tr}(\mathbf{A}^*\mathbf{A}) - |\lambda_1|^2 - |\lambda_2|^2)^{1/2}$. If a matrix $\mathbf{A}$ is normal, it can be unitary equivalent to a diagonal matrix whose diagonal elements are the eigenvalues λ_1 and λ_2. In this case, $s = 0$ and the ellipse degenerate into a line segment connecting λ_1 and λ_2. On the other hand, for $\mathbf{A}$ with coinciding eigenvalues, the ellipse $W(\mathbf{A})$ degenerates into a circle.

3. Main Results

This section begins with an introduction to the Schur decomposition theorem for the $n \times n$ complex matrix.

Theorem 3.1. *Let* $\mathbf{A}$ *be an* $n \times n$ *complex matrix. Then the matrix* $\mathbf{A}$ *has the Schur decomposition as the following:*

$$\mathbf{A} = \mathbf{UTU}^*,$$

where $\mathbf{U}$ *is an unitary matrix,* $\mathbf{U}^*$ *is a conjugate transpose of* $\mathbf{U}$*, and* $\mathbf{T}$ *is an upper triangular matrix.*

Now, in accordance with this theorem, we will find the Schur decomposition of the 2×2 complex matrix $\mathbf{A}$ given by

$$\mathbf{A} \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.1}$$

First, let λ_1 be an eigenvalue of the matrix $\mathbf{A}$ and $\mathbf{x}$ be an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ_1. Then we have

$$\mathbf{Ax} \;=\; \lambda_1 \mathbf{x}, \qquad \mathbf{x} \neq \mathbf{0}. \tag{3.2}$$

From the equation (3.2), we get the equation $ax + by = \lambda_1 x$, for the eigenvector $\mathbf{x} = (x, y)^T \in \mathbb{C}^2$. So we can choose an eigenvector $\mathbf{x}$ as following:

$$\mathbf{x} \;=\; \begin{pmatrix} b \\ \lambda_1 - a \end{pmatrix} \tag{3.3}$$

Here we can find the QR-factorization of $\mathbf{x}$ such as

$$\mathbf{x} \;=\; \begin{pmatrix} b/R & -e^{-i\theta}(\bar{\lambda_1} - \bar{a})/R \\ (\lambda_1 - a)/R & e^{-i\theta}\bar{b}/R \end{pmatrix} \begin{pmatrix} R \\ 0 \end{pmatrix}, \tag{3.4}$$

where $R = \sqrt{|b|^2 + |\lambda_1 - a|^2}$. For the sake of convenience, we take $R = 1$, so the QR-factorization of $\mathbf{x}$, i.e. $\mathbf{x} = \mathbf{UR}$, is rewritten by

$$\mathbf{x} \;=\; \mathbf{UR} \;:=\; \begin{pmatrix} b & -e^{-i\theta}(\bar{\lambda_1} - \bar{a}) \\ \lambda_1 - a & e^{-i\theta}\bar{b} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{3.5}$$

By direct computation, we can check the property of the matrix $\mathbf{U}$ such that $\mathbf{UU}^* = \mathbf{I}$. So, the matrix $\mathbf{U}$ is unitary. Also, by substituting the equation (3.5) into (3.2), we have

$$\mathbf{AUR} \;=\; \lambda_1 \mathbf{UR} \tag{3.6}$$

or

$$\mathbf{U}^*\mathbf{AU}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \;=\; \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix}. \tag{3.7}$$

In order to satisfy the equation (3.7), the elements of the first column of $\mathbf{U}^*\mathbf{AU}$ must be λ_1 and 0.

Using the unitary matrix $\mathbf{U}$ in equation (3.5) and the given matrix $\mathbf{A}$, we get the very meaningful equation by direct computation as following:

$$\mathbf{U}^*\mathbf{A}\mathbf{U} \;=\; \begin{pmatrix} \lambda_1 & \xi \\ 0 & \lambda_2 \end{pmatrix}. \tag{3.8}$$

Here λ_2 is the another eigenvalue of $\mathbf{A}$ and ξ is the complex number satisfying $|\xi|^2 = \mathrm{tr}(\mathbf{A}^*\mathbf{A}) - |\lambda_1|^2 - |\lambda_2|^2$.

Hence we get the following theorem that obtain Schur decomposition of the 2×2 complex matrix.

Theorem 3.2. *Let $\mathbf{A}$ be a 2×2 complex matrix given by (3.1). Then the matrix $\mathbf{A}$ has the Schur decomposition as the following:*

$$\mathbf{A} = \mathbf{U}\mathbf{T}\mathbf{U}^*, \tag{3.9}$$

where the unitary matrix $\mathbf{U}$ and the upper triangular matrix $\mathbf{T}$ is given as

$$\mathbf{U} = \frac{1}{R}\begin{pmatrix} b & -e^{-i\theta}(\bar{\lambda_1} - \bar{a}) \\ \lambda_1 - a & e^{-i\theta}\bar{b} \end{pmatrix}, \qquad \mathbf{T} = \begin{pmatrix} \lambda_1 & \xi \\ 0 & \lambda_2 \end{pmatrix}.$$

So, many authors show that the numerical range of a 2×2 complex matrix $\mathbf{A}$ is ellipse whose foci are eigenvalues of the matrix $\mathbf{A}$, by using the unitary similar form (3.9) and the numerical range of the upper triangular matrix $\mathbf{T}$ [2, 4, 8, 12, 13].

Now we will find a relation between the numerical ranges $W(\mathbf{A}^2)$ and $W(\mathbf{A})$ for the 2×2 complex matrix $\mathbf{A}$ by using the Theorem 3.2. We obtain the following theorem.

Theorem 3.3. *Let $\mathbf{A}$ be a 2×2 complex matrix given by (3.1). Let λ_1 and λ_2 be the eigenvalues of the matrix $\mathbf{A}$. Then we have*

$$W(\mathbf{A}^2) \;=\; (\lambda_1 + \lambda_2)W(\mathbf{A}) - \lambda_1\lambda_2. \tag{3.10}$$

Proof. First, from Theorem 3.2, we have shown that a 2×2 complex matrix $\mathbf{A}$ has Schur decomposition $\mathbf{A} = \mathbf{U}\mathbf{T}\mathbf{U}^*$, where the upper triangular matrix $\mathbf{T}$ is given as follows

$$\mathbf{T} = \begin{pmatrix} \lambda_1 & \xi \\ 0 & \lambda_2 \end{pmatrix},$$

for some complex number ξ satisfying $|\xi|^2 = \mathrm{tr}(\mathbf{A}^*\mathbf{A}) - |\lambda_1|^2 - |\lambda_2|^2$. Since $\mathbf{U}$ is the unitary matrix, by Proposition 2.1 (c), we have $W(\mathbf{A}) = W(\mathbf{T})$.

Now, we investigate the numerical range of $\mathbf{A}^2$. By the Schur decomposition of $\mathbf{A}$, we have

$$\mathbf{A}^2 = (\mathbf{U}\mathbf{T}\mathbf{U}^*)(\mathbf{U}\mathbf{T}\mathbf{U}^*) = \mathbf{U}\mathbf{T}^2\mathbf{U}^*. \tag{3.11}$$

By Proposition 2.1 (c), we have $W(\mathbf{A}^2) = W(\mathbf{U}\mathbf{T}^2\mathbf{U}^*) = W(\mathbf{T}^2)$.

Since

$$\mathbf{T}^2 \;=\; \begin{pmatrix} \lambda_1 & \xi \\ 0 & \lambda_2 \end{pmatrix}^2 = \begin{pmatrix} \lambda_1^2 & \xi(\lambda_1 + \lambda_2) \\ 0 & \lambda_2^2 \end{pmatrix}$$

$$\begin{aligned} &= (\lambda_1+\lambda_2)\begin{pmatrix} \lambda_1 & \xi \\ 0 & \lambda_2 \end{pmatrix} - \lambda_1\lambda_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= (\lambda_1+\lambda_2)\mathbf{T} - \lambda_1\lambda_2\mathbf{I}, \end{aligned}$$

we have, by Proposition 2.1 (a),

$$\begin{aligned} W(\mathbf{T}^2) &= W\left((\lambda_1+\lambda_2)\mathbf{T} - \lambda_1\lambda_2\mathbf{I}\right) \\ &= (\lambda_1+\lambda_2)W(\mathbf{T}) - \lambda_1\lambda_2. \end{aligned}$$

Since $W(\mathbf{A}) = W(\mathbf{T})$ and $W(\mathbf{A}^2) = W(\mathbf{T}^2)$, we have

$$W(\mathbf{A}^2) = (\lambda_1+\lambda_2)W(\mathbf{A}) - \lambda_1\lambda_2.$$

□

The next corollary is stated without proof.

Corollary 3.4. *Let* $\mathbf{A}$ *be a* 2×2 *complex matrix given by (3.1). Let* λ_1 *and* λ_2 *be the eigenvalues of the matrix* $\mathbf{A}$. *Then we have*

$$\begin{aligned} W(\mathbf{A}^2) &= (\lambda_1+\lambda_2)\cdot W(\mathbf{A}) - \lambda_1\lambda_2 \\ &= (a+d)\cdot W(\mathbf{A}) - (ad-bc) \\ &= \operatorname{tr}(\mathbf{A})\cdot W(\mathbf{A}) - \det(\mathbf{A}) \end{aligned}$$

Finally, we will find a relation between the numerical ranges $W(\mathbf{A}^n)$ and $W(\mathbf{A})$ for the 2×2 complex matrix $\mathbf{A}$ by using the Theorem 3.2 and the Theorem 3.3.

Theorem 3.5. *Let* $\mathbf{A}$ *be a* 2×2 *complex matrix. Let* λ_1 *and* λ_2 *be the eigenvalues of the matrix* $\mathbf{A}$. *Then we have, for every* $n \in \mathbb{N}$,

$$W(\mathbf{A}^{n+1}) = f_n W(\mathbf{A}) - g_n, \tag{3.12}$$

where the sequence (f_n) *and* (g_n) *are defined by*

$$\begin{cases} f_1 = \lambda_1+\lambda_2, & g_1 = \lambda_1\lambda_2 \\ f_{n+1} = (\lambda_1+\lambda_2)f_n - g_n, & g_{n+1} = \lambda_1\lambda_2 f_n, \quad n \geq 1 \end{cases}.$$

Proof. By Theorem 3.2, for the 2×2 complex matrix $\mathbf{A}$, we have the Schur decomposition $\mathbf{A} = \mathbf{U}\mathbf{T}\mathbf{U}^*$, where the unitary matrix $\mathbf{U}$ and the upper triangular matrix $\mathbf{T}$.

Then we have

$$\mathbf{T} = \begin{pmatrix} \lambda_1 & \xi \\ 0 & \lambda_2 \end{pmatrix} = \mathbf{U}^*\mathbf{A}\mathbf{U},$$

and $W(\mathbf{A}) = W(\mathbf{T})$. Since $\mathbf{T}^2 = \mathbf{U}^*\mathbf{A}^2\mathbf{U}$ and $\mathbf{T}^2 = (\lambda_1+\lambda_2)\mathbf{T} - \lambda_1\lambda_2\mathbf{I}$, we have $W(\mathbf{A}^2) = W(\mathbf{T}^2)$ and

$$W(\mathbf{A}^2) = (\lambda_1+\lambda_2)W(\mathbf{A}) - \lambda_1\lambda_2.$$

We define $f_1 = \lambda_1+\lambda_2$ and $g_1 = \lambda_1\lambda_2$. Then we have $\mathbf{T}^2 = f_1\mathbf{T} - g_1\mathbf{I}$. At first, by mathematical induction we prove that for every $n \in \mathbb{N}$,

$$\mathbf{T}^{n+1} = f_n\mathbf{T} - g_n\mathbf{I}.$$

We assume that $\mathbf{T}^{k+1} = f_k\mathbf{T} - g_k\mathbf{I}$, for $k \in \mathbb{N}$, Then we have

$$\begin{aligned}
\mathbf{T}^{k+2} &= \mathbf{T}^{k+1}\mathbf{T} \\
&= (f_k\mathbf{T} - g_k\mathbf{I})\,\mathbf{T} \\
&= f_k\mathbf{T}^2 - g_k\mathbf{T} \\
&= f_k\left\{(\lambda_1+\lambda_2)\mathbf{T} - \lambda_1\lambda_2\mathbf{I}\right\} - g_k\mathbf{T} \\
&= \left\{(\lambda_1+\lambda_2)f_k - g_k\right\}\mathbf{T} - \lambda_1\lambda_2 f_k\mathbf{I} \\
&= f_{k+1}\mathbf{T} - g_{k+1}\mathbf{I}
\end{aligned}$$

Hence we proved that $\mathbf{T}^{n+1} = f_n\mathbf{T} - g_n\mathbf{I}$, for every $n \in \mathbb{N}$. Therefore we have

$$W(\mathbf{A}^{n+1}) = W(\mathbf{T}^{n+1}) = f_nW(\mathbf{T}) - g_n = f_nW(\mathbf{A}) - g_n,$$

for every $n \in \mathbb{N}$. □

4. Concluding Remarks

So far, we studied the numerical range $W(\mathbf{A}^n)$ of the power of the matrix $\mathbf{A}$ for the case of the 2×2 complex matrix. Based on the property $W(\mathbf{U}^*\mathbf{A}\mathbf{U}) = W(\mathbf{A})$, we find the Schur decomposition of $\mathbf{A}$, in other words, the unitary matrix $\mathbf{U}$ and upper triangular matrix $\mathbf{T}$ satisfying $\mathbf{A} = \mathbf{U}\mathbf{T}\mathbf{U}^*$. Also we represent the numerical range $W(\mathbf{A}^n)$ in terms of both the numerical range $W(\mathbf{A})$ and the eigenvalues of $\mathbf{A}$. For the further study, in the case of the $m\times m$ complex matrix, we may expect these similar properties.

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