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ON REFORMULATED INJECTIVE CHROMATIC INDEX OF GRAPHS

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Abstract. For a graph $G = (V, E)$, a vertex coloring (or, simply, a coloring) of G is a function $C : V(G) \to \{1, 2, ..., k\}$ (using the non-negative integers $\{1, 2, ..., k\}$ as colors). We say that a coloring of a graph G is injective if for every vertex $v \in V(G)$, all the neighbors of v are assigned with distinct colors. The injective chromatic number $\chi_i(G)$ of a graph G is the least k such that there is an injective $k-$coloring [6]. In this paper, we study a natural variation of the injective coloring problem: coloring the edges of a graph under the same constraints (alternatively, to investigate the injective chromatic number of line graphs), we define the $k-$ injective edge coloring of a graph G as a mapping $C : E(G) \to \{1, 2, ..., k\}$, such that for every edge $e \in E(G)$, all the neighbors edges of e are assigned with distinct colors. The injective chromatic index $\chi'_{in}(G)$ of G is the least positive integer k such that G has $k-$ injective edge coloring, exact values of the injective chromatic index of different families of graphs are obtained, some related results and bounds are established. Finally, we define the injective clique number ω_{in} and state a conjecture, that, for any graph G, $\omega_{in} \le \chi'_{in}(G) \le \omega_{in} + 2$.

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1. Introduction

Throughout this research work, by a graph we mean finite graph without loops and parallel edges. Any notations or terminology not specifically defined here, we refer the book [7]. More details about originality of the coloring and the history of the famous Four-Colour Problem and its related are reported in [5]. One of the application of coloring is the satellite range scheduling [12]. The open neighborhood of an edge $e \in E$ is denoted as $N(e)$ and it is the set of all edges adjacent to e in G. Further, $N[e] = N(e) \cup \{e\}$ is the closed neighborhood

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of e in G. An edge coloring of a graph G is a function $f : E(G) \to C$, where C is a set of distinct colors. For any positive integer k, a k-edge coloring is an edge coloring that uses exactly k distinct colors. A proper edge coloring of a graph G is an edge coloring such that no two adjacent edges are assigned the same color. Thus a proper edge coloring f of G is a function $f : E(G) \to C$ such that $f(e_i) \neq f(e_j)$ whenever edges e_i and e_j are adjacent in G. The edge chromatic number $\chi^{'}(G)$ of G is the smallest integer k such that G has an edge coloring. A proper edge coloring of a graph G is called a strong edge coloring if no edge $e \in E(G)$ is adjacent to two edges of the same color. The strong chromatic index of G, denoted by $\chi_s^{'}(G)$, is the least integer k for which G is strongly edge k-colorable. The strong edge coloring with its applications is studied in [8].

An injective k-coloring of a graph G is an assignment of at most k colors to the vertices of G such that two vertices sharing a common neighbor must have distinct colors. The injective chromatic number $\chi_i(G)$ of a graph G is the minimum integer k such that G has an injective k-coloring. This concept was introduced and studied for the first time in [6]. In 2015, [4] Cardoso et al. have introduced the notion of injective edge coloring as follows : An injective edge coloring of a graph G is an edge coloring of G such that if e_1, e_2 and e_3 are consecutive edges in G , then e_1 and e_3 receive the different colors. The injective edge chromatic number of a graph G is the minimum number of colors permitted in an injective edge coloring of G.

In this paper, we study a natural variation of the injective coloring problem: coloring the edges of a graph under the same constraints (alternatively, to investigate the injective chromatic number of line graphs).

The line graph of a graph G is denoted by $L(G)$ is a graph H whose vertex set is equal to the edge set of G, with two vertices in H being adjacent if the corresponding edges in G are adjacent (i.e., have a common vertex) [7]. Let G be a simple graph with vertex set $V(G)$. Alwardi et al in ([1], [2], [3]) have introduced and studied the common neighborhood property between the vertices by defining a new graph called common neighborhood graph and a new matrix called common neighborhood matrix. The common neighborhood graph (or, shorter congraph) of G, denoted by $con(G)$, is the graph with $V(con(G)) = V(G)$, in which two vertices are adjacent if they have a common neighbor in G, [1]. By the definitions of congraph and injective chromatic number of a graph G, it is easy to see that $\chi_i(G) = \chi(con(G))$.

In ([9, 10, 11]) the authors studied some types of graph energies and labeling which give interesting methodology for studying graph parameters. As usual P_n, C_n, K_n and W_n are the n-vertex path, cycle, complete, and wheel graph, respectively, $K_{m,n}$ is the complete bipartite graph on $m + n$ vertices and S_n is the star with n vertices.

The concept of common neighborhood [1, 2], injective chromatic number [6] and the huge application of edge chromatic number of a graph motivated us to introduce and study the reformulated injective chromatic index of graphs.

2. Basic results

In this section, we define the reformulated injective chromatic index of a graph and give several preliminary results and straightforward facts regarding the injective chromatic index of graphs.

Definition 2.1. A k-edge coloring of a graph $G = (V, E)$ is a mapping $C : E(G) \to \{1, 2, ..., k\}$. The edge coloring C is called injective edge coloring of a graph $G = (V, E)$ if, for every edge $e \in E(G)$, all the neighbors edges of e are assigned with distinct colors. The injective chromatic index $\chi'_{in}(G)$ of G is the least positive integer k such that G has a k–injective edge coloring. Note that the injective coloring is not necessarily a proper coloring.

The proofs of the following propositions are straightforward.

Proposition 2.2. *For any path P_n on $n \geq 4$ vertices, $\chi'_{in}(P_n) = 2$.*

Proposition 2.3. *For any cycle C_n on $n \geq 3$ vertices,*

$$\chi'_{in}(C_n) = \begin{cases} 2, & \text{if } n = 4k,\ k \geq 1; \\ 3, & \text{otherwise.} \end{cases}$$

Proposition 2.4. *For a wheel graph W_n with $n \geq 5$ vertices,*

$$\chi'_{in}(W_n) = \begin{cases} n+4, & \text{if } n \equiv 0 \,(mod\, 3); \\ n+2, & \text{if } n \equiv 1 \,(mod\, 3); \\ n+3, & \text{if } n \equiv 2 \,(mod\, 3). \end{cases}$$

The graph obtained from the wheel graph W_{n+1} by adding pendent edge at each vertex of the cycle called helm graph H_n [9].

Proposition 2.5. *Let G be the helm graph H_n with $n \geq 4$ vertices. Then*

$$\chi'_{in}(G) = \begin{cases} n+5, & \text{if } n \equiv 0 \,(mod\, 3); \\ n+6, & \text{if } n \equiv 1 \,(mod\, 3); \\ n+7, & \text{if } n \equiv 2 \,(mod\, 3). \end{cases}$$

Proposition 2.6. *If $L(G)$ is the line graph of G. Then $\chi'_{in}(G) = \chi_i(L(G))$.*

Proposition 2.7. *Let $G = \bigcup_{j=1}^{m}(G_j)$. Then $\chi'_{in}(G) = \max\{\chi'_{in}(G_j) : j = 1, 2, .., m\}$.*

Proposition 2.8. *For any graph G,*

$$\chi'_{in}(G) = \chi(con(L(G))).$$

Proof. By the definitions of the line graph and the common neighborhood graph (congraph) of a graph G, we have $\chi'(G) = \chi(L(G))$ and $\chi_i(G) = \chi(con(G))$. Therefore,

$$\chi'_{in}(G) = \chi_i(L(G)) = \chi(con(L(G))).$$

□

Proposition 2.9. *Let G be a connected graph with $n \geq 4$ vertices. Then*

$$\chi^{'}(G) \leq \chi^{'}_{in}(G).$$

Further, equality holds if $G \cong P_n$, $K_{1,n}$ or C_{4k} where $k \geq 1$.

Proof. Let G be a connected graph with $n \geq 4$ vertices. Then $L(G)$ is connected graph and not isomorphic to K_2. Therefore, $\chi(L(G)) \leq \Delta^{'}(G)$ and by the definition of the injective chromatic index of a graph, we have $\Delta^{'}(G) \leq \chi^{'}_{in}(G)$. Thus, $\chi(L(G)) \leq \chi^{'}_{in}(G)$. Hence, $\chi^{'}(G) \leq \chi^{'}_{in}(G)$.
Obviously, if $G \cong P_n$ or C_{4k}, $k \geq 1$, then $\chi^{'}(G) = \chi^{'}_{in}(G) = 2$ and if $G \cong K_{1,n}$, then $\chi^{'}(G) = \chi^{'}_{in}(G) = n$. □

Proposition 2.10. *If H is a subgraph of a connected graph G, then $\chi^{'}_{in}(H) \leq \chi^{'}_{in}(G)$.*

The square of a simple graph G is also a simple graph denoted by G^2 has the same vertices as G in which any two vertices u and v are adjacent in G^2 if and only if $d(u,v) \leq 2$, where $d(u,v)$ is the distance between u and v in G [7]. According to Proposition 2.10, we have the following corollary.

Corollary 2.11. *For any connected graph G, $\chi^{'}_{in}(G) \leq \chi^{'}_{in}(G^2)$.*

In the following theorem, we will determine the necessary and sufficient condition to $\chi^{'}_{in}(G) = q$ for a connected graph G with $q \geq 3$ edges. Actually, for any graph G of diameter greater than or equal four ($diam(G) \geq 4$) $\chi^{'}_{in}(G) < q$, so we will discuss the graphs of diameter less than or equal three ($diam(G) \leq 3$).

Theorem 2.12. *Let G be a connected graph with $q \geq 3$ edges and $diam(G) \leq 3$. Then $\chi^{'}_{in}(G) = q$ if and only if for any two edges e and g in G there exists an edge adjacent to both of e and g.*

Proof. Let G be a connected graph with $q \geq 3$ edges and $diam(G) \leq 3$. Suppose that any two edges e and g in G have a common edge. Then $con(L(G)) \cong K_q$. Hence by Proposition 2.8, $\chi^{'}_{in}(G) = q$.
Conversely, let G be a connected graph with $q \geq 3$ edges and $diam(G) \leq 3$. Suppose that $\chi^{'}_{in}(G) = q$. Then by Proposition 2.8, $\chi(con(L(G))) = q$. Therefore, $con(L(G)) \cong K_q$ that means any two vertices in $L(G)$ having a common neighbor. Hence, any two edges in G have a common edge neighbor. □

Corollary 2.13. *Let G be a complete graph with $n \geq 3$ vertices. Then*

$$\chi^{'}_{in}(G) = \frac{n(n-1)}{2}.$$

Corollary 2.14.

(1) *For any bi-star graph $G \cong B(m,n)$,*

$$\chi^{'}_{in}(G) = m + n + 1.$$

(2) *For any strongly regular graph G with the parameters (n, k, λ, μ), where $\lambda,\ \mu \geq 1$,*

$$\chi'_{in}(G) = \frac{nk}{2}.$$

(3) *For any complete bipartite graph $G \cong K_{m,n}$, where $3 \leq m \leq n$,*

$$\chi'_{in}(G) = mn.$$

(4) *For any multi complete bipartite graph $G \cong K_{n_1,n_2,\dots,n_t}$, where $n_i \geq 3$, $i = 1, 2, \dots, t$, we have,*

$$\chi'_{in}(K_{n_1,n_2,\dots,n_t}) = \prod_{i=1}^{t} n_i.$$

Recall that in [7], the **join** $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2.

In the following, we obtain the exact value of χ'_{in} for the join graph $G = G_1 + G_2$ of any two graphs G_1 and G_2, the firefly graph $F_{s,t,n-2s-2t-1}$, the corona product $C_4 \circ \overline{K_t}$, the grid graphs $P_t \,\Box\, K_2$, $P_t \,\Box\, P_s$ and $C_t \,\Box\, K_2$.

Proposition 2.15. *Let G_1 and G_2 with $n_1, n_2 \geq 2$ vertices, respectively, be two connected graphs. Then $\chi'_{in}(G_1 + G_2) = \chi'_s(G_1) + \chi'_s(G_2) + n_1 n_2$.*

Proof. Let G_1 and G_2 with $n_1, n_2 \geq 2$ vertices, respectively, be two connected graphs. From the definition of the join $G = G_1 + G_2$, we have $E(G) = E_1 \cup E_2 \cup B$, where $B = \{uv : u \in V_1 \, and \, v \in V_2\}$. Since G_1 and G_2 are connected, then any two edges of B having a common edge neighbor, thus all the edges of B should assigned by different colors, also every edge in E_1 has a common edge neighbor with every edge in E_2 so we need to color the edges of E_1 by colors different of the colors of the edges of B and E_2 (and the same for E_2). But, how we can color the edges of E_1 and E_2?. Since $n_1, n_2 \geq 2$ and each vertex of G_1 adjacent to all the vertices of G_2 and vice versa (definition of $G = G_1 + G_2$), then we should color the edges of E_1 (and E_2) such that no two adjacent edges sharing a same color (i.e strong edge coloring). Hence, $\chi'_{in}(G_1 + G_2) = \chi'_s(G_1) + \chi'_s(G_2) + n_1 n_2$. □

We recall that in [10], A firefly graph $F_{s,t,n-2s-2t-1}$, where $s \geq 0$, $t \geq 0$, $n - 2s - 2t - 1 \geq 0$ is a graph of order n that consists of s triangles, t pendent paths of length 2 and $n - 2s - 2t - 1$ pendant edges sharing a common vertex (see Figure 1.).

Let $\mathfrak{F}n$ be the set of all firefly graphs $F_{s,t,n-2s-2t-1}$. Note that, $\mathfrak{F}n$ contains the stars S_n ($\cong F_{0,0,n-1}$), stretched stars ($\cong F_{0,t,n-2t-1}$), friendship graphs ($\cong F_{\frac{n-1}{2},0,0}$) and butterfly graphs ($\cong F_{s,0,n-2s-1}$).

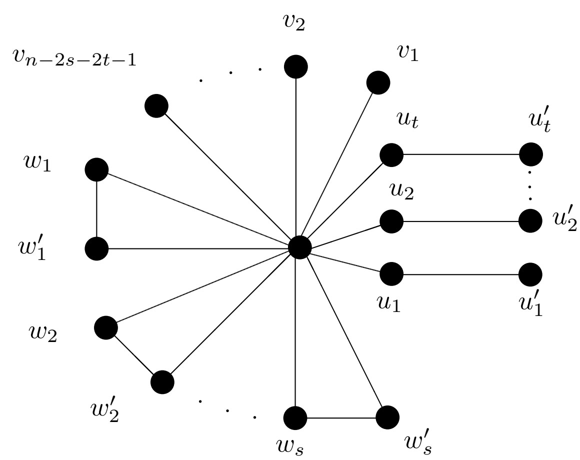

FIGURE 1. Firefly graph $F_{s,t,n-2s-2t-1}$

Proposition 2.16. *For any firefly graph $G \cong F_{s,t,n-2s-2t-1}$,*

$$\chi'_{in}(G) = n - t = \Delta(G) + 1.$$

Proof. Let $G \cong F_{s,t,n-2s-2t-1}$ be a firefly graph, where $s \geq 0$, $t \geq 0$, $n - 2s - 2t - 1 \geq 0$. Let the s triangles and t pendent paths of length 2 and $n-2s-2t-1$ pendant edges sharing a common vertex v in G. Clearly, $\Delta(G) = deg(v) = n - t - 1$. The line graph of G contains a clique isomorphic to K_{n-t-1} and s triangles, every triangle attached with one edge and t edges attached. Now, it is easy to see that the congraph of the line graph $L(G)$ is $H \cong \overline{K_{s+t}} + K_{n-t-1}$ and hence, $\chi(H) = n-t$. Hence by Proposition 2.8, we have $\chi'_{in}(G) = n-t = \Delta(G)+1$. $\square$

The corona product $G_1 \circ G_2$ of two graphs G_1 and G_2, where $V(G_1)$, $V(G_2)$ are the set of vertices of G_1, G_2, respectively, is the graph obtained by taking $|V(G_1)|$ copies of G_2 and joining each vertex of the i-th copy with the corresponding vertex $u \in V(G_1)$ [13].

Proposition 2.17. *For any positive integer $t \geq 1$, $\chi'_{in}(C_4 \circ \overline{K_t}) = 2t + 4$.*

Proof. Let $G = C_4 \circ \overline{K_t}$ and let H be the subgraph of G induced by the set of vertices $V' \subseteq V(G)$, where $V' = \{v_1, v_2, v_3, v_4, a_1, \ldots, a_t, b_1, \ldots, b_t\}$, [see Figure 2.]. In the graph H any two edges sharing a common edge accept two pairs of edges those are v_1v_3, v_3v_4 and v_2v_4, v_4v_3. Therefore, all the edges of H should be colored by different colors accept the edge v_3v_4. Hence, $\chi'_{in}(H) = 2t + 3$. Now, by Proposition 2.10, we know that $\chi'_{in}(H) \leq \chi'_{in}(G)$. But in G, since $deg(v_3)$, $deg(v_4) \geq 3$, then we should color the edge v_3v_4 by one more different color from the edges in H and we can repeat the colors of H for the remaining edges in G. Thus, $\chi'_{in}(G) > \chi'_{in}(H)$ and $\chi'_{in}(G) \leq \chi'_{in}(H)+1$. Hence, $\chi'_{in}(C_4 \circ \overline{K_t}) = 2t+4$. $\square$

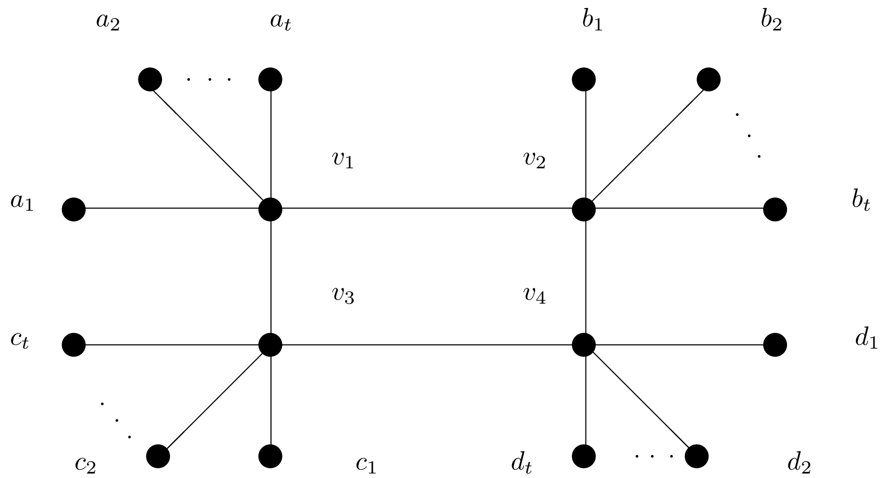

FIGURE 2. Graph $C_4 \circ \overline{K_t}$

The Cartesian product $G_1 \Box G_2$ of two graphs G_1 and G_2, where $V(G_1)$, $E(G_1)$ and $V(G_2)$, $E(G_2)$ are the sets of vertices and edges of G_1 and G_2, respectively, has the vertex set $V(G_1) \times V(G_2)$ and two vertices (u, u') and (v, v') are connected by an edge if and only if either ($u = v$ and $u'v' \in E(G_2)$) or ($u' = v'$ and $uv \in E(G_1)$) [7].

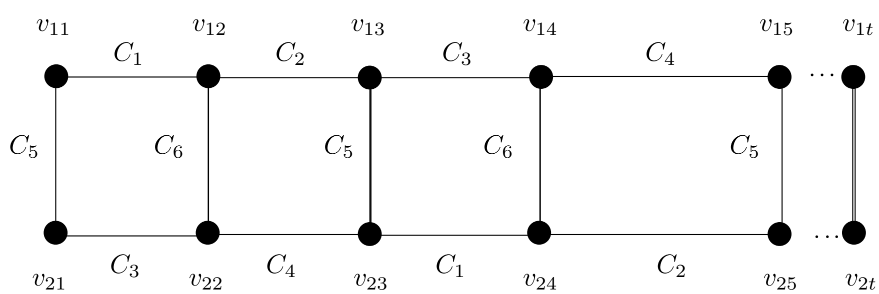

FIGURE 3. Graph $P_t \Box K_2$

Proposition 2.18. *Let $G \cong P_t \,\square\, K_2$, where $n \geq 4$. Then $\chi'_{in}(G) = 6$.*

Proof. Color the edges $v_{11}v_{12}, v_{12}v_{13}, \ldots, v_{1(t-1)}v_{1t}$ by the sequence of colors $C_1, C_2, C_3, C_4, C_1, C_2, C_3, C_4, C_1, ...$ and the edges $v_{21}v_{22}, v_{22}v_{23}, \ldots, v_{2(t-1)}v_{2t}$ by the sequence of colors $C_3, C_4, C_1, C_2, C_3, C_4, C_1, C_2, C_3, ...$, respectively, and color the edges $v_{11}v_{21}, v_{12}v_{22}, \ldots, v_{1t}v_{2t}$ by the sequence of colors $C_5, C_6, C_5, C_6, ...$ (see Figure 3.). It is easy to check that, this coloring is 6-injective edge coloring for the graph $G \cong P_t \,\square\, K_2$. Therefore,

$$\chi'_{in}(G) \leq 6 \tag{1}$$

Now, from Figure 3., clearly that there is at least a subgraph H in G isomorphic to $C_4 \circ K_1$ and by Proposition 2.17, $\chi'_{in}(H) = 6$. Also, we have by Proposition 2.10,

$$\chi'_{in}(H) \leq \chi'_{in}(G) \tag{2}$$

From equations (1) and (2), the proof is complete. □

By the same argument in the proof of Proposition 2.18, we have the following result for $C_t \,\square\, K_2$.

Proposition 2.19. *For any integer $t \geq 3$,*

$$\chi'_{in}(C_t \,\square\, K_2) = \begin{cases} 9, & \text{if } t = 3; \\ 8, & \text{if } t = 5, 6; \\ 6, & \text{if } t = 4k,\ k \geq 1; \\ 7, & \text{otherwise.} \end{cases}$$

Proposition 2.20. *Let $G \cong P_t \,\square\, P_s$, where $t \geq 4$ and $s \geq 3$. Then $\chi'_{in}(G) = 8$.*

Proof. Let us consider the graph G as horizontal paths from up to down labeling $L_1, L_2, \ldots, L_s$ and vertical paths from the left to the right $M_1, M_2, \ldots, M_t$. color the edges in the horizontal paths, alternately, by the sequences of colors $S_1 = \{C_1, C_2, C_3, C_4, C_1, C_2, C_3, C_4, C_1, ...\}$ and $S_2 = \{C_3, C_4, C_1, C_2, C_3, C_4, C_1, C_2, C_3, ...\}$, that means color L_1 by S_1 and L_2 by S_2 and so on.

Similarly, color the edges in the vertical paths by the sequences of colors $S_3 = \{C_5, C_7, C_6, C_8, C_5, C_7, C_6, C_8, C_5, ...\}$ and $S_4 = \{C_6, C_8, C_5, C_7, C_6, C_8, C_5, C_7, C_6, ...\}$, alternately. It is not difficult to check that, any two edges sharing a common edge have different colors which means that this coloring is 8-edge injective coloring for G. Therefore,

$$\chi'_{in}(G) \leq 8 \tag{3}$$

Now, $H = C_4 \circ \overline{K_2}$ is a subgraph of G, then by Proposition 2.17, $\chi'_{in}(H) = 8$. Since $\chi'_{in}(H) \leq \chi'_{in}(G)$, we get

$$\chi'_{in}(G) \geq 8, \tag{4}$$

then by (3) and (4), we get $\chi'_{in}(G) = 8$. □

Discussion of the section:
In the above section, motivated by the huge application of the edge chromatic number and injective chromatic number of graph, we have introduced and defined the injective chromatic index of graphs $\chi_{in}^{'}$ and due to the importance of the families and the operations of graphs and their various applications, we have obtained the exact values of $\chi_{in}^{'}$ for the very important graphs like paths, cycles, wheel, Helm, bi-star, complete bipartite and strongly regular graphs, also, we calculate the same index for the join graph $G = G_1 + G_2$ of any two graphs G_1 and G_2, the firefly graph $F_{s,t,n-2s-2t-1}$, the corona product $C_4 \circ \overline{K_t}$, the grid graphs $P_t \square K_2$, $P_t \square P_s$ and $C_t \square K_2$ which importance in social and computer networks. Also, we have got some general properties of $\chi_{in}^{'}$ like, its value for the union of some graphs and the sufficient and necessary condition for $\chi_{in}^{'}$ to be equal the total number of edges, the relation between, a connected graph G and its subgraphs, a graph G and its square graph G^2, a graph G and its common neighborhood graph $con(G)$ with respect to $\chi_{in}^{'}$. Finally, we have got the relation between $\chi_{in}^{'}$ and the chromatic number χ, the edge chromatic number $\chi^{'}$ and the injective chromatic number χ_i.

It is easy to see that, the injective edge coloring that have introduced in [4] is not an injective coloring of the line graph, so, we introduce the injective chromatic index to study a natural variation of the injective coloring problem to investigate the injective chromatic number of line graphs.

3. Certain values and bounds

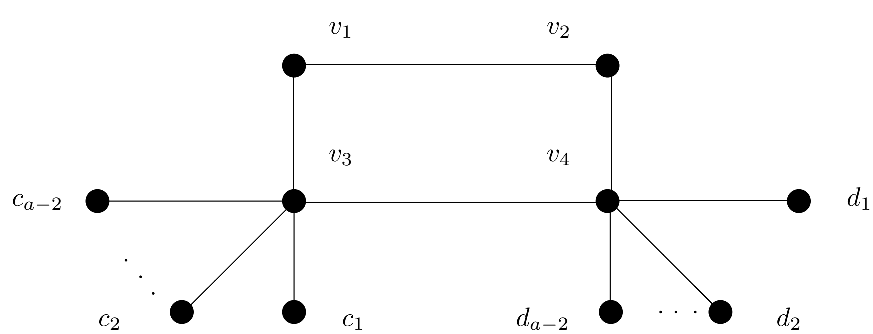

FIGURE 4. Graph with $\chi^{'}(G) = a$ and $\chi_{in}^{'}(G) = 2a - 1$

Proposition 3.1. *For any positive integer $a > 2$, there exists a graph G with $2a$ vertices such that $\chi^{'}(G) = a$ and $\chi^{'}_{in}(G) = 2a - 1$.*

Proof. Let G be a graph which obtained from the cycle C_4 by attaching $a - 2$ pendent edges to each vertex of only two adjacent vertices of C_4 (see Figure 4.). Let H be the bi-star $B(a-1, a-1)$ with the centers v_3 and v_4, it is a subgraph of the graph G and by Corollary 2.14, $\chi^{'}_{in}(H) = 2a - 1$. By Proposition 2.10, we get,

$$\chi^{'}_{in}(G) \geq 2a - 1. \tag{5}$$

By $2a - 1$ colors for the edges of the bi-star H and making the color of the edge v_1v_2 as the color of v_1v_3 or v_2v_4, we will get $(2a - 1)$–injective edge coloring for G that means,

$$\chi^{'}_{in}(G) \leq 2a - 1 \tag{6}$$

By inequalities (5) and (6), we get $\chi^{'}_{in}(G) = 2a - 1$. □

Theorem 3.2. *Let G be a connected graph. Then $\chi^{'}_{in}(G) = 1$ if and only if $G \cong P_2$ or P_3.*

Proof. Let G be a connected graph with $\chi^{'}_{in}(G) = 1$. Then, we need only one color to coloring the edges of G such that the neighborhood edges of any edge have different colors and so there are two cases here.
Case 1. There is only one edge in G and since G is connected. Then $G \cong P_2$.
Case 2. There exist more than one edge in G and no edge has neighborhood more than one and G is connected . Hence, $G \cong P_3$.
The other side is obvious. □

Corollary 3.3. *$\chi^{'}_{in}(G) = 1$ if and only if $G \cong aK_1 \cup bK_2 \cup K_{1,2}$ where $a, b, c \in \{0, 1, 2, ...\}$ and $b + c \geq 1$.*

Theorem 3.4. *Let G be a connected graph. Then $\chi^{'}_{in}(G) = 2$ if and only if $G \cong P_n$ or C_{4k}, $k \geq 1$.*

Proof. Let $\chi^{'}_{in}(G) = 2$. Then by Proposition 2.8, $\chi^{'}_{in}(G) = \chi(con(L(G))) = 2$ which implies that $con(L(G))$ is a bipartite graph. But it is known that, for any connected graph G, $con(G)$ is a bipartite graph if and only if $G \cong P_n$ or C_{4k}, $k \geq 1$ [3]. Therefore, $L(G) \cong P_n$ or $L(G) \cong C_{4k}$, $k \geq 1$. Hence, $G \cong P_n$ or C_{4k}, $k \geq 1$.
The other side is obvious. □

Theorem 3.5. *Let G be a graph with $q \geq 2$ edges and without isolated vertex. Then*

$$2\delta(G) - 2 \leq \chi^{'}_{in}(G) \leq q.$$

Proof. The upper bound holds trivially, by q colors, we can color the graph G such that any two edges sharing a common neighbor have different colors by signing color for each edge.
For the lower bound. Let e be any edge in G such that $deg(e) = \Delta^{'}(G)$. Then e is a common neighbor for $\Delta^{'}(G)$ edges in G, and if G has minimum degree $\delta(G) \geq 1$, then $\Delta^{'}(G) \geq 2\delta(G) - 2$. Therefore, the edge e is common neighbor for $2\delta(G) - 2$ edges, that means $\chi^{'}_{in}(G)$ at least $2\delta(G) - 2$. Hence, $2\delta - 2 \leq \chi^{'}_{in}(G) \leq q$.
For sharpness, for the lower bound $G \cong C_4$ and for the upper bound $G \cong K_{1,n}$. □

One natural question will arise, for which graph G, $\chi^{'}_{in}(G) = 2\delta(G) - 2$?. Part of the answer is in the following results.

Proposition 3.6. *For any k-regular graph G with $\chi^{'}_{in}(G) = 2k - 2$, the number of edges q is even and $q \equiv 0 \,(mod\,(k-1))$.*

Proof. Let G be k-regular graph. Then $L(G)$ is $(2k-2)$-regular graph. To finding the number of edges with the same color in G clearly every color appear once in every neighborhood of any edge and it counted exactly $(2k-2)$ times in $\sum_{e \in E(G)} 1 = q$. Therefore, each color appear exactly in $\dfrac{q}{2k-2}$ edges. That means $q = 2t(k-1)$ for some positive integer t. Hence, q is even and $q \equiv 0 \,(mod\,(k-1))$. □

Corollary 3.7. *Let G be a connected graph with $q \geq 2$ edges and with maximum degree $\Delta(G) = \Delta$. Then*

$$2(n - \Delta - 1) \leq \chi^{'}_{in}(G) + \chi^{'}_{in}(\overline{G}) \leq \frac{n(n-1)}{2}.$$

A clique of a graph G is a complete subgraph of G. A clique of G is a maximal clique of G if it is not properly contained in another clique of G. The clique number of a graph G, denoted by $\omega(G)$, is the number of vertices in a maximum clique of G. The number of edges in a maximum clique of G is denoted by $\omega^{'}(G)$. It is clear that, if G has size $q \geq 1$, $\omega^{'}(G) = \dfrac{\omega(G)(\omega(G)-1)}{2}$.

Proposition 3.8. *Let G be a connected graph of order $n \geq 2$ and $G \neq K_n$. Then $\chi^{'}_{in}(G) \geq \omega^{'}(G) + 1$.*

Proposition 3.9. *Let G be any k-regular graph with clique number $\omega(G) = \omega$. Then $\chi^{'}_{in}(G) \geq \dfrac{\omega(2k - \omega + 1)}{2}$.*

Proof. Let G be any k-regular graph with clique number $\omega(G) = \omega$. Suppose that H is the subgraph which contains the maximum clique of G and the attached edges in each vertex of the clique, it is easy to check that, any two edges in

H has a common edges and the number of edges in H is $\dfrac{\omega(2k-\omega+1)}{2}$, so $\chi_{in}^{'}(H) = \dfrac{\omega(2k-\omega+1)}{2}$. Therefore, $\chi_{in}^{'}(G) \geq \dfrac{\omega(2k-\omega+1)}{2}$. □

Proposition 3.10. *For any connected graph G of size $q \geq 3$, $\chi_{in}^{'}(G) \leq q - diam(G) + 3$. Further, equality holds if $G \cong B(m, n)$.*

Proof. Let P_d be a diametral path in G. We can color P_d by three colors because we cannot guarantee that no vertices in P_d have degree more than two. By coloring all the other edges $(q - diam(G))$ with different colors we get the bound. Clearly, if $G \cong B(m, n)$, then $\chi_{in}^{'}(G) = m+n+1-diam(G)+3 = m+n+1$. □

Theorem 3.11. *Let G and H with n_1, n_2 vertices, respectively, be non trivial connected graphs such that at least n_1 or n_2 does not equal two and $G \square H \neq P_3 \square K_2$. Then*

$$\chi_{in}^{'}(G \square H) \geq \Delta(G \square H) + \max\Big\{deg_G(w) + \Delta(H), deg_H(w') + \Delta(G)\Big\},$$

where $deg_G(w) = \max\limits_{v \in N_G(u)} deg(v)$, $deg_H(w') = \max\limits_{v' \in N_H(u')} deg(v')$ and $deg_G(u) = \Delta(G)$, $deg_H(u') = \Delta(H)$.

Proof. Let $deg_G(u) = \Delta(G)$, $deg_H(u') = \Delta(H)$ and $deg_G(w) = \max\limits_{v \in N_G(u)} deg(v)$, $deg_H(w') = \max\limits_{v' \in N_H(u')} deg(v')$. From the definition of the Cartesian product of two graphs we conclude that, at least one from the two edges $e = (u, u')(w, u')$, $f = (u, u')(u, w')$ has a maximum degree in $G \square H$ and since n_1 or n_2 is different from two and $G \square H \neq P_3 \square K_2$, then the graph W in Figure 5. is a subgraph in $G \square H$ which needs number of colors equal to

$$\Delta(G \square H) + \max\Big\{deg_G(w) + \Delta(H), deg_H(w') + \Delta(G)\Big\}$$

to color all of its edges. Hence by Proposition 2.10, the proof is complete. □

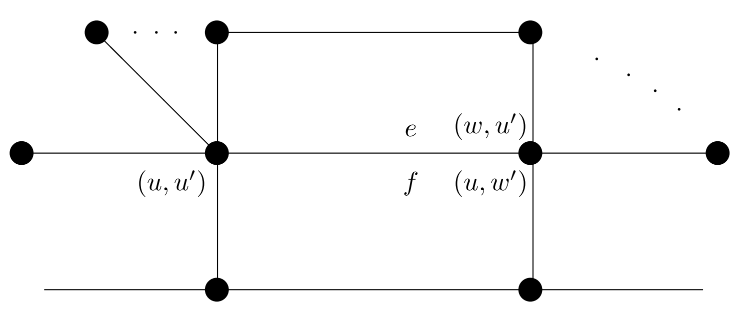

FIGURE 5. Subgraph W of $G \square H$

Proposition 3.12. *Let $G \cong P_t \,\square\, S_m$, where $t \geq 4$ and $m \geq 3$. Then $\chi'_{in}(G) = 2m+2$.*

Proof. Suppose $V(P_t) = \{v_1, v_2, \ldots, v_t\}$ and $V(S_m) = \{u_1, u_2, \ldots, u_m\}$, where u_1 is the center vertex. Let us denote to a vertex in $P_t \,\square\, S_m$ by v_iu_j instead of (v_i, u_j) and an edge by $v_iu_j - v_iu_k$ or $v_iu_j - v_lu_j$ instead of $(v_i, u_j)(v_i, u_k)$ or $(v_i, u_j)(v_l, u_j)$, respectively [see Figure 6.]. In the graph $P_t \,\square\, S_m$, color the edges of the path $v_1u_1 - v_2u_1 - \cdots - v_tu_1$ by the sequence of colors $S_2 = \{C_3, C_4, C_1, C_2, C_3, C_4, C_1, C_2, C_3, ...\}$, respectively, and color the edges of all the parallel paths $v_1u_2 - v_2u_2 - \cdots - v_tu_2$,..., $v_1u_m - v_2u_m - \cdots - v_tu_m$ by the sequence of colors $S_1 = \{C_1, C_2, C_3, C_4, C_1, C_2, C_3, C_4, C_1, ...\}$, respectively. Now, let us divide the remaining edges in $P_t \,\square\, S_m$ as rows,
$R_1 = \{v_1u_1 - v_1u_2, v_2u_1 - v_2u_2, \ldots, v_tu_1 - v_tu_2\}$,
$R_2 = \{v_1u_1 - v_1u_3, v_2u_1 - v_2, u_3, \ldots, v_tu_1 - v_tu_3\}$,...,
$R_{m-1} = \{v_1u_1 - v_1u_m, v_2u_1 - v_2u_m, \ldots, v_tu_1 - v_tu_m\}$, so, we need to color each row by two more different colors, alternately. Therefore, we have

$$\chi'_{in}(G) \leq 2(m-1) + 4 = 2m + 2. \tag{7}$$

Also, from Theorem 3.11, we have

$$\chi'_{in}(G) \geq m + 1 + max\{2 + m - 1, 1 + 2\} = 2m + 2. \tag{8}$$

This complete the proof. $\square$

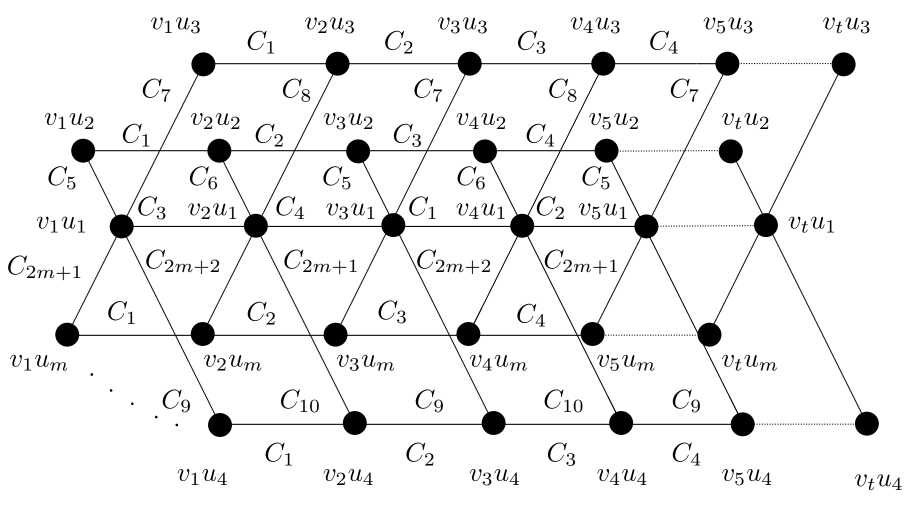

FIGURE 6. Graph $P_t \,\square\, S_m$

Let F_1, F_2 and F_3 be the graphs shown in Figure 7.

Theorem 3.13 ([11])**.** *If $diam(G) \leq 2$ and if none of the three graphs F_1, F_2 and F_3 of Figure 7. is an induced subgraph of G, then $diam(L(G)) \leq 2$.*

Theorem 3.14. *Let G be a diameter 2 graph with matching number m and none of the three graphs F_1, F_2 and F_3 of Figure 7. is an induced subgraph of G. Then*

$$\chi_{in}^{'}(G) \geq m.$$

Furthermore, the equality holds if $G \cong P_4$.

Proof. Let F be a maximum matching set of G. By Theorem 3.13, clearly for any two edges in F there is one common neighbor edge between them as the $diam(L(G)) \leq 2$. Therefore, any edge injective coloring for G required at least m colors. Hence, $\chi_{in}^{'}(G) \geq m$. Also, if $G \cong P_4$, then $\chi_{in}^{'}(P_4) = m = 2$. □

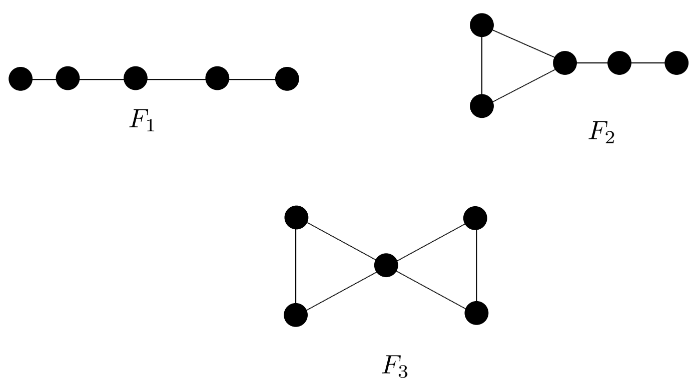

FIGURE 7. Graphs F_1, F_2 and F_3

Now, it is the turn to discuss the injective chromatic index for any tree T_n of n vertices. Clearly that case $n \leq 3$ is trivial so we concern here about $n \geq 4$.

Theorem 3.15. *For any tree T_n with $n \geq 4$ vertices,*

$$\chi_{in}^{'}(T_n) \leq 4(\chi^{'}(T_n))^2 - 10\chi^{'}(T_n) + 6.$$

Furthermore, the equality holds for any path P_n of n vertices.

Proof. Let G be a tree T_n with n vertices and maximum degree $\Delta(T_n) = \Delta$. The line graph $L(T_n)$ has $n - \Delta$ cliques and the maximum degree of $L(G)$ is at most $2\Delta - 2$. By the definition of congraph of $L(G)$, we have the maximum degree of $con(L(G))$ is at most $(2\Delta - 2)(2\Delta - 3) = 4\Delta^2 - 10\Delta + 6$ and it well known that for any tree $\chi^{'}(T_n) = \Delta(T_n)$ and also $\chi(G) \geq \Delta(G)$. Therefore by

Proposition 2.8, we get $\chi'_{in}(T_n) \leq 4(\chi'(T_n))^2 - 10\chi'(T_n) + 6$.
Also, if $T_n \cong P_n$, then $\chi'_{in}(T_n) = 4(\chi'(T_n))^2 - 10\chi'(T_n) + 6 = 2$. □

Theorem 3.16. *For any tree T_n with $n \geq 4$ vertices, $\chi'_{in}(T_n) = \Delta'(T_n)$ or $\Delta'(T_n) + 1$, where $\Delta'(T_n)$ is the maximum edge degree in T_n.*

Proof. Let T_n be a tree on $n \geq 4$ vertices and $e = uv$ be a maximum edge in T_n. Then $e = uv$ has at least $\Delta'(T_n)$ common edges in T_n, so $\chi'_{in}(T_n) \geq \Delta'(T_n)$.
Suppose $g = uw$ and $h = vx$ be two arbitrary edges in T_n incident to u and v, respectively. Then $deg(w) \leq deg(v)$ and $deg(x) \leq deg(u)$ because if not (means $deg(w) > deg(v)$ or $deg(x) > deg(u)$), then we get a contradiction with the maximality of degree $e = uv$ in T_n. Hence, all the edges incident to w will assign by the same colors as the edges incident to v and all the edges incident to x will assign by the same colors as the edges incident to u. Also, by taking in account a different color for the edge $e = uv$, then $\chi'_{in}(T_n)$ does not exceed $\Delta'(T_n)+1$. □

In the following, we determine the necessary and sufficient conditions for any tree T_n, with $n \geq 4$, the equality $\chi'_{in}(T_n) = \chi'(T_n)$ is satisfied.

Theorem 3.17. *Let T_n, with $n \geq 4$, be a tree of maximum degree $\Delta'(T_n)$. Then $\chi'_{in}(T_n) = \chi'(T_n)$ if and only if the following conditions are satisfied.*

(1) *Every edge e in T_n with $deg(e) = \Delta'(T_n)$ must be incident to vertices of degree two or one,*
(2) *if two edges of maximum degree in T_n connected by a path of vertices of degree two, then the length of that path must be even.*

Proof. Let T_n, with $n \geq 4$, be a tree of maximum degree $\Delta'(T_n)$. It is easy to see that, if condition (i) is satisfied, then $\Delta(T_n) = \Delta'(T_n)$. Thus, it is enough here if we proof that for any tree T_n, with $n \geq 4$, $\chi'_{in}(T_n) = \Delta'(T_n)$ if and only if conditions (i) and (ii) are both satisfied.
Let $\chi'_{in}(T_n) = \Delta'(T_n)$. Suppose condition (i) does not hold. Then there exists at least an edge $e = uv$ with $deg(e) = \Delta'(T_n)$ such that $deg(u)$ and $deg(v)$ do not equal two or one. Therefore, the edge $e = uv$ must assign by a different color as their edge neighbors. Thus, $\chi'_{in}(T_n) = \Delta'(T_n) + 1$, a contradiction. Hence, condition (i) must be satisfied. Suppose now condition (ii) does not hold. This means that there exists at least two edges e_1 and e_2 with $deg(e_1) = deg(e_2) = \Delta'(T_n)$ satisfying condition (i) and connecting by a path P_s of vertices of degree two such that the length of P_s is odd. Clearly that, in this case the edges e_1, e_2 and the edges of P_s must assign by three different colors or we need one more color to coloring the edge neighbors of either e_1 or e_2. Therefore, $\chi'_{in}(T_n) = \Delta'(T_n) + 1$, a contradiction. Hence, condition (ii) also must be satisfied.
The converse is obvious. □

Definition 3.18. An injective clique of a graph G is the subgraph H of G such that any two edges e ,$f \in E(H)$ have a common edge adjacent to both of e and

f. The largest number of edges of injective clique is called the injective clique number of G and denoted by $\omega_{in}(G)$ or in short ω_{in}.

Proposition 3.19.

(1) *For any complete bipartite graph* $K_{m,n}$,$\omega_{in}(K_{m.n}) = mn$.
(2) *For any tree* T_n , $\omega_{in}(T_n) = \Delta^{'}(T_n)$.
(3) *For any cycle* C_n , $\omega_{in}(C_n) = 2$.

Conjecture: For any graph G,

$$\omega_{in} \leq \chi^{'}_{in}(G) \leq \omega_{in} + 2.$$

Discussion of the section:
In section 3, due to the several importance of the study of bounds in graph theory, we have determined some bounds of $\chi^{'}_{in}$ in terms of the minimum degree δ, total number of edges, maximum degree Δ, total number of vertices, diameter, matching number and the clique number ω of graph. And for the trees, which having a huge applications in graph theory, we have shown that $\chi^{'}_{in}(T_n) = \Delta^{'}(T_n)$ or $\Delta^{'}(T_n) + 1$, where $\Delta^{'}(T_n)$ is the maximum edge degree in T_n and determined the necessary and sufficient conditions for any tree T_n with $n \geq 4$ the equality $\chi^{'}_{in}(T_n) = \chi^{'}(T_n)$ is satisfied. Finally, after studied our new parameter for many graphs, we have defined the injective clique number ω_{in} and conjectured that, for any graph G with at least one edge, $\omega_{in} \leq \chi^{'}_{in}(G) \leq \omega_{in}+2$.

4. Conclusion

In this research work, we have demonstrated new results concerning the reformulated injective chromatic index of different families of graphs. In particular, we have obtained exact values for cycles, complete graph, complete bipartite graph, some grids graphs, some cases of corona product, join graph and trees. Also, we have determined bounds for the injective chromatic index in terms of, number of edges, maximum degree, minimum degree, matching number and edge clique number. Also, we have shown several more general properties concerning the injective chromatic number. We have got the necessary and sufficient conditions for the injective chromatic index to be equal to one, two and total number of the edges. Also, we have determined the necessary and sufficient conditions for any tree T_n with $n \geq 4$ the equality $\chi^{'}_{in}(T_n) = \chi^{'}(T_n)$ is satisfied. Finally, we have defined the injective clique number ω_{in} and conjectured that, for any graph G with at least one edge, $\omega_{in} \leq \chi^{'}_{in}(G) \leq \omega_{in} + 2$. Many problems remain open, such as proving or disproving our conjecture and studying the new classifications of graphs by using the proposed conjecture. The conditions for which the injective index of graph is equal to ω_{in} or $\omega_{in} + 1$ is a challenging problem.

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