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STUDIES ON PROPERTIES AND CHARACTERISTICS OF TWO NEW TYPES OF q-GENOCCHI POLYNOMIALS†

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Abstract. In this paper, we construct q-cosine and sine Genocchi polynomials using q-analogues of addition, subtraction, and q-trigonometric function. From these polynomials, we obtain some properties and identities. We investigate some symmetric properties of q-cosine and sine Genocchi polynomials. Moreover, we find relations between these polynomials and others polynomials.

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1. Introduction

We would like to begin by introducing several definitions related to q-numbers used in this paper(see [1, 3-8, 10, 15, 18]). For any $n \in \mathbb{N}$, the q-number can be defined as follows.

$$[n]_q = \frac{1-q^n}{1-q}, \quad \text{where} \quad q \neq 1. \tag{1.1}$$

Definition 1.1. The Gaussian binomial coefficients are defined by

$$\begin{bmatrix} m \\ r \end{bmatrix}_q = \begin{cases} 0 & \text{if } r > m \\ \frac{(1-q^m)(1-q^{m-1})\cdots(1-q^{m-r+1})}{(1-q)(1-q^2)\cdots(1-q^r)} & \text{if } r \leq m, \end{cases} \tag{1.2}$$

where m and r are non-negative integers.

For $r = 0$, the value is 1 since the numerator and the denominator are both empty products. Like the classical binomial coefficients, the Gaussian binomial

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coefficients are center-symmetric. There are analogues of the binomial formula and this definition has a number of properties(see [3, 10]).

Definition 1.2. The q-analogues of $(x-a)^n$ and $(x+a)^n$ are defined by

$$\text{(i)}\quad (x\ominus a)_q^n=\begin{cases}1 & \text{if } n=0\\ (x-a)(x-qa)\cdots(x-q^{n-1}a) & \text{if } n\geq 1\end{cases},$$
$$\text{(ii)}\quad (x\oplus a)_q^n=\begin{cases}1 & \text{if } n=0\\ (x+a)(x+qa)\cdots(x+q^{n-1}a) & \text{if } n\geq 1\end{cases}, \text{respectively.} \tag{1.3}$$

Definition 1.3. Let z be any complex numbers with $|z|<1$. Two forms of q-exponential functions can be expressed as

$$e_q(z)=\sum_{n=0}^{\infty}\frac{z^n}{[n]_q!},\qquad E_q(z)=\sum_{n=0}^{\infty}q^{\binom{n}{2}}\frac{z^n}{[n]_q!}. \tag{1.4}$$

From Definition 1.3, we note that (1) $e_q(x)e_q(y)=e_q(x+y)$ if $yx=qxy$, (2) $e_q(x)E_q(-x)=1$, and (3) $e_{q^{-1}}(x)=E_q(x)$.

Definition 1.4. The definition of the q-derivative operator of any function f follows that

$$D_qf(x)=\frac{f(x)-f(qx)}{(1-q)x},\qquad x\neq 0, \tag{1.5}$$

and $D_qf(0)=f'(0)$.

Definition 1.5. The q-trigonometric functions are

$$\begin{aligned}\sin_q(x)&=\frac{e_q(ix)-e_q(-ix)}{2i}, & \mathrm{SIN}_q(x)&=\frac{E_q(ix)-E_q(-ix)}{2i}\\ \cos_q(x)&=\frac{e_q(ix)+e_q(-ix)}{2}, & \mathrm{COS}_q(x)&=\frac{E_q(ix)+E_q(-ix)}{2},\end{aligned} \tag{1.6}$$

where $\mathrm{SIN}_q(x)=\sin_{q^{-1}}(x), \mathrm{COS}_q(x)=\cos_{q^{-1}}(x)$.

In various mathematics applications which include number theory, combinatorial analysis, and other fields, the Bernoulli, Euler, Genocchi, tangent polynomials are widely studied. Acknowledging their significance, many mathematicians are familiar with these numbers and polynomials and it has been studied for a long time. The previous Definitions and Theorems are also applied to polynomials and their properties are studied in various ways in combination with Bernoulli, Euler, Genocchi, and tangent polynomials, which are considered important(see [2, 9, 11-14, 16-17]). The definition of q-cosine and q-sine Bernoulli polynomials are as follows:

Definition 1.6. Let x, y be real numbers. Then, q-cosine Bernoulli polynomials and q-sine Bernoulli polynomials are defined by:

$$\begin{aligned}\sum_{n=0}^{\infty} {}_{C}B_{n,q}(x,y)\frac{t^n}{[n]_q!} &= \frac{t}{e_q(t)-1}e_q(tx)COS_q(ty),\\ \sum_{n=0}^{\infty} {}_{S}B_{n,q}(x,y)\frac{t^n}{[n]_q!} &= \frac{t}{e_q(t)-1}e_q(tx)SIN_q(ty),\end{aligned} \tag{1.7}$$

respectively.

Recently, in [9], we confirmed the properties of q-cosine Bernoulli polynomials and q-sine Bernoulli polynomials. The definitions and representative properties of cosine Euler polynomials and sine Euler polynomials are the following.

Definition 1.7. The generating functions of q-cosine Euler polynomials and q-sine Euler polynomials are correspondingly defined by

$$\begin{aligned}\sum_{n=0}^{\infty} {}_{C}\mathcal{E}_{n,q}(x,y)\frac{t^n}{[n]_q!} &= \frac{2}{e_q(t)+1}e_q(tx)COS_q(ty),\\ \sum_{n=0}^{\infty} {}_{S}\mathcal{E}_{n,q}(x,y)\frac{t^n}{[n]_q!} &= \frac{2}{e_q(t)+1}e_q(tx)SIN_q(ty).\end{aligned} \tag{1.8}$$

Based on the above, many studies can confirm various polynomials and their properties(see [9, 11, 14, 16]).

Definition 1.8. q-Genocchi polynomials are defined by

$$\sum_{n=0}^{\infty} G_{n,q}(x)\frac{t^n}{[n]_q!} = \frac{2t}{e_q(t)+1}e_q(tx). \tag{1.9}$$

The main aim of this paper is to define the q-cosine Genocchi polynomials and q-sine Genocchi polynomials. In Section 2, we construct q-Genocchi polynomials combined with q-trigonometric functions and confirm various properties and identities of these polynomials. Here, we use the properties and exponential functions associated with the q-number and q-analogues of subtraction and addition. In Section 3, we find interesting and symmetric properties for q-cosine Genocchi polynomials and q-sine Genocchi polynomials.

2. Some properties of q-cosine Genocchi, q-sine Genocchi polynomials

In this section, we construct q-cosine Genocchi polynomials and q-sine Genocchi polynomials using q-trigonometric functions. From these polynomials, we find some identities and properties. Furthermore, we investigate some relations between these polynomials and others polynomials.

Theorem 2.1. *Let $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$. Then we investigate*

$$\begin{aligned}
&(i) \quad \sum_{n=0}^{\infty} \frac{G_{n,q}((x \oplus iy)_q) + G_{n,q}((x \ominus iy)_q)t^n}{2[n]_q!} = \frac{2t}{e_q(t)+1} e_q(tx) COS_q(ty), \\
&(ii) \quad \sum_{n=0}^{\infty} \frac{G_{n,q}((x \oplus iy)_q) - G_{n,q}((x \ominus iy)_q)t^n}{2i[n]_q!} = \frac{2t}{e_q(t)+1} e_q(tx) SIN_q(ty).
\end{aligned} \tag{2.1}$$

Proof. (i) By substituting $(x \oplus iy)_q$ into z in the generating function of q-Genocchi polynomials and using a property of q-exponential function, $e_q(t(x \oplus iy)_q) = e_q(tx)E_q(ity)$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} G_{n,q}((x \oplus iy)_q) \frac{t^n}{[n]_q!} &= \frac{2t}{e_q(t)+1} e_q(t(x \oplus iy)_q) \\
&= \frac{2t}{e_q(t)+1} e_q(tx) \left(COS_q(ty) + iSIN_q(ty) \right).
\end{aligned} \tag{2.2}$$

In a similar method of (2.2), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} G_{n,q}((x \ominus iy)_q) \frac{t^n}{[n]_q!} &= \frac{2t}{e_q(t)+1} e_q(t(x \ominus iy)_q) \\
&= \frac{2t}{e_q(t)+1} e_q(tx) \left(COS_q(ty) - iSIN_q(ty) \right).
\end{aligned} \tag{2.3}$$

From the Equations (2.2) and (2.3), we find the desired results.
(ii) From (2.2) and (2.3), we also have the desired results. □

Definition 2.2. The q-cosine Genocchi polynomials and q-sine Genocchi polynomials are defined respectively

$$\begin{aligned}
\sum_{n=0}^{\infty} {}_{C}G_{n,q}(x,y) \frac{t^n}{[n]_q!} &= \frac{2t}{e_q(t)+1} e_q(tx) COS_q(ty), \\
\sum_{n=0}^{\infty} {}_{S}G_{n,q}(x,y) \frac{t^n}{[n]_q!} &= \frac{2t}{e_q(t)+1} e_q(tx) SIN_q(ty).
\end{aligned} \tag{2.4}$$

From the Definition 2.2, we note that ${}_{C}G_{n,q}(x,y) = {}_{C}G_n(x,y)$ when $q \to 1$.

Remark 2.1. From the Theorem 2.1 and Definition 2.2, we have

$$\begin{aligned}
&(i) \quad 2{}_{C}G_{n,q}(x,y) = G_{n,q}((x \oplus iy)_q) + G_{n,q}((x \ominus iy)_q), \\
&(ii) \quad 2i{}_{S}G_{n,q}(x,y) = G_{n,q}((x \oplus iy)_q) - G_{n,q}((x \ominus iy)_q),
\end{aligned} \tag{2.5}$$

where $G_{n,q}(x)$ are the q-Genocchi polynomials.

In [9], Ryoo and Kang defined $C_{n,q}(x,y)$ and $S_{n,q}(x,y)$ as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} C_{n,q}(x,y)\frac{t^n}{[n]_q!} &= e_q(tx)COS_q(ty),\\ \sum_{n=0}^{\infty} S_{n,q}(x,y)\frac{t^n}{[n]_q!} &= e_q(tx)SIN_q(ty).\end{aligned} \tag{2.6}$$

Moreover, we can remark that $COS_q(x) = \sum_{n=0}^{\infty}(-1)^n q^{(2n-1)n}\frac{x^{2n}}{[2n]_q!}$ and $SIN_q(x) = \sum_{n=0}^{\infty}(-1)^n q^{(2n+1)n}\frac{x^{2n+1}}{[2n+1]_q!}$ in [8]. From Equation (2.6), we can find some relations between the q-cosine, sine Genocchi polynomials and the $C_{n,q}(x,y)$ and $S_{n,q}(x,y)$.

Theorem 2.3. *Let $|q| < 1$ and k be a nonnegative integer. Then, we derive*

$$\begin{aligned}&(i)\quad {}_CG_{n,q}(x,y) = \sum_{k=0}^{[\frac{n}{2}]}\begin{bmatrix} n\\ 2k\end{bmatrix}_q(-1)^k q^{(2k-1)k}y^{2k}G_{n-2k,q}(x),\\ &(ii)\quad {}_SG_{n,q}(x,y) = \sum_{k=0}^{[\frac{n-1}{2}]}\begin{bmatrix} n\\ 2k+1\end{bmatrix}_q(-1)^k q^{(2k+1)k}y^{2k+1}G_{n-(2k+1),q}(x),\end{aligned} \tag{2.7}$$

where $G_{n,q}(x)$ are the q-Genocchi polynomials and $[x]$ is the great integer not exceeding x.

Proof. (i) By using the power series of $COS_q(x)$ and q-Genocchi polynomials in the generating function of q-cosine Genocchi polynomials, we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_CG_{n,q}(x,y)\frac{t^n}{[n]_q!} &= \frac{2t}{e_q(t)+1}e_q(tx)COS_q(ty)\\ &= \sum_{n=0}^{\infty} G_{n,q}(x)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}(-1)^n q^{(2n-1)n}y^{2n}\frac{t^{2n}}{[2n]_q!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n+k\\ 2k\end{bmatrix}_q(-1)^k q^{(2k-1)k}y^{2k}G_{n-k,q}(x)\right)\frac{t^{n+k}}{[n+k]_q!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{[\frac{n}{2}]}\begin{bmatrix} n\\ 2k\end{bmatrix}_q(-1)^k q^{(2k-1)k}y^{2k}G_{n-2k,q}(x)\right)\frac{t^n}{[n]_q!}.\end{aligned} \tag{2.8}$$

From Equation (2.8), we have the required result.
(ii) We obtain the result (ii) using the power series of $SIN_q(x)$ and q-Genocchi polynomials in q-sine Genocchi polynomials. □

Corollary 2.4. *Let $y = 1$ in Theorem 2.3. Then, the following holds*

$$\begin{aligned}
&(i) \quad {}_{C}G_{n,q}(x,1) = \sum_{k=0}^{[\frac{n}{2}]} \begin{bmatrix} n \\ 2k \end{bmatrix}_q (-1)^k q^{(2k-1)k} G_{n-2k,q}(x), \\
&(ii) \quad {}_{S}G_{n,q}(x,1) = \sum_{k=0}^{[\frac{n-1}{2}]} \begin{bmatrix} n \\ 2k+1 \end{bmatrix}_q (-1)^k q^{(2k+1)k} G_{n-(2k+1),q}(x),
\end{aligned} \tag{2.9}$$

where $G_{n,q}(x)$ are the q-Genocchi polynomials and $[x]$ is the great integer not exceeding x.

Theorem 2.5. *Let $|q| < 1$, $e_q(t) \neq -1$ and $x, y \in \mathbb{R}$. Then, we investigate*

$$\begin{aligned}
&(i) \quad \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q {}_{C}G_{k,q}(x,y) + {}_{C}G_{n,q}(x,y) = 2[n]_q C_{n-1,q}(x,y), \\
&(ii) \quad \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q {}_{S}G_{k,q}(x,y) + {}_{S}G_{n,q}(x,y) = 2[n]_q S_{n-1,q}(x,y).
\end{aligned} \tag{2.10}$$

Proof. (i) In the generating function of the q-cosine Genocchi polynomials, we can consider $e_q(t) \neq -1$. Then, we have

$$\sum_{n=0}^{\infty} {}_{C}G_{n,q}(x,y) \frac{t^n}{[n]_q!} \left(\sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} + 1 \right) = 2t \sum_{n=0}^{\infty} C_{n,q}(x,y) \frac{t^n}{[n]_q!}. \tag{2.11}$$

The left-hand side in Equation (2.11) is transformed as

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_{C}G_{n,q}(x,y) \frac{t^n}{[n]_q!} \left(\sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} + 1 \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q {}_{C}G_{k,q}(x,y) + {}_{C}G_{n,q}(x,y) \right) \frac{t^n}{[n]_q!}
\end{aligned} \tag{2.12}$$

and the right-hand side of (2.11) is changed as follows.

$$2t \sum_{n=0}^{\infty} C_{n,q}(x,y) \frac{t^n}{[n]_q!} = 2 \sum_{n=0}^{\infty} [n]_q C_{n-1,q}(x,y) \frac{t^n}{[n]_q!}. \tag{2.13}$$

By comparing the coefficients of Equations (2.12) and (2.13), we find the required result.
(ii) In a similar method as in the proof of (i), we finish the proof of Theorem 2.6 (ii). □

Corollary 2.6. *Consider $q \to 1$ in Theorem 2.6. Then, one holds*

$$\begin{aligned} &(i) \quad nC_{n-1}(x,y) = \frac{1}{2}\left(\sum_{k=0}^{n}\binom{n}{k}{}_{C}G_k(x,y) + {}_{C}G_n(x,y)\right), \\ &(ii) \quad nS_{n-1}(x,y) = \frac{1}{2}\left(\sum_{k=0}^{n}\binom{n}{k}{}_{S}G_k(x,y) + {}_{S}G_n(x,y)\right), \end{aligned} \tag{2.14}$$

where ${}_{C}G_n(x,y)$ are the cosine Genocchi polynomials and ${}_{S}G_n(x,y)$ are the sine Genocchi polynomials in two parameters.

Theorem 2.7. *For $|q| < 1$ and any real numbers x, y, we have*

$$\begin{aligned} &(i) \quad {}_{C}G_{n,q}(1,y) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}}(-x)^{n-k}\left(2[k]_q C_{k-1,q}(x,y) - {}_{C}G_{k,q}(x,y)\right), \\ &(ii) \quad {}_{S}G_{n,q}(1,y) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}}(-x)^{n-k}\left(2[k]_q S_{k-1,q}(x,y) - {}_{S}G_{k,q}(x,y)\right). \end{aligned} \tag{2.15}$$

Proof. (i) We consider the generating function of q-cosine Genocchi polynomials when $x = 1$. Then, we have

$$\begin{aligned} \sum_{n=0}^{\infty} {}_{C}G_{n,q}(1,y)\frac{t^n}{[n]_q!} &= \frac{2t}{e_q(t)+1}(e_q(t)+1-1)COS_q(ty) \\ &= 2tCOS_q(ty) - \frac{2t}{e_q(t)+1}COS_q(ty). \end{aligned} \tag{2.16}$$

By using $e_q(x)E_q(-x) = 1$ and a property of q-factorial for q-number, the left-hand side of Equation (2.16) can be transformed as the following.

$$\begin{aligned} &\sum_{n=0}^{\infty} {}_{C}G_{n,q}(1,y)\frac{t^n}{[n]_q!} \\ &= \left(2te_q(tx)COS_q(ty) - \frac{2t}{e_q(t)+1}e_q(tx)COS_q(ty)\right)E_q(-tx). \end{aligned} \tag{2.17}$$

Here, we can note that

$$2te_q(tx)COS_q(ty) = 2\sum_{n=0}^{\infty} C_{n,q}(x,y)\frac{t^{n+1}}{[n]_q!} = 2\sum_{n=0}^{\infty} C_{n-1,q}(x,y)\frac{t^n}{[n]_q!}. \tag{2.18}$$

From Equation (2.18), (2.17) can be rewritten as the following.

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_{C}G_{n,q}(1,y)\frac{t^n}{[n]_q!}\\
&=\sum_{n=0}^{\infty}\left(2[n]_qC_{n-1,q}(x,y)-{}_{C}G_{n,q}(x,y)\right)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}q^{\binom{n}{2}}(-x)^n\frac{t^n}{[n]_q!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{\binom{n-k}{2}}(-x)^{n-k}(2[k]_qC_{k-1,q}(x,y)-{}_{C}G_{k,q}(x,y)\right)\frac{t^n}{[n]_q!}.
\end{aligned}\tag{2.19}$$

By comparing the coefficient of both sides in Equation (2.19), we finish the proof of Theorem 2.7 (i).
(ii) We also take the desired result for q-sine Genocchi polynomials using the similar method of (i). □

Corollary 2.8. *Setting $q \to 1$ in Theorem 2.7, the following holds*

$$\begin{aligned}
&(i)\quad {}_{C}G_n(1,y)=\sum_{k=0}^{n}\binom{n}{k}(-x)^{n-k}\left(2kC_{k-1}(x,y)-{}_{C}G_k(x,y)\right),\\
&(ii)\quad {}_{S}G_{n,q}(1,y)=\sum_{k=0}^{n}\binom{n}{k}(-x)^{n-k}\left(2kS_{k-1}(x,y)-{}_{S}G_k(x,y)\right),
\end{aligned}\tag{2.20}$$

where ${}_{C}G_n(x,y)$ are the cosine Genocchi polynomials and ${}_{S}G_n(x,y)$ are the sine Genocchi polynomials.

Theorem 2.9. *For a non-negative integer n and $|q|<1$, we obtain*

$$\begin{aligned}
&(i)\quad {}_{C}G_{n,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q G_{k,q}C_{n-k,q}(x,y),\\
&(ii)\quad {}_{S}G_{n,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q G_{k,q}S_{n-k,q}(x,y),
\end{aligned}\tag{2.21}$$

where $G_{n,q}$ are the q-Genocchi numbers.

Proof. (i) From the generating function of q-cosine Genocchi polynomials, there is a relation which is related to q-Genocchi numbers such as

$$\begin{aligned}
\sum_{n=0}^{\infty} {}_{C}G_{n,q}(x,y)&=\sum_{n=0}^{\infty}G_{n,q}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}C_{n,q}(x,y)\frac{t^n}{[n]_q!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q G_{k,q}C_{n-k,q}(x,y)\right)\frac{t^n}{[n]_q!}.
\end{aligned}\tag{2.22}$$

From the Equation (2.22), we obtain the result (i).
(ii) We also complete the proof of Theorem 2.9 (ii) using the q-Genocchi numbers and $S_{n,q}(x,y)$. □

Corollary 2.10. *When $q \to 1$ in Theorem 2.9, one holds*

$$\begin{aligned} &(i) \quad {}_CG_n(x,y) = \sum_{k=0}^{n} \binom{n}{k} G_k C_{n-k}(x,y), \\ &(ii) \quad {}_SG_n(x,y) = \sum_{k=0}^{n} \binom{n}{k} G_k S_{n-k}(x,y). \end{aligned} \tag{2.23}$$

Theorem 2.11. *Let $|q| < 1$ and $x, y \in \mathbb{R}$. Then, we obtain*

$$\begin{aligned} &(i) \quad \frac{\partial}{\partial x} {}_CG_{n,q}(x,y) = [n]_q {}_CG_{n-1,q}(x,y), \quad \frac{\partial}{\partial x} {}_SG_{n,q}(x,y) = [n]_q {}_SG_{n-1,q}(x,y), \\ &(ii) \quad \frac{\partial}{\partial y} {}_CG_{n,q}(x,y) = -[n]_q {}_SG_{n-1,q}(x,qy), \quad \frac{\partial}{\partial y} {}_SG_{n,q}(x,y) = [n]_q {}_CG_{n-1,q}(x,qy). \end{aligned} \tag{2.24}$$

Proof. (i) For any real number x, we derive the q-partial derivative for q-cosine Genocchi polynomials using the q-derivative of the q-cosine function as

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\partial}{\partial x} {}_CG_{n,q}(x,y) \frac{t^n}{[n]_q!} &= \sum_{n=0}^{\infty} {}_CG_{n,q}(x,y) \frac{t^{n+1}}{[n]_q!} \\ &= \sum_{n=0}^{\infty} [n]_q {}_CG_{n-1,q}(x,y) \frac{t^n}{[n]_q!}. \end{aligned} \tag{2.25}$$

Therefore, we obtain the required results.
(ii) In a similar method as in the proof of (i), we also consider the q-derivative of q-cosine Genocchi polynomials for any real number y such as

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\partial}{\partial y} {}_CG_{n,q}(x,y) \frac{t^n}{[n]_q!} &= -\sum_{n=0}^{\infty} {}_SG_{n,q}(x,qy) \frac{t^{n+1}}{[n]_q!} \\ &= -\sum_{n=0}^{\infty} [n]_q {}_SG_{n-1,q}(x,qy) \frac{t^n}{[n]_q!}. \end{aligned} \tag{2.26}$$

Then, we investigate the results of Theorem 2.11. □

Now, we find some relations between the q-cosine, sine Genocchi polynomials and others polynomials.

Theorem 2.12. *Let $x, y \in \mathbb{R}$. Then, we have*

$$\begin{aligned} &(i) \quad {}_CG_{n,q}(x,y) = [n]_q {}_C\mathcal{E}_{n-1,q}(x,y), \\ &(ii) \quad {}_SG_{n,q}(x,y) = [n]_q {}_S\mathcal{E}_{n-1,q}(x,y), \end{aligned} \tag{2.27}$$

where ${}_C\mathcal{E}_{n,q}(x,y)$ are the q-cosine Euler polynomials and ${}_S\mathcal{E}_{n,q}(x,y)$ are the q-sine Euler polynomials.

Proof. (*i*) From the generating function of q-cosine Genocchi polynomials, we derive a relation as

$$\begin{aligned}\sum_{n=0}^{\infty} {}_CG_{n,q}(x,y)\frac{t^n}{[n]_q!} &= \sum_{n=0}^{\infty} {}_C\mathcal{E}_{n,q}(x,y)\frac{t^{n+1}}{[n]_q!} \\ &= \sum_{n=0}^{\infty} [n]_q {}_C\mathcal{E}_{n-1,q}(x,y)\frac{t^n}{[n]_q!}.\end{aligned} \tag{2.28}$$

Therefore, we find the required relation.
(*ii*) From q-sine Genocchi polynomials, we also have the required relation (*ii*). □

Corollary 2.13. *Consider $q \to 1$ in Theorem 2.12. Then one holds*

$$\begin{aligned}&(i) \quad {}_CG_n(x,y) = n {}_C\mathcal{E}_{n-1}(x,y), \\ &(ii) \quad {}_SG_n(x,y) = n {}_S\mathcal{E}_{n-1,q}(x,y),\end{aligned} \tag{2.29}$$

where ${}_CG_n(x,y)$ are the cosine Genocchi polynomials, ${}_SG_n(x,y)$ are the sine Genocchi polynomials, ${}_C\mathcal{E}_n(x,y)$ are the cosine Euler polynomials, and ${}_S\mathcal{E}_n(x,y)$ are the sine Euler polynomials.

Theorem 2.14. *Let $x, y \in \mathbb{R}$. Then, we investigate*

$$\begin{aligned}&(i) \quad {}_CG_{n,q}(x,y) + 2{}_CB_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(2{}_CB_{k,q}(x,y) - {}_CG_{k,q}(x,y)\right), \\ &(ii) \quad {}_SG_{n,q}(x,y) + 2{}_SB_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(2{}_SB_{k,q}(x,y) - {}_SG_{k,q}(x,y)\right),\end{aligned} \tag{2.30}$$

where ${}_CB_{n,q}(x,y)$ are the q-cosine Bernoulli polynomials and ${}_SB_{n,q}(x,y)$ are the q-sine Bernoulli polynomials.

Proof. (*i*) To find a relation between q-cosine Genocchi polynomials and q-Bernoulli polynomials, we transform the q-cosine Genocchi polynomials as

$$\sum_{n=0}^{\infty} {}_CG_{n,q}(x,y)\frac{t^n}{[n]_q!} = \frac{2(e_q(t)-1)}{e_q(t)+1} \sum_{n=0}^{\infty} {}_CB_{n,q}(x,y)\frac{t^n}{[n]_q!}. \tag{2.31}$$

When $e_q(t) \neq -1$, we find

$$\sum_{n=0}^{\infty} {}_CG_{n,q}(x,y)\frac{t^n}{[n]_q!}\left(\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!} + 1\right) = 2\sum_{n=0}^{\infty} {}_CB_{n,q}(x,y)\frac{t^n}{[n]_q!}\left(\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!} - 1\right). \tag{2.32}$$

By using Cauchy' product in Equation (2.32), we derive

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q {}_CG_{k,q}(x,y)+{}_CG_{n,q}(x,y)\right)\frac{t^n}{[n]_q!}\\&=2\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q {}_CB_{k,q}(x,y)-{}_CB_{n,q}(x,y)\right)\frac{t^n}{[n]_q!}.\end{aligned}\tag{2.33}$$

Therefore, we find the required result.
(ii) By using the same method in q-sine Genocchi polynomials, we obtain the desired result. □

Corollary 2.15. *Consider $q \to 1$ in Theorem 2.14. Then, the following holds*

$$\begin{aligned}&(i)\quad {}_CG_n(x,y)+2{}_CB_n(x,y)=\sum_{k=0}^{n}\binom{n}{k}\left(2{}_CB_k(x,y)-{}_CG_k(x,y)\right),\\&(ii)\quad {}_SG_n(x,y)+2{}_SB_n(x,y)=\sum_{k=0}^{n}\binom{n}{k}\left(2{}_SB_k(x,y)-{}_SG_k(x,y)\right),\end{aligned}\tag{2.34}$$

where ${}_CB_n(x,y)$ are the cosine Bernoulli polynomials and ${}_SB_n(x,y)$ are the sine Bernoulli polynomials.

3. Some properties using q-analogues of subtraction and some symmetric properties

In this section, we introduce some special properties of q-cosine and q-sine Genocchi polynomials using q-analogues of subtraction and addition. We also find several symmetric properties of these polynomials taking forms.

Lemma 3.1. *Let $|q| < 1$ and $r \in \mathbb{R}$. Then, we have*

$$\begin{aligned}&(i)\quad {}_CG_{n,q}((x\oplus r)_q,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{\binom{n-k}{2}}{}_CG_{k,q}(x,y)r^{n-k},\\&(ii)\quad {}_CG_{n,q}((x\ominus r)_q,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q (-1)^{n-k}q^{\binom{n-k}{2}}{}_CG_{k,q}(x,y)r^{n-k}.\\&(iii)\quad {}_SG_{n,q}((x\oplus r)_q,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{\binom{n-k}{2}}{}_SG_{k,q}(x,y)r^{n-k},\\&(iv)\quad {}_SG_{n,q}((x\ominus r)_q,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q (-1)^{n-k}q^{\binom{n-k}{2}}{}_SG_{k,q}(x,y)r^{n-k}.\end{aligned}\tag{3.1}$$

Proof. (i) By substituting q-analogues of addition instead of x in the generating function of q-cosine Genocchi polynomials, we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_{C}G_{n,q}((x\oplus r)_q, y)\frac{t^n}{[n]_q!} &= \frac{2t}{e_q(t)+1}e_q(tx)COS_q(ty)E_q(tr)\\ &= \sum_{n=0}^{\infty} {}_{C}G_{n,q}(x,y)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty} q^{\binom{n}{2}}r^n\frac{t^n}{[n]_q!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q q^{\binom{n-k}{2}}{}_{C}G_{k,q}(x,y)r^{n-k}\right)\frac{t^n}{[n]_q!}.\end{aligned} \tag{3.2}$$

By using q-exponential functions and comparing the coefficients of both sides in the Equation (3.2), we derive the required result.
(ii) We can consider the q-analogues of subtraction instead of x in the q-cosine Genocchi polynomials, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} {}_{C}G_{n,q}((x\ominus r)_q, y)\frac{t^n}{[n]_q!} &= \frac{2t}{e_q(t)+1}e_q((x\ominus r)_q, t)COS_q(ty)\\ &= \sum_{n=0}^{\infty} {}_{C}G_{n,q}(x,y)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty} q^{\binom{n}{2}}(-r)^n\frac{t^n}{[n]_q!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q (-1)^{n-k}q^{\binom{n-k}{2}}{}_{C}G_{k,q}(x,y)r^{n-k}\right)\frac{t^n}{[n]_q!}.\end{aligned} \tag{3.3}$$

From Equation (3.3), we find the desired result.
$(iii), (iv)$ By using the similar manner as in the proof of (i) and (ii) for q-sine Genocchi polynomials, we find Lemma 3.1 (iii) and (iv), respectively. □

Theorem 3.2. *For $|q| < q$ and real numbers r, x, y, we derive*

$$\begin{aligned}&(i)\quad {}_{C}G_{n,q}((x\oplus r)_q, y) + {}_{C}G_{n,q}((x\ominus r)_q, y)\\ &= \begin{cases} 2\sum_{k=0}^{n}\begin{bmatrix} n\\ 2k+1\end{bmatrix}_q q^{\binom{n-(2k+1)}{2}}{}_{C}G_{2k+1,q}(x,y)r^{n-(2k+1)}, & \text{if } \; n : odd\\ 2\sum_{k=0}^{n}\begin{bmatrix} n\\ 2k\end{bmatrix}_q q^{\binom{n-2k}{2}}{}_{C}G_{2k,q}(x,y)r^{n-2k}, & \text{if } \; n : even.\end{cases}\\ &(ii)\quad {}_{S}G_{n,q}((x\oplus r)_q, y) + {}_{S}G_{n,q}((x\ominus r)_q, y)\\ &= \begin{cases} 2\sum_{k=0}^{n}\begin{bmatrix} n\\ 2k+1\end{bmatrix}_q q^{\binom{n-(2k+1)}{2}}{}_{S}G_{2k+1,q}(x,y)r^{n-(2k+1)}, & \text{if } \; n : odd\\ 2\sum_{k=0}^{n}\begin{bmatrix} n\\ 2k\end{bmatrix}_q q^{\binom{n-2k}{2}}{}_{S}G_{2k,q}(x,y)r^{n-2k}, & \text{if } \; n : even.\end{cases}\end{aligned} \tag{3.4}$$

Proof. (i) From Lemma 3.1 (i) and (ii), we investigate

$$\begin{aligned} &{}_{C}G_{n,q}((x\oplus r)_q,y)+{}_{C}G_{n,q}((x\ominus r)_q,y)\\ &=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q q^{\binom{n-k}{2}}({}_{C}G_{k,q}(x,y)+(-1)^{n-k}{}_{C}G_{k,q}(x,y))r^{n-k}. \end{aligned} \tag{3.5}$$

Therefore, we finish the proof of Theorem 3.2. (i).
(ii) By using Lemma 3.1 (iii) and (iv) for q-sine Genocchi polynomials, we find the required result. □

Corollary 3.3. *Consider $r=1$ in Theorem 3.2. Then, the following holds*

$$\begin{aligned} &(i)\quad {}_{C}G_{n,q}((x\oplus 1)_q,y)+{}_{C}G_{n,q}((x\ominus 1)_q,y)\\ &=\begin{cases} 2\sum_{k=0}^{n}\begin{bmatrix} n\\ 2k+1\end{bmatrix}_q q^{\binom{n-(2k+1)}{2}}{}_{C}G_{2k+1,q}(x,y), & \textit{if}\quad n\,:\,\textit{odd}\\[2ex] 2\sum_{k=0}^{n}\begin{bmatrix} n\\ 2k\end{bmatrix}_q q^{\binom{n-2k}{2}}{}_{C}G_{2k,q}(x,y), & \textit{if}\quad n\,:\,\textit{even}. \end{cases}\\ &(ii)\quad {}_{S}G_{n,q}((x\oplus 1)_q,y)+{}_{S}G_{n,q}((x\ominus 1)_q,y)\\ &=\begin{cases} 2\sum_{k=0}^{n}\begin{bmatrix} n\\ 2k+1\end{bmatrix}_q q^{\binom{n-(2k+1)}{2}}{}_{S}G_{2k+1,q}(x,y), & \textit{if}\quad n\,:\,\textit{odd}\\[2ex] 2\sum_{k=0}^{n}\begin{bmatrix} n\\ 2k\end{bmatrix}_q q^{\binom{n-2k}{2}}{}_{S}G_{2k,q}(x,y), & \textit{if}\quad n\,:\,\textit{even}. \end{cases} \end{aligned} \tag{3.6}$$

Corollary 3.4. *From Lemma 3.1, we have*

$$\begin{aligned} &(i)\quad {}_{C}G_{n,q}((x\oplus r)_q,y)+{}_{S}G_{n,q}((x\oplus r)_q,y)\\ &=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q q^{\binom{n-k}{2}}\left({}_{C}G_{k,q}(x,y)+{}_{S}G_{k,q}(x,y)\right)r^{n-k},\\ &(ii)\quad {}_{C}G_{n,q}((x\ominus r)_q,y)+{}_{S}G_{n,q}((x\ominus r)_q,y)\\ &=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q (-1)^{n-k}q^{\binom{n-k}{2}}\left({}_{C}G_{k,q}(x,y)+{}_{S}G_{k,q}(x,y)\right)r^{n-k}. \end{aligned} \tag{3.7}$$

Theorem 3.5. *For any integers a, b and real numbers x, y, we derive*

$$\begin{aligned}
&(i) \quad \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^{n-k} b^k {}_C G_{n-k,q}(bx, by) {}_C G_{k,q}(ax, ay) \\
&\quad = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b^{n-k} a^k {}_C G_{n-k,q}(ax, ay) {}_C G_{k,q}(bx, by), \\
&(ii) \quad \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^{n-k} b^k {}_S G_{n-k,q}(bx, by) {}_S G_{k,q}(ax, ay) \\
&\quad = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b^{n-k} a^k {}_S G_{n-k,q}(ax, ay) {}_S G_{k,q}(bx, by).
\end{aligned} \tag{3.8}$$

Proof. (i) To find a symmetric property which is related to q-cosine Genocchi polynomials, we set up form A as the following.

$$A := \frac{4\left(te_q(abtx)COS_q(abty)\right)^2}{(e_q(at)+1)(e_q(bt)+1)}. \tag{3.9}$$

From A, we find the following equation:

$$\begin{aligned}
A &= \frac{2t}{e_q(at)+1} e_q(abtx) COS_q(abty) \frac{2t}{e_q(bt)+1} e_q(abtx) COS_q(abty) \\
&= \sum_{n=0}^{\infty} {}_C G_{n,q}(bx, by) \frac{(at)^n}{[n]_q!} \sum_{n=0}^{\infty} {}_C G_{n,q}(ax, ay) \frac{(bt)^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^{n-k} b^k {}_C G_{n-k,q}(bx, by) {}_C G_{k,q}(ax, ay) \right) \frac{t^n}{[n]_q!},
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
A &= \sum_{n=0}^{\infty} {}_C G_{n,q}(ax, ay) \frac{(bt)^n}{[n]_q!} \sum_{n=0}^{\infty} {}_C G_{n,q}(bx, by) \frac{(at)^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b^{n-k} a^k {}_C G_{n-k,q}(ax, ay) {}_C G_{k,q}(bx, by) \right) \frac{t^n}{[n]_q!}.
\end{aligned} \tag{3.11}$$

From (3.10) and (3.11), we immediately finish the proof of (i).
(ii) If we use the similar method as in the proof of (i) in q-sine Genocchi polynomials, then we obtain the required result. $\square$

Corollary 3.6. *For $q \to 1$ in Theorem 3.5, one holds*

$$\begin{aligned} (i) \quad & \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k}{}_{C}G_{n-k}(bx, by)_{C}G_{k}(ax, ay) \\ &= \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^{k}{}_{C}G_{n-k,q}(ax, ay)_{C}G_{k}(bx, by), \\ (ii) \quad & \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k}{}_{S}G_{n-k,q}(bx, by)_{S}G_{k}(ax, ay) \\ &= \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^{k}{}_{S}G_{n-k,q}(ax, ay)_{S}G_{k}(bx, by), \end{aligned} \tag{3.12}$$

where ${}_{C}G_{n}(x, y)$ are the cosine Genocchi polynomials and ${}_{S}G_{n}(x, y)$ are the sine Genocchi polynomials.

Corollary 3.7. *Let $a = 1$ in Theorem 3.5. Then, the following holds*

$$\begin{aligned} (i) \quad & \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q} b^{k}{}_{C}G_{n-k,q}(bx, by)_{C}G_{k,q}(x, y) \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q} b^{n-k}{}_{C}G_{n-k,q}(x, y)_{C}G_{k,q}(bx, by), \\ (ii) \quad & \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q} b^{k}{}_{S}G_{n-k,q}(bx, by)_{S}G_{k,q}(x, y) \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q} b^{n-k}{}_{S}G_{n-k,q}(x, y)_{S}G_{k,q}(bx, by). \end{aligned} \tag{3.13}$$

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