J. Appl. Math. & Informatics Vol. **39***(2021), No. 1 - 2, pp. 105 - 116*
https://doi.org/10.14317/jami.2021.105

DISTANCE SPACES, ALEXANDROV PRETOPOLOGIES AND JOIN-MEET OPERATORS†

YOUNG-HEE KIM, YONG CHAN KIM* AND JONGSUNG CHOI

Abstract. Information systems and decision rules with imprecision and uncertainty in data analysis are studied in complete residuated lattices. In this paper, we introduce the notions of distance spaces, Alexandrov pretopology (precotopology) and join-meet (meet-join) operators in complete co-residuated lattices. We investigate their relations and properties. Moreover, we give their examples.

AMS Mathematics Subject Classification : 03E72, 54A40, 54B10.
Key words and phrases : Complete co-residuated lattice, distance spaces, Alexandrov pretopology (precotopology), join-meet (meet-join) operators.

1. Introduction

Ward et al.[24] introduced a complete residuated lattice which is an important mathematical tool for many valued logics [1-12,20,21]. Pawlak [16,17] introduced the rough set theory as a formal tool to deal with imprecision and uncertainty in the data analysis. For an extension of Pawlak's rough sets, many researchers[1-12, 20,21] developed L-lower and L-upper approximation operators in complete residuated lattices.

Zheng et al.[25] introduced a complete co-residuated lattice as the generalization of t-conorm. Junsheng et al.[7] investigated $(\odot, \&)$-generalized fuzzy rough set on $(L, \vee, \wedge, \odot, \&, 0, 1)$ where $(L, \vee, \wedge, \&, 0, 1)$ is a complete residuated lattice and $(L, \vee, \wedge, \odot, 0, 1)$ is complete co-residuated lattice in a sense [13].

An interesting and natural research topic in rough set theory is the study topological structures. Lai [13] and Ma [14] investigated the Alexandrov L-topology and lattice structures on L-fuzzy rough sets determined by lower and upper sets.

Received October 25, 2020. Revised December 19, 2020. Accepted December 29, 2020.
*Corresponding author.
† The present Research has been conducted by the Research Grant of Kwangwoon University in 2019.

Kim et al. [8-12] studied the properties of fuzzy join and meet completeness, L-fuzzy upper and lower approximation spaces and Alexandrov L-topologies with fuzzy partially ordered spaces in complete residuated lattices.

In this paper, we introduce the notions of distance spaces, Alexandrov pre-topology (precotopology) and join-meet (meet-join) operators in complete co-residuated lattices. We investigate their relations and properties. Moreover, we give their examples.

2. Preliminaries

Definition 2.1. [7,26] An algebra $(L, \wedge, \vee, \oplus, 0, 1)$ is called a *complete co-residuated lattice* if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, 0, 1)$ is a complete lattice where 0 is the bottom element and 1 is the top element.

(C2) $a = a \oplus 0$, $a \oplus b = b \oplus a$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ for all $a, b, c \in L$.

(C3) $(\bigwedge_{i\in\Gamma} a_i) \oplus b = \bigwedge_{i\in\Gamma}(a_i \oplus b)$.

Let $(L, \leq, \oplus)$ be a complete co-residuated lattice. For each $x, y \in L$, we define

$$x \ominus y = \bigwedge\{z \in L \mid y \oplus z \geq x\}.$$

Then $(x \oplus y) \geq z$ iff $x \geq (z \ominus y)$.

For $\alpha \in L, A \in L^X$, we denote $(\alpha \ominus A), (\alpha \oplus A), \alpha_X \in L^X$ as $(\alpha \ominus A)(x) = \alpha \ominus A(x)$, $(\alpha \oplus A)(x) = \alpha \oplus A(x)$, $\alpha_X(x) = \alpha$.

Put $n(x) = 1 \ominus x$. The condition $n(n(x)) = x$ for each $x \in L$ is called a *double negative law*.

Remark 2.1. (1) An infinitely distributive lattice $(L, \leq, \vee, \wedge, \oplus = \vee, 0, 1)$ is a complete co-residuated lattice. In particular, the unit interval $([0, 1], \leq, \vee, \wedge, \oplus = \vee, 0, 1)$ is a complete co-residuated lattice where

$$\begin{aligned} x \ominus y &= \bigwedge\{z \in L \mid y \vee z \geq x\} \\ &= \begin{cases} 0, & \text{if } y \geq x, \\ x, & \text{if } y \not\geq x. \end{cases} \end{aligned}$$

Put $n(x) = 1 \ominus x = 1$ for $x \neq 1$ and $n(1) = 0$. Then $n(n(x)) = 0$ for $x \neq 1$ and $n(n(1)) = 1$. Hence n does not satisfy a double negative law.

(2) The unit interval with a right-continuous t-conorm $\oplus$, $([0, 1], \leq, \oplus)$, is a complete co-residuated lattice [23].

(3) $([1, \infty], \leq, \vee, \oplus = \cdot, \wedge, 1, \infty)$ is a complete co-residuated lattice where

$$\begin{aligned} x \ominus y &= \bigwedge\{z \in [1, \infty] \mid yz \geq x\} \\ &= \begin{cases} 1, & \text{if } y \geq x, \\ \frac{x}{y}, & \text{if } y \not\geq x. \end{cases} \end{aligned}$$

$$\infty \cdot a = a \cdot \infty = \infty, \forall a \in [1, \infty], \infty \ominus \infty = 1.$$

Put $n(x) = \infty \ominus x = \infty$ for $x \neq \infty$ and $n(\infty) = 1$. Then $n(n(x)) = 1$ for $x \neq \infty$ and $n(n(\infty)) = \infty$. Hence n does not satisfy a double negative law.

(4) $([0,\infty],\leq,\vee,\oplus=+,\wedge,0,\infty)$ is a complete co-residuated lattice where

$$\begin{aligned}&y\ominus x=\bigwedge\{z\in[0,\infty]\mid x+z\geq y\}\\&=\bigwedge\{z\in[0,\infty]\mid z\geq -x+y\}=(y-x)\vee 0,\\&\infty+a=a+\infty=\infty,\forall a\in[0,\infty],\infty\ominus\infty=0.\end{aligned}$$

Put $n(x)=\infty\ominus x=\infty$ for $x\neq\infty$ and $n(\infty)=0$. Then $n(n(x))=0$ for $x\neq\infty$ and $n(n(\infty))=\infty$. Hence n does not satisfy a double negative law.

(5) $([0,1],\leq,\vee,\oplus,\wedge,0,1)$ is a complete co-residuated lattice where

$$\begin{aligned}&x\oplus y=(x^p+y^p)^{\frac{1}{p}}\wedge 1,\ \ 1\leq p<\infty,\\&x\ominus y=\bigwedge\{z\in[0,1]\mid (z^p+y^p)^{\frac{1}{p}}\geq x\}\\&=\bigwedge\{z\in[0,1]\mid z\geq (x^p-y^p)^{\frac{1}{p}}\}=(x^p-y^p)^{\frac{1}{p}}\vee 0,\end{aligned}$$

Put $n(x)=1\ominus x=(1-x^p)^{\frac{1}{p}}$ for $1\leq p<\infty$. Then $n(n(x))=x$ for $x\in[0,1]$. Hence n satisfies a double negative law.

(6) Let $P(X)$ be the collection of all subsets of X. Then $(P(X),\subset,\cup,\cap,\oplus=\cup,\emptyset,X)$ is a complete co-residuated lattice where

$$\begin{aligned}&A\ominus B=\bigwedge\{C\in P(X)\mid B\cup C\supset A\}\\&=A\cap B^c=A-B.\end{aligned}$$

Put $n(A)=X\ominus A=A^c$ for each $A\subset X$. Then $n(n(A))=A$. Hence n satisfies a double negative law.

Lemma 2.2. [11] Let $(L,\wedge,\vee,\oplus,\ominus,0,1)$ be a complete co-residuated lattice. For each $x,y,z,x_i,y_i\in L$, we have the following properties.

(1) If $y\leq z$, $x\oplus y\leq x\oplus z$, $y\ominus x\leq z\ominus x$ and $x\ominus z\leq x\ominus y$.

(2) $(\bigvee_{i\in\Gamma}x_i)\ominus y=\bigvee_{i\in\Gamma}(x_i\ominus y)$ and $x\ominus(\bigwedge_{i\in\Gamma}y_i)=\bigvee_{i\in\Gamma}(x\ominus y_i)$.

(3) $(\bigwedge_{i\in\Gamma}x_i)\ominus y\leq\bigwedge_{i\in\Gamma}(x_i\ominus y)$

(4) $x\ominus(\bigvee_{i\in\Gamma}y_i)\leq\bigwedge_{i\in\Gamma}(x\ominus y_i)$.

(5) $x\ominus x=0$, $x\ominus 0=x$ and $0\ominus x=0$. Moreover, $x\ominus y=0$ iff $x\leq y$.

(6) $y\oplus(x\ominus y)\geq x$, $y\geq x\ominus(x\ominus y)$ and $(x\ominus y)\oplus(y\ominus z)\geq x\ominus z$.

(7) $x\ominus(y\oplus z)=(x\ominus y)\ominus z=(x\ominus z)\ominus y$.

(8) $x\ominus y\geq(x\oplus z)\ominus(y\oplus z)$, $y\ominus x\geq(z\ominus x)\ominus(z\ominus y)$ and $(x\oplus y)\ominus(z\oplus w)\leq(x\ominus z)\oplus(y\ominus w)$.

(9) $x\oplus y=0$ iff $x=0$ and $y=0$.

(10) $(x\oplus y)\ominus z\leq x\oplus(y\ominus z)$ and $(x\ominus y)\oplus z\geq x\ominus(y\ominus z)$.

(11) If L satisfies a double negative law and $n(x)=1\ominus x$, then $n(x\oplus y)=n(x)\ominus y=n(y)\ominus x$ and $x\ominus y=n(y)\ominus n(x)$.

Definition 2.3. [11] Let $(L,\wedge,\vee,\oplus,\ominus,0,1)$ be a complete co-residuated lattice. Let X be a set. A function $d_X:X\times X\to L$ is called a *distance function* if it satisfies the following conditions:

(M1) $d_X(x,x)=0$ for all $x\in X$,

(M2) $d_X(x,y)\oplus d_X(y,z)\geq d_X(x,z)$, for all $x,y,z\in X$.

The pair (X,d_X) is called a *distance space*.

Remark 2.2. [11] (1) We define a distance function $d_X : X \times X \to [0, \infty]$. Then (X, d_X) is called a pseudo-quasi-metric space.

(2) Let $(L, \wedge, \vee, \oplus, \ominus, 0, 1)$ be a complete co-residuated lattice. Define a function $d_L : L \times L \to L$ as $d_L(x, y) = x \ominus y$. By Lemma 2.3 (5) and (6), (L, d_L) is a distance space.

3. Distance spaces, Alexandrov pretopologies and join-meet operators

In this section, we assume $(L, \wedge, \vee, \oplus, \ominus, 0, 1)$ is a complete co-residuated lattice with a double negative law $n(x) = 1 \ominus x$.

Definition 3.1. (1) A subset $\tau \subset L^X$ is called an *Alexandrov pretopology* on X iff it satisfies the following conditions:

(O1) $\alpha_X \in \tau$.

(O2) If $A_i \in \tau$ for all $i \in I$, then $\bigvee_{i \in I} A_i \in \tau$.

(O3) If $A \in \tau$ and $\alpha \in L$, then $A \ominus \alpha \in \tau$.

(2) A subset $\eta \subset L^X$ is called an *Alexandrov precotopology* on X iff it satisfies the following conditions:

(CO1) $\alpha_X \in \eta$.

(CO2) If $A_i \in \eta$ for all $i \in I$, then $\bigwedge_{i \in I} A_i \in \eta$.

(CO3) If $A \in \eta$ and $\alpha \in L$, then $\alpha \oplus A \in \eta$.

A subset $\tau \subset L^X$ is called an *Alexandrov topology* on X iff it is both Alexandrov pretopology and Alexandrov precotopology on X.

Definition 3.2. A map $\mathcal{K} : L^X \to L^X$ is called a *meet-join operator* if it satisfies the following conditions:

(K1) $\mathcal{K}(\alpha_X) = n(\alpha_X)$,

(K2) $\mathcal{K}(A) \leq n(A)$, for $A \in L^X$,

(K3) $\mathcal{K}(A \oplus \alpha) \geq \mathcal{K}(A) \ominus \alpha$ for each $\alpha \in L, A \in L^X$ and $\mathcal{K}(B) \leq \mathcal{K}(A)$ for $A \leq B$.

The pair $(X, \mathcal{K})$ is called a *meet-join space*.

Definition 3.3. A map $\mathcal{D} : L^X \to L^X$ is called a *join-meet operator* if it satisfies the following conditions:

(D1) $\mathcal{D}(\alpha_X) = n(\alpha_X)$,

(D2) $n(A) \leq \mathcal{D}(A)$, for $A \in L^X$,

(D3) $\alpha \oplus \mathcal{D}(A) \geq \mathcal{D}(A \ominus \alpha)$ for each $\alpha \in L, A \in L^X$ and $\mathcal{D}(A) \geq \mathcal{D}(B)$ for $A \leq B$.

The pair $(X, \mathcal{D})$ is called a *join-meet space*.

Theorem 3.4. *Let $\mathcal{K}_X : L^X \to L^X$ be a meet-join operator. Then the following properties hold.*

(1) *Define $\tau_{\mathcal{K}_X} = A \in L^X \mid A = \mathcal{K}_X(n(A))\}$. Then $\tau_{\mathcal{K}_X}$ is an Alexandrov pretopology on X.*

(2) *Define $d_{\mathcal{K}_X}(x, y) = \bigvee_{A \in L^X} (\mathcal{K}_X(n(A))(y) \ominus \mathcal{K}_X(n(A))(x))$. Then $d_{\mathcal{K}_X}$ is a distance function.*

(3) *If d_X is a distance function. Define $\mathcal{K}_{d_X}(A)(y) = \bigwedge_{z\in X}(d_X(z,y) \oplus n(A)(z))$. Then $\mathcal{K}_{d_X}$ is a meet-join operator. Moreover, $\mathcal{K}_{d_{\mathcal{K}_X}}(A) \geq \mathcal{K}_X(A)$ and $d_{\mathcal{K}_{d_X}} = d_X$.*

Proof. (1) (O1) Since $\mathcal{K}_X(n(\alpha_X)) = n(n(\alpha_X)) = \alpha_X$, $\alpha_X \in \tau_{\mathcal{K}_X}$.

(O2) If $A_i \in \tau_{\mathcal{K}_X}$ for all $i \in I$, by (K2), then $\mathcal{K}_X(n(\bigvee_{i\in I} A_i)) \leq n(n(\bigvee_{i\in I} A_i)) = \bigvee_{i\in I} A_i$. By Lemma 2.3(2), $n(\bigwedge_{i\in I} y_i) = 1\ominus\bigwedge_{i\in I} y_i = \bigvee_{i\in I}(1\ominus y_i) = \bigvee_{i\in I} n(y_i)$. Put $y_i = n(x_i)$. Then $n(\bigwedge_{i\in I} n(x_i)) = \bigvee_{i\in I} n(n(x_i)) = \bigvee_{i\in I} x_i$. Thus $n(\bigvee_{i\in I} x_i) = \bigwedge_{i\in I} n(x_i)$. By (K3), $\mathcal{K}_X(n(\bigvee_{i\in I} A_i)) = \mathcal{K}_X(\bigwedge_{i\in I} n(A_i)) \geq \bigvee_{i\in I} \mathcal{K}_X(n(A_i)) = \bigvee_{i\in I} A_i$.

So, $\bigvee_{i\in I} A_i \in \tau_{\mathcal{K}_X}$.

(O3) Let $A \in \tau_{\mathcal{K}_X}$ and $\alpha \in L$. Then $A \ominus \alpha \in \tau_{\mathcal{K}_X}$ from:

$$\begin{aligned}
&A \ominus \alpha \geq \mathcal{K}_X(n(A \ominus \alpha)) = \mathcal{K}_X(n(A) \oplus \alpha)\\
&\text{(by Lemma 2.3(11) and (K3))}\\
&\geq \mathcal{K}_X(n(A)) \ominus \alpha = A \ominus \alpha.
\end{aligned}$$

(2) (M1) $d_{\mathcal{K}_X}(x,x) = \bigvee_{A\in L^X}(\mathcal{K}_X(n(A))(x) \ominus \mathcal{K}_X(n(A))(x)) = 0$.

(M2) For each $x, y, z \in X$,

$$\begin{aligned}
&d_{\mathcal{K}_X}(x,z) = \bigvee_{A\in L^X}(\mathcal{K}_X(n(A))(z) \ominus \mathcal{K}_X(n(A))(x))\\
&\leq \bigvee_{A\in L^X}\Big((\mathcal{K}_X(n(A))(z) \ominus \mathcal{K}_X(n(A))(y))\\
&\oplus(\mathcal{K}_X(n(A))(y) \ominus \mathcal{K}_X(n(A))(x))\Big)\\
&\leq \bigvee_{A\in L^X}(\mathcal{K}_X(n(A))(z) \ominus \mathcal{K}_X(n(A))(y))\\
&\oplus \bigvee_{A\in L^X}(\mathcal{K}_X(n(A))(y) \ominus \mathcal{K}_X(n(A))(x))\\
&= d_{\mathcal{K}_X}(y,z) \oplus d_{\mathcal{K}_X}(x,y).
\end{aligned}$$

(3) (K1) Since $d_X(z,y) \oplus n(\alpha_X)(y) \geq n(\alpha_X)(y)$, $\mathcal{K}_{d_X}(\alpha_X)(y) \geq n(\alpha_X)(y)$. $\mathcal{K}_{d_X}(\alpha_X)(y) = \bigwedge_{z\in X}(d_X(z,y) \oplus n(A)(z)) \leq d_X(y,y) \oplus n(A)(y) = n(A)(y)$. Hence $\mathcal{K}_{d_X}(\alpha_X) = n(\alpha_X)$.

(K2) $\mathcal{K}_{d_X}(A)(y) = \bigwedge_{z\in X}(d_X(z,y)\oplus n(A)(z)) \leq d_X(y,y)\oplus n(A)(y) = n(A)(y)$.

(K3) If $A \leq B$, then $n(A) \geq n(B)$. Thus $\mathcal{K}_{d_X}(A) \geq \mathcal{K}_{d_X}(B)$. Moreover,

$$\begin{aligned}
&\mathcal{K}_{d_X}(\alpha_X \oplus A) = \bigwedge_{z\in X}(d_X(z,y) \oplus n(\alpha_X \oplus A)(z))\\
&= \bigwedge_{z\in X}(d_X(z,y) \oplus (n(A)(z) \ominus \alpha))\\
&\text{(by Lemma 2.3(11))}\\
&\geq (\bigwedge_{z\in X}(d_X(z,y) \oplus n(A)(z))) \ominus \alpha = \mathcal{K}_{d_X}(A) \ominus \alpha\\
&\text{(by Lemma 2.3(3)).}
\end{aligned}$$

For $A \in L^X$ and $y \in X$,

$$\begin{aligned}
&\mathcal{K}_{d_{\mathcal{K}_X}}(A)(y) = \bigwedge_{z\in X}(d_{\mathcal{K}_X}(z,y) \oplus n(A)(z))\\
&= \bigwedge_{z\in X}(\bigvee_{B\in L^X}(\mathcal{K}_X(n(B))(y)\\
&\ominus\mathcal{K}_X(n(B))(z)) \oplus n(A)(z)) \text{ (put } B = n(A))\\
&\geq \bigwedge_{z\in X}((\mathcal{K}_X(A)(y) \ominus \mathcal{K}_X(A)(z)) \oplus n(A)(z))\\
&\text{(by } \mathcal{K}_X(A) \leq n(A))\\
&\geq \bigwedge_{z\in X}((\mathcal{K}_X(A)(y) \ominus n(A)(z)) \oplus n(A)(z)) \geq \mathcal{K}_X(A)(y)\\
&\text{(by Lemma 2.3(6)).}
\end{aligned}$$

Since $\bigwedge_{z\in X}(d_X(z,y)\oplus d_X(p,z))\geq d_X(p,y)$ from (M2) and $\bigwedge_{z\in X}(d_X(z,y)\oplus d_X(p,z))\leq d_X(y,y)\oplus d_X(p,y)=d_X(p,y)$, $\bigwedge_{z\in X}(d_X(z,y)\oplus d_X(p,z))=d_X(p,y)$.

Moreover, $\bigvee_{p\in X}(d_X(p,y)\ominus d_X(p,x))\geq d_X(x,y)\ominus d_X(x,x)=d_X(x,y)$. Since $d_X(p,y)\ominus d_X(p,x)\leq d_X(x,y)$, $\bigvee_{p\in X}(d_X(p,y)\ominus d_X(p,x))\geq d_X(x,y)$.

For $x,y\in X$,

$$\begin{array}{l} d_{\mathcal{K}_{d_X}}(x,y)=\bigvee_{A\in L^X}(\mathcal{K}_{d_X}(n(A))(y)\ominus \mathcal{K}_{d_X}(n(A))(x)) \\ =\bigvee_{A\in L^X}(\bigwedge_{z\in X}(d_X(z,y)\oplus A(z))\ominus \bigwedge_{w\in X}(d_X(w,x)\oplus A(w))) \\ (\text{ Put } A=d_X(p,-)\in L^X) \\ \geq \bigvee_{p\in X}(\bigwedge_{z\in X}(d_X(z,y)\oplus d_X(p,z))\ominus \bigwedge_{w\in X}(d_X(w,x)\oplus d_X(p,w))) \\ =\bigvee_{p\in X}(d_X(p,y)\ominus d_X(p,x))=d_X(x,y), \end{array}$$

$$\begin{array}{l} d_{\mathcal{K}_{d_X}}(x,y)=\bigvee_{A\in L^X}(\mathcal{K}_{d_X}(n(A))(y)\ominus \mathcal{K}_{d_X}(n(A))(x)) \\ =\bigvee_{A\in L^X}(\bigwedge_{z\in X}(d_X(z,y)\oplus A(z))\ominus \bigwedge_{w\in X}(d_X(w,x)\oplus A(w))) \\ \leq \bigvee_{A\in L^X}(\bigvee_{z\in X}(d_X(z,y)\oplus A(z))\ominus (d_X(z,x)\oplus A(z))) \\ (\text{by Lemma 2.3(8)}) \\ \leq \bigvee_{z\in X}(d_X(z,y)\ominus d_X(z,x))=d_X(x,y). \end{array}$$

□

Theorem 3.5. *Let $\mathcal{D}_X:L^X\to L^X$ be a join-meet operator. Then the following properties hold.*

(1) *Define $\eta_{\mathcal{D}_X}=A\in L^X\mid A=\mathcal{D}_X(n(A))\}$. Then $\eta_{\mathcal{D}_X}$ is an Alexandrov precotopology on X.*

(2) *Define $d_{\mathcal{D}_X}(x,y)=\bigvee_{A\in L^X}(\mathcal{D}_X(n(A))(y)\ominus \mathcal{D}_X(n(A))(x))$. Then $d_{\mathcal{D}_X}$ is a distance function.*

(3) *If d_X is a distance function. Define $\mathcal{D}_{d_X}(A)(y)=\bigvee_{z\in X}(n(A)(z)\ominus d_X(y,z))$. Then $\mathcal{D}_{d_X}$ is a join-meet operator. Moreover, $\mathcal{D}_{d_{\mathcal{D}_X}}(A)\leq \mathcal{D}_X(A)$ and $d_{\mathcal{D}_{d_X}}=d_X$.*

Proof. (1) (CO1) Since $\mathcal{D}_X(n(\alpha_X))=n(n(\alpha_X))=\alpha_X$, $\alpha_X\in\eta_{\mathcal{D}_X}$.

(CO2) If $A_i\in\eta_{\mathcal{D}_X}$ for all $i\in I$, then $\bigwedge_{i\in I}A_i\leq \mathcal{D}_X(n(\bigwedge_{i\in I}A_i))=\mathcal{D}_X(\bigvee_{i\in I}n(A_i))$ $\leq \bigwedge_{i\in I}\mathcal{D}_X(n(A_i))=\bigwedge_{i\in I}A_i$. So, $\bigwedge_{i\in I}A_i\in\eta_{\mathcal{D}_X}$.

(CO3) Let $A\in\eta_{\mathcal{D}_X}$ and $\alpha\in L$. Then $A\oplus\alpha\in\eta_{\mathcal{D}_X}$ from:

$$\begin{array}{l} A\oplus\alpha\leq \mathcal{D}_X(n(A\oplus\alpha))=\mathcal{D}_X(n(A)\ominus\alpha) \\ \leq \mathcal{D}_X(n(A))\oplus\alpha=A\oplus\alpha. \end{array}$$

(2) It is similarly proved as Theorem 3.4(2).

(3) (D1) and (D2) are easily proved. If $A\leq B$, $\mathcal{D}_{d_X}(A)\geq \mathcal{D}_{d_X}(B)$. Moreover,

$$\begin{array}{l} \mathcal{D}_{d_X}(A\ominus\alpha)=\bigvee_{z\in X}(n(A\ominus\alpha)(z)\ominus d_X(y,z)) \\ =\bigvee_{z\in X}((n(A)\oplus\alpha)\ominus d_X(y,z)) \ (\text{ by Lemma 2.3(11)}) \\ \leq \bigvee_{z\in X}(\alpha\oplus(n(A)\ominus d_X(y,z)) \ \ (\text{by Lemma 2.3(10)}) \\ \leq \alpha\oplus\bigvee_{z\in X}((n(A)\ominus d_X(y,z))=\alpha\oplus\mathcal{D}_{d_X}(A). \end{array}$$

For $A \in L^X$ and $y \in X$,

$$\begin{aligned}&\mathcal{D}_{d_{\mathcal{D}_X}}(A)(y) = \bigvee_{z\in X}(n(A)(z) \ominus d_{\mathcal{K}_X}(y,z))\\ &= \bigvee_{z\in X}(n(A)(z) \ominus \bigvee_{A\in L^X}(\mathcal{D}_X(n(A))(z) \ominus \mathcal{D}_X(n(A))(y)))\\ &(\text{by } \mathcal{D}_X(n(A)) \geq n(A))\\ &\leq \bigvee_{z\in X}(n(A)(z) \ominus (n(A)(z) \ominus \mathcal{D}_X(A)(y))) \leq \mathcal{D}_X(A)(y)\\ &(\text{by Lemma 2.3(6)}).\end{aligned}$$

For $x, y \in X$,

$$\begin{aligned}&d_{\mathcal{D}_{d_X}}(x,y) = \bigvee_{A\in L^X}(\mathcal{D}_{d_X}(n(A))(y) \ominus \mathcal{D}_{d_X}(n(A))(x))\\ &= \bigvee_{A\in L^X}(\bigvee_{z\in X}(A(z) \ominus d_X(y,z)) \ominus \bigvee_{w\in X}(A(w) \ominus d_X(x,w)))\\ &(\text{ Put } A = d_X(p,-) \in L^X)\\ &\geq \bigvee_{p\in X}(\bigvee_{z\in X}(d_X(p,z) \ominus d_X(y,z)) \ominus \bigvee_{w\in X}(d_X(p,w) \ominus d_X(x,w)))\\ &= \bigvee_{p\in X}(d_X(p,y) \ominus d_X(p,x)) = d_X(x,y),\end{aligned}$$

$$\begin{aligned}&d_{\mathcal{D}_{d_X}}(x,y) = \bigvee_{A\in L^X}(\mathcal{D}_{d_X}(n(A))(y) \ominus \mathcal{D}_{d_X}(n(A))(x))\\ &= \bigvee_{A\in L^X}(\bigvee_{z\in X}(A(z) \ominus d_X(y,z)) \ominus \bigvee_{w\in X}(A(w) \ominus d_X(x,w)))\\ &\leq \bigvee_{A\in L^X}(\bigvee_{z\in X}(A(z) \ominus d_X(y,z)) \ominus (A(z) \ominus d_X(x,z)))\\ &(\text{by Lemma 2.3(8)})\\ &\leq \bigvee_{z\in X}(d_X(x,z) \ominus d_X(y,z)) = d_X(x,y).\end{aligned}$$

□

Theorem 3.6. *Let (X, τ) be an Alexandrov pretopological space. Then the following properties hold.*

(1) *Define $\mathcal{K}_\tau(A) = \bigvee\{B \in \tau \mid B \leq n(A)\}$. Then $\mathcal{K}_\tau$ is a meet-join operator.*

(2) *Define $d_\tau(x,y) = \bigvee_{A\in\tau}(A(y) \ominus A(x))$. Then d_τ is a distance function with $\mathcal{K}_{\tau_{d_\tau}}(A) \geq \mathcal{K}_\tau(A)$ and $\tau \subset \tau_{d_\tau}$ where $\tau_{d_\tau} = \{B \in L^X \mid B(x) \oplus d_\tau(x,y) \geq B(y)\}$.*

(3) *If τ is an Alexandrov topology on X, then $\mathcal{K}_{\tau_{d_\tau}}(A) = \mathcal{K}_\tau(A)$ and $\tau = \tau_{d_\tau}$.*

Proof. (1) (K1) For each $x \in X$,

$$\begin{aligned}\mathcal{K}_\tau(\alpha_X)(x) &= \bigvee\{B \in \eta \mid B \leq n(\alpha_X)\\ &= n(\alpha_X) = n(\alpha)_X.\end{aligned}$$

(K2) For each $A \in L^X$,$\mathcal{K}_\tau(A) = \bigvee\{B \in \tau \mid B \leq n(A)\} \leq n(A)$.

(K3) For each $A, C \in L^X$,

$$\begin{aligned}&\mathcal{K}_\tau(A) \ominus \alpha = \bigvee\{B_i \in \tau \mid B_i \leq n(A)\} \ominus \alpha\\ &= \bigvee\{B_i \ominus \alpha \in \tau \mid B_i \leq n(A)\}\\ &\leq \bigvee\{B_i \ominus \alpha \in \tau \mid B_i \ominus \alpha \leq n(A) \ominus \alpha = n(A \oplus \alpha)\}\\ &\leq \mathcal{K}_\tau(A \oplus \alpha).\end{aligned}$$

Hence $\mathcal{K}_\tau$ is a meet-join operator.

(2) We easily prove that d_τ is a distance function from:

$$\begin{aligned}&d_\tau(x,y) \oplus d_\tau(y,z)\\ &= \bigvee_{A\in\tau}(A(y) \ominus A(x)) \oplus \bigvee_{A\in\tau}(A(z) \ominus A(y))\\ &\geq \bigvee_{A\in\tau}((A(y) \ominus A(x)) \oplus (A(z) \ominus A(y)))\\ &\geq \bigvee_{A\in\tau}(A(z) \ominus A(x)) = d_\tau(x,z).\end{aligned}$$

For $B \in \tau$, $B(x) \oplus d_\tau(x,y) = B(x) \oplus \bigvee_{A\in\tau}(A(y) \ominus A(x)) \geq B(x) \oplus (B(y) \ominus B(x)) \geq B(y)$. Hence $B \in \tau_{d_\tau}$. Moreover $\mathcal{K}_\tau(A) = \bigvee_{i\in\Gamma}\{A_i \mid A_i \leq A, A_i \in \tau\} \leq \mathcal{K}_{\tau_{d_\tau}}(A)$.

(3) If τ is an Alexandrov topology on X, for $B \in \tau_{d_\tau}$, $B(x) \oplus d_\tau(x,y) \geq B(y)$ and $\bigwedge_{x\in X}(B(x) \oplus d_\tau(x,y)) \leq B(y) \oplus d_\tau(y,y) = B(y)$. Thus $B(y) = \bigwedge_{x\in X}(B(x) \oplus d_\tau(x,y))$. Since $\bigvee_{A\in\tau}(A(-) \ominus A(x))) \in \tau$, $B = \bigwedge_{x\in X}(B(x) \oplus d_\tau(x,-)) = \bigwedge_{x\in X}(B(x) \oplus \bigvee_{A\in\tau}(A(-) \ominus A(x))) \in \tau$. Hence $B \in \tau$. Thus, by (2), $\tau = \tau_{d_\tau}$ and $\mathcal{K}_{\tau_{d_\tau}}(A) = \mathcal{K}_\tau(A)$. □

Theorem 3.7. *Let (X,η) be an Alexandrov precotopological space. Define $d_\eta(x,y)$
$= \bigvee_{A\in\eta}(A(y) \ominus A(x))$. Then*

(1) Define $\mathcal{D}_\eta(A) = \bigwedge\{B \in \eta \mid n(A) \leq B\}$. Then $\mathcal{D}_\eta$ is a join-meet operator.

(2) Define $d_\eta(x,y) = \bigvee_{A\in\eta}(A(y) \ominus A(x))$. Then d_η is a distance function with $\mathcal{D}_{\eta_{d_\eta}}(A) \leq \mathcal{D}_\eta(A)$ and $\eta \subset \eta_{d_\eta}$ where $\eta_{d_\eta} = \{B \in L^X \mid B(x) \oplus d_\eta(x,y) \geq B(y)\}$.

(3) If η is an Alexandrov topology on X, then $\mathcal{D}_{\eta_{d_\eta}}(A) = \mathcal{D}_\eta(A)$ and $\eta = \eta_{d_\eta}$.

Proof. (1) (D1) For all $x \in X$, we have

$$\mathcal{D}_\eta(\alpha_X)(x) = \bigwedge\{B \in \eta \mid n(\alpha_X) \leq B\}$$
$$= n(\alpha_X) = n(\alpha)_X.$$

(D2) For each $A \in L^X$,$\mathcal{D}_\eta(A) = \bigwedge\{B \in \eta \mid n(A) \leq B\} \geq n(A)$.

(D3) If $A \leq B$, then $\mathcal{D}_\eta(A) \geq \mathcal{D}_\eta(B)$.

$$\mathcal{D}_\eta(A) \oplus \alpha = \bigwedge\{B_i \in \eta \mid B_i \geq n(A)\} \oplus \alpha$$
$$= \bigwedge\{B_i \oplus \alpha \in \eta \mid B_i \geq n(A)\}$$
$$\geq \bigwedge\{B_i \oplus \alpha \in \eta \mid B_i \oplus \alpha \leq n(A) \oplus \alpha = n(A \ominus \alpha)\}$$
$$\geq \mathcal{D}_\eta(A \ominus \alpha).$$

Hence $\mathcal{D}_\eta$ is a join-meet operator.

(2) We similarly prove that d_η is a distance function from Theorem 3.6(2). For $B \in \eta$, $B(x) \oplus d_\eta(x,y) = B(x) \oplus \bigvee_{A\in\eta}(A(y) \ominus A(x)) \geq B(x) \oplus (B(y) \ominus B(x)) \geq B(y)$. Hence $B \in \eta_{d_\eta}$. Moreover $\mathcal{D}_\eta(A) = \bigwedge_{i\in\Gamma}\{A_i \mid A \leq A_i, A_i \in \eta\} \geq \mathcal{D}_{\eta_{d_\eta}}(A)$.

Since $B(y) \leq \bigwedge_{x\in X}(B(x) \oplus \bigvee_{A\in\eta}(A(y) \ominus A(x))) \leq B(y) \oplus \bigvee_{A\in\eta}(A(y) \ominus A(y)) = B(y)$, $B = \bigwedge_{x\in X}(B(x) \oplus \bigvee_{A\in\eta}(A(-) \ominus A(x)))$.

(3) If η is an Alexandrov topology on X, for $B \in \eta_{d_\eta}$, $B = \bigwedge_{x\in X}(B(x) \oplus \bigvee_{A\in\eta}(A(-) \ominus A(x))) \in \eta$. Hence $B \in \eta$. Thus, $\eta = \eta_{d_\eta}$ and $\mathcal{D}_{\eta_{d_\eta}}(A) = \mathcal{D}_\eta(A)$.
□

Example 3.8. Let $X = \{x,y,z\}$ and $([0,1], \leq, \vee, \wedge, \oplus, \ominus, 0, 1)$ be a complete co-residuated lattice defined as $n(x) = 1 - x$,

$$x \oplus y = (x+y) \wedge 1, \quad x \ominus y = (x-y) \vee 0.$$

(1) Define $\mathcal{D}_X : [0,1]^X \to [0,1]^X$ as

$$\mathcal{D}_X(A) = \begin{cases} n(\alpha_X), & \text{if} A = \alpha_X, \\ (n(A) \oplus 0.1) \wedge \sup n(A), & \text{otherwise.} \end{cases}$$

(D1) and (D2) are easily proved.

(D3) If $A \leq B$, then $\mathcal{D}_X(A) \geq \mathcal{D}_X(B)$.

$$\begin{array}{l} \alpha \oplus \mathcal{D}_X(A) = \alpha \oplus ((n(A) \oplus 0.1) \wedge \sup n(A)) \\ \geq (n(A \ominus \alpha) \oplus 0.1) \wedge \sup n(A \ominus \alpha) \\ \geq (n(A) \oplus \alpha) \oplus 0.1) \wedge \sup n(A \ominus \alpha) \end{array}$$

Put $A \in [0,1]^X$ with $A(x) = 0.6, A(y) = 0.3, A(z) = 0.5$. Then $\mathcal{D}_X(A) = (n(A) \oplus 0.1) \wedge \sup n(A) = (0.5, 0.8, 0.6) \wedge 0.7_X = (0.5, 0.7, 0.6)$. Since $\eta_{\mathcal{D}_X} = \{\alpha_X \mid \alpha \in [0,1]\}$, $\mathcal{D}_{\eta_{\mathcal{D}_X}}(A) = 0.7_X$. Moreover, $\mathcal{D}_{\eta_{\mathcal{D}_X}}(B) = \sup n(B) \geq \mathcal{D}_X(B)$ for each $B \in L^X$.

For $0_x \in L^X$ with $0_x(y) = 0$, for $x = y$ and $0_x(y) = 1$,for $x \neq y$,

$$\begin{array}{l} d_{\mathcal{D}_X}(x,y) = \bigvee_{A \in L^X} (\mathcal{D}_X(A)(y) \ominus \mathcal{D}_X(A)(x)) \\ = \begin{cases} 0, & \text{if } x = y, \\ (0_x(y) \oplus 0.1) \ominus (0_x(x) \oplus 0.1) = 0.9, & \text{if } x \neq y \end{cases} \end{array}$$

(2) Define $\mathcal{K}_X : [0,1]^X \to [0,1]^X$ as

$$\mathcal{K}_X(A) = \begin{cases} n(\alpha_X), & \text{if} A = \alpha_X, \\ (n(A) \ominus 0.1) \vee \inf n(A), & \text{otherwise.} \end{cases}$$

(K1) and (K2) are easily proved.

(K3) If $A \leq B$, then $\mathcal{K}_X(A) \geq \mathcal{K}_X(B)$.

$$\begin{array}{l} \mathcal{K}_X(A) \ominus \alpha = ((n(A) \ominus 0.1) \vee \inf n(A)) \ominus \alpha \\ = ((n(A) \ominus 0.1) \ominus \alpha) \vee (\inf n(A) \ominus \alpha) \\ \leq (n(A \oplus \alpha) \ominus 0.1) \vee \inf n(A \oplus \alpha) \\ = \mathcal{K}_X(A \oplus \alpha). \end{array}$$

Put $A \in [0,1]^X$ with $A(x) = 0.6, A(y) = 0.3, A(z) = 0.5$. Then $\mathcal{K}_X(A) = (n(A) \ominus 0.1) \vee \inf n(A) = (0.3, 0.6, 0.4) \vee 0.4_X = (0.4, 0.6, 0.4)$. Since $\tau_{\mathcal{K}_X} = \{\alpha_X \mid \alpha \in [0,1]\}$, $\mathcal{K}_{\tau_{\mathcal{K}_X}}(A) = 0.4_X$. Moreover, $\mathcal{K}_{\tau_{\mathcal{D}_X}}(B) = \inf n(B) \leq \mathcal{K}_X(B)$ for each $B \in L^X$.

For $0_x \in L^X$ with $0_x(y) = 0$, for $x = y$ and $0_x(y) = 1$,for $x \neq y$,

$$\begin{array}{l} d_{\mathcal{K}_X}(x,y) = \bigvee_{A \in L^X} (\mathcal{K}_X(A)(y) \ominus \mathcal{K}_X(A)(x)) \\ = \begin{cases} 0, & \text{if } x = y, \\ (0_x(y) \ominus 0.1) \ominus (0_x(x) \ominus 0.1) = 0.9, & \text{if } x \neq y \end{cases} \end{array}$$

(3) Define an Alexandrov pretopology

$$\tau_X = \{(A \ominus \alpha) \vee \beta_X) \mid \alpha, \beta \in L\}.$$

For $B = (0.2, 0.4, 0.3) \in [0,1]^X$,

$$\mathcal{K}_{\tau_X}(B) = \bigvee \{A_i \in \tau_X \mid A_i \leq n(B)\} = 0.6_X.$$

(4) Define an Alexandrov precotopology

$$\eta_X = \{(A \oplus \alpha) \wedge \beta_X) \mid \alpha, \beta \in L\}.$$

For $B = (0.2, 0.4, 0.3) \in [0,1]^X$,

$$\begin{aligned} &\mathcal{D}_{\eta_X}(B) = \bigwedge\{A_i \in \eta_X \mid n(B) \le A_i\} \\ &= (0.9, 0.6, 0.8) \wedge 0.8_X = (0.8, 0.6, 0.8). \end{aligned}$$

Example 3.9. (1) Define maps $d^i : [0,1] \times [0,1] \to [0,1]$ for $i = 0, 1, 2, 3$ as follows:

$$d^0(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \ne y, \end{cases} \quad d^1(x,y) = \begin{cases} 0, & \text{if } x \ge y, \\ 1, & \text{if } x \not\ge y, \end{cases}$$

$$d^2(x,y) = \begin{cases} 0, & \text{if } x \le y, \\ 1, & \text{if } x \not\le y, \end{cases} \quad d^3(x,y) = 0.$$

Since $\mathcal{K}_{d_X}(A)(y) = \bigwedge_{x \in [0,1]} (n(A)(x) \oplus d_X(x,y))$ for each $A \in [0,1]^{[0,1]}$, we can obtain

$$\begin{aligned} &\mathcal{K}_{d^0}(A)(y) = \bigwedge_{x \in [0,1]} (n(A)(x) \oplus d^0_X(x,y)) = n(A)(y), \\ &\mathcal{K}_{d^1}(A) = \bigwedge_{x \ge y} n(A)(x), \\ &\mathcal{K}_{d^2}(A) = \bigwedge_{x \le y} n(A)(x), \\ &\mathcal{K}_{d^3}(A) = \bigwedge_{x \in [0,1]} n(A)(x). \end{aligned}$$

$$\begin{aligned} \tau_{d^0} &= [0,1]^{[0,1]}, \\ \tau_{d^1} &= \{A \in [0,1]^{[0,1]} \mid A(x) \le A(y) \text{ if } x \le y\}, \\ \tau_{d^2} &= \{A \in [0,1]^{[0,1]} \mid A(x) \ge A(y) \text{ if } x \le y\}, \\ \tau_{d^3} &= \{\alpha_X \in [0,1]^{[0,1]} \mid \alpha \in [0,1]\}. \end{aligned}$$

4. Conclusion

In this paper, we investigate between the topological structures on fuzzy sets and fuzzy join and meet complete lattices with distance spaces in complete co-residuated lattices.

In the future, as extensions of fuzzy rough sets, by using the concepts of distance spaces in complete co-residuated lattices, fuzzy concepts, information systems and decision rules are investigated.

References

1. R. Bělohlávek, *Fuzzy Relational Systems*, Kluwer Academic Publishers, New York, 2002.
2. P. Chen, D. Zhang, *Alexandroff co-topological spaces* Fuzzy Sets and Systems **161** (2010), 2505-2514.
3. P. Hájek, *Metamathematices of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht, 1998.
4. U. Höhle, E.P. Klement, *Non-classical logic and their applications to fuzzy subsets*, Kluwer Academic Publishers, Boston, 1995.
5. U. Höhle, S.E. Rodabaugh, *Mathematics of Fuzzy Sets, Logic, Topology and Measure Theory*, The Handbooks of Fuzzy Sets Series, Kluwer Academic Publishers, Dordrecht, 1999.
6. F. Jinming, *I-fuzzy Alexandrov topologies and specialization orders*, Fuzzy Sets and Systems **158** (2007), 2359-2374.

7. Q. Junsheng, H.B. Qing, *On* $(\odot, \&)$ *-fuzzy rough sets based on residuated and co-residuated lattices*, Fuzzy Sets and Systems **336** (2018), 54-86.
8. Y.C. Kim, *Join-meet preserving maps and Alexandrov fuzzy topologies*, Journal of Intelligent and Fuzzy Systems **28** (2015), 457-467.
9. Y.C. Kim, *Join-meet preserving maps and fuzzy preorders*, Journal of Intelligent and Fuzzy Systems **28** (2015), 1089-1097.
10. Y.C. Kim, *Categories of fuzzy preorders, approximation operators and Alexandrov topologies*, Journal of Intelligent and Fuzzy Systems **31** (2016), 1787-1793.
11. Y.C. Kim, J.M Ko, *Preserving maps and approximation operators in complete co-residuated lattices*, Journal of the korean Insitutute of Intelligent Systems **30** (2020), 389-398.
12. Y.C. Kim, J.M Ko, *Fuzzy complete lattices, Alexandrov L-fuzzy topologies and fuzzy rough sets*, Journal of Intelligent and Fuzzy Systems **38** (2020), 3253-3266.
13. H. Lai, D. Zhang, *Fuzzy preorder and fuzzy topology*, Fuzzy Sets and Systems **157** (2006), 1865-1885.
14. Z.M. Ma, B.Q. Hu, *Topological and lattice structures of L-fuzzy rough set determined by lower and upper sets*, Information Sciences **218** (2013), 194-204.
15. J.S. Mi, Y. Leung, H.Y. Zhao, T. Feng, *Generalized fuzzy rough sets determined by a trianglar norm*, Information Sciences **178** (2008), 3203-3213.
16. Z. Pawlak, *Rough sets*, Internat. J. Comput. Inform. Sci. **11** (1982), 341-356.
17. Z. Pawlak, *Rough sets: Theoretical Aspects of Reasoning about Data, System Theory, Knowledge Engineering and Problem Solving*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1991.
18. A.M. Radzikowska, E.E. Kerre, *A comparative study of fuzy rough sets*, Fuzzy Sets and Systems **126** (2002), 137-155.
19. A.M. Radzikowska, E.E. Kerre, *Characterisation of main classes of fuzzy relations using fuzzy modal operators*, Fuzzy Sets and Systems **152** (2005), 223-247.
20. S.E. Rodabaugh, E.P. Klement, *Topological and Algebraic Structures In Fuzzy Sets*, The Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Kluwer Academic Publishers, Boston, Dordrecht, London, 2003.
21. Y.H. She, G.J. Wang, *An axiomatic approach of fuzzy rough sets based on residuated lattices*, Computers and Mathematics with Applications **58** (2009), 189-201.
22. S.P. Tiwari, A.K. Srivastava, *Fuzzy rough sets, fuzzy preorders and fuzzy topologies*, Fuzzy Sets and Systems **210** (2013), 63-68.
23. E. Turunen, *Mathematics Behind Fuzzy Logic,* A Springer-Verlag Co., 1999.
24. M. Ward, R.P. Dilworth, *Residuated lattices*, Trans. Amer. Math. Soc. **45** (1939), 335-354.
25. W.Z. Wu, Y. Leung, J.S. Mi, *On charterizations of* (Φ, T)*-fuzzy approximation operators*, Fuzzy Sets and Systems **154** (2005), 76-102.
26. M.C. Zheng, G.J. Wang, *Coresiduated lattice with applications*, Fuzzy systems and Mathematics **19** (2005), 1-6.

Young-Hee Kim received M.Sc. and Ph.D. from Yonsei University. Since 2003 she has been at Kwangwoon University. Her research interests are numerical analysis, information systems, fuzzy sets and logic.

Ingenium College of Liberal Arts-Mathematics, Kwangwoon University, Seoul, 01897, Korea.
e-mail: yhkim@kw.ac.kr

Yong Chan Kim received M.Sc. and Ph.D. from Yonsei University. Since 1991 he has been at Gangneung-Wonju National University. His research interests are fuzzy logics, rough sets and fuzzy topology.

Department of Mathematics, Gangneung-Wonju University, Gangneung, Gangwondo 25457, Korea.
e-mail: yck@gwnu.ac.kr

Jongsung Choi received M.Sc. and Ph.D. from Pusan and Tokyo University. Since 2005 he has been at Kwangwoon University. Her research interests are inverse problems, p-adic analysis, fuzzy analysis.

Ingenium College of Liberal Arts-Mathematics, Kwangwoon University, Seoul, 01897, Korea.
e-mail: jeschoi@kw.ac.kr