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LINEARLY DEPENDENT AND CONCISE SUBSETS OF A SEGRE VARIETY DEPENDING ON k FACTORS

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Abstract. We study linearly dependent subsets with prescribed cardinality s of a multiprojective space. If the set S is a circuit, there is an upper bound on the number of factors of the minimal multiprojective space containing S. B. Lovitz gave a sharp upper bound for this number. If S has higher dependency, this may be not true without strong assumptions (and we give examples and suitable assumptions). We describe the dependent subsets S with $\#S = 6$.

1. Introduction

Take k non-zero finite dimensional vector spaces $V_1, \dots, V_k$ and consider $V_1 \otimes \cdots \otimes V_k$. An element $u \in V_1 \otimes \cdots \otimes V_k$ is called a k-tensor with format $(\dim V_1, \dots, \dim V_k)$ ([9, p. 33]). Two non-zero proportional tensors share many properties. Thus often the right object to study is the projectivization $\mathbb{P}^r$ of $V_1 \otimes \cdots \otimes V_k$, where $r := -1 + \dim V_1 \times \cdots \times \dim V_k$. Set $n_i := \dim V_i - 1$ and consider the multiprojective space $Y := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Let $\nu : Y \hookrightarrow \mathbb{P}^r$ denote the Segre embedding. Many properties of a non-zero tensor u (e.g., the tensor rank and the tensor border rank) may be describe in how its equivalence class $[u] \in \mathbb{P}^r$ sits with respect to the Segre variety $\nu(Y)$ (see [9, Def. 4.3.5.1] for the definition of Segre variety). For instance, the tensor rank $r_Y([u])$ (as defined in [9, Def. 2.4.1.2]) of u is the minimal cardinality of a finite set $S \subset Y$ such that $\nu(S)$ spans $[u]$. We call $\mathcal{S}(Y, [u])$ the set of all $S \subset Y$ with minimal cardinality such that $\nu(S)$ spans $[u]$. Using subsets of Y instead of ordered sets of points and $\mathbb{P}^r$ instead of $V_1 \otimes \cdots \otimes V_k$ we take care of the obvious non-uniqueness in a finite decomposition $u = \sum_i v_{i1} \otimes \cdots v_{ik}$, $v_{ij} \in V_j$, of a tensor.

Fix an equivalence class $q = [u] \in \mathbb{P}^r$ of non-zero tensors. Let $\pi_i : Y \to \mathbb{P}^{n_i}$, $1 \le i \le k$, denote the projection of Y onto its i-th factor. The *width* $w(q)$ of q is the minimal number of non-trivial factors of the minimal multiprojective subspace $Y' \subseteq Y$ such that $q \in \langle \nu(Y') \rangle$, where $\langle \ \rangle$ denote the linear span. For

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any finite set $A \subset Y$ the *width* $w(A)$ of A is the number of integers $i \in \{1, \dots, k\}$ such that $\#\pi_i(A) > 1$, where $\#E$ denotes the cardinality of the finite set E. By concision we have $w(q) = w(A)$ if $A \in \mathcal{S}(Y, q)$ ([9, Proposition 3.1.3.1]).

The non-uniqueness of tensor decompositions, i.e., the fact that $\mathcal{S}(Y, [u])$ may have more than one element, may be rephrased as the linear dependency of certain subsets of Y ([5]). For any finite set $S \subset Y$ set $e(S) := h^1(\mathcal{I}_S(1, \dots, 1))$. By the definition of Segre embedding and the Grassmann's formula we have $e(S) = \#S - 1 - \dim\langle\nu(S)\rangle$. We say that a non-empty finite set $S \subset Y$ (or that the finite set $\nu(S) \subset \mathbb{P}^r$) is *equally dependent* if $\dim\langle\nu(S)\rangle \le \#S - 2$ and $\langle\nu(S')\rangle = \langle\nu(S)\rangle$ for all $S' \subset S$ such that $\#S' = \#S - 1$. Note that S is equally dependent if and only if $e(S) > 0$ and $e(S') < e(S)$ for all $S' \subset S$, $S' \ne S$, i.e., if and only if $S \ne \emptyset$ and $e(S') < e(S)$ for all $S' \subset S$, $S' \ne S$. We say that S is *uniformly dependent* if $e(S') = \max\{0, e(S) - \#S + \#S'\}$ for all $S' \subset S$. A uniformly dependent subset is equally dependent, but when $e(S) \ge 2$ the two notions are different (the key Examples 3.1 and 3.2 are equally dependent, but not uniformly dependent). When $e(S) = 1$ equal and uniform dependence coincide. An equally dependent subset with $e(S) = 1$ is often called a *circuit*. Fix an integer $e > 0$. Let S be a finite subset of a multiprojective space. We say that S is an *e-circuit* if $e(S) = e$ and there is a subset $S' \subseteq S$ such that S' is a circuit and $\#S - \#S' = e - 1$. A uniformly dependent set S is an $e(S)$-circuit, but the converse does not hold (Example 3.4).

The following result is an immediate corollary of [10, Corollary 14].

Proposition 1.1. *Let $S \subset Y$ be an e-circuit. Then $w(S) \le \#S - e - 1$.*

We give examples for any integer $s \ge 6$ of an equally dependent set S with $e(S) > 1$, $\#S = s$ and $w(S)$ arbitrarily large (Example 3.3). This example shows there is no upper bound for $w(S)$ in term of $\#S$ for all equally dependent sets if $e(S) > 1$.

The main result of this paper is the classification of all equally dependent subsets S of a Segre variety with $\#S = 6$ and $w(S) > 4$. We prove the following result.

Theorem 1.2. *Let $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, $n_1 \ge \cdots \ge n_k > 0$ be a multiprojective space and $S \subset Y$ a concise and equally dependent set with $\#S = 6$. Then either $e(S) \ge 2$ and (Y, S) is in one of Examples 3.1 and 3.2 or $w(Y) \le 4$ and $Y = (\mathbb{P}^1)^4$ if $w(Y) = 4$.*

The families in Examples 3.1, 3.2 have arbitrarily large width. The case $Y = (\mathbb{P}^1)^4$ and $e(S) = 1$ occurs ([5, Case 3 of Theorem 7.1]). In several cases we could give a more precise description of the pairs (Y, S), but using too much ink.

For any $q \in \mathbb{P}^r$ and any finite set $S \subset Y$ we say that S *irredundantly spans* q if $q \in \langle\nu(S)\rangle$ and $q \notin \langle\nu(S')\rangle$ for any $S' \subset S$, $S' \ne S$. As a byproduct of a small part of the proof of Theorem 1.2 we also classify the set of all rank 2 tensors which may be irredundantly spanned by a set of 3 points (Proposition 4.3).

We work over a field K, since for the examples we only use that $\mathbb{P}^1(K)$ has at least 3 points. For the proofs which require cohomology of coherent algebraic sheaves (like in the quotations of [2, Lemma 5.1] or [4, Lemmas 2.4 and 2.5] or [5]) it is sufficient to work over the algebraic closure $\overline{K}$ of K, because dimensions of cohomology of algebraic sheaves on projective varieties (and in particular the definition of $e(S)$) are invariants under the extension $K \hookrightarrow \overline{K}$ ([6, Proposition III.9.3]). We use Landsberg's book [9] for essential properties on Segre varieties related to tensors (e.g., the notion of concision), in particular concision is [9, Proposition 3.1.3.1] and [9, Ch. 5] contains many results and references on the secant varieties of the Segre varieties . This book contains many applications of tensors ([9, Ch. 11, 12, 13, 14]) and additive tensor decompositions are just a way to state linear combinations of elements of the Segre variety $\nu(Y)$. The elementary properties of the Segre varieties that we use do not depend on the base field. For an in-depth study of them over a finite field, see [7, Ch. 25].

1.1. Motivations for this paper

(a) There is no need to stress the importance of tensors and tensor decompositions for the applications of mathematics. Hence the importance of the solution sets $\mathcal{S}(Y,q)$, $q \in \mathbb{P}^r$. Outside Kruskal's bound it is very difficult to prove that an irredundant decomposition of a tensor T associated to q, say $q \in \langle \nu(S) \rangle$, evinces the tensor rank of T, i.e., $r_Y(q) = \#S$. Thus it seems important to study irredundant decompositions without assuming that they evince the tensor rank, i.e., to study all solution sets $\mathcal{S}(Y,q,t)$, $t \geq r_Y(q)$, i.e., all $S \subset Y$ such that $\#S = t$ and $\nu(S)$ irredundantly spans q. It is known that even if Y is minimal for S, q may not be concise for Y ([3, Theorem 3.8]). Proposition 4.3 classifies all triples (Y,q,S) with $r_Y(q) = 2$, $\#S = 3$ and Y minimal for S, but not always for q. This result is proved studying dependent subsets with cardinality 5.

(b) Take as K a finite field, $\mathbb{F}_q$. Any $S \subset \mathbb{P}^{k-1}(\mathbb{F}_q)$ such that $\langle S \rangle = \mathbb{P}^{k-1}(\mathbb{F}_q)$ gives an $[n,k]$-code $\mathcal{C}$ over $\mathbb{F}_q$, where $n := \#S$. Circuits $S' \subset S$ arise in the computation of the minimum distance of S. Equally defined sets $S' \subset S$ with $e(S') \geq 2$ arise in the computation of the generalized Hamming weights of $\mathcal{C}$ introduced by Wei ([8, §7.10]).

(c) In the proofs in [1] we needed to classify some rational normal curves contained in a Segre variety X. These curves occur implicitly when we quote [1] and explicitly (plus degenerations/variations of them like reducible conics or unions of 2 disjoint lines) in Example 3.2 and Remarks 5.1 and 5.2. It is easy to see that being contained in the linear span of a certain curve $C \subset X$ often gives that $\#\mathcal{S}(Y,q,t) > 1$ for some small t. When C is irreducible it is often easy to construct e-circuits $S \subset C$. More general curves, e.g. elliptic curves, should occur for larger t, but a full classification of the set S should be too long. In our opinion the classification of the curves (and if K is finite the computation of their number) seems to be interesting.

1.2. Outline of the proof of Theorem 1.2

In Section 3 we describe the examples mentioned in the statement of Theorem 1.2. Take $S \subset Y$ such that $\#S = 6$ and S is equally dependent. We fix a partition $S = A \cup B$ with $\#A = \#B = 3$ and hence $A \cap B = \emptyset$. In Section 5 we assume that at least one among $\nu(A)$ and $\nu(B)$ is linearly dependent. In that section we get Examples 3.1 and 3.2. Then we assume $\nu(A)$ and $\nu(B)$ linearly independent. Since $A \cap B = \emptyset$, the Grassmann's formula gives $\dim(\langle\nu(A)\rangle \cap \langle\nu(B)\rangle) = e(S) - 1$. Thus $\langle\nu(A)\rangle \cap \langle\nu(B)\rangle \neq \emptyset$. We fix a general $q \in \langle\nu(A)\rangle \cap \langle\nu(B)\rangle$. Since $q \in \langle\nu(A)\rangle$, we have $r_Y(q) \leq 3$. We discuss the cases $r_Y(q) = 1$, $r_Y(q) = 2$, $r_Y(q) = 3$ in Sections 6, 7 and 8, respectively. For the case $r_Y(q) = 3$ we use [5, Theorem 7.1].

Remark 1.3. In the set-up of Theorem 1.2 the case $k = 1$ is possible with $Y = \mathbb{P}^n$ for any $2 \leq n \leq 4$ (any 6 points spanning $\mathbb{P}^n$ partitioned in two sets of 3 elements no 3 of them collinear). The case $Y = \mathbb{P}^1$ was obtained when $e(A) > 0$ and $e(B) > 0$. When $Y = \mathbb{P}^n$ we have $e(S) = 6 - n - 1$.

Thus in Sections 5, 6, 7 and 8 we silently assume $k > 1$.

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2. Preliminaries, notation and the proof of Proposition 1.1

For any subset E of any projective space let $\langle E\rangle$ denote the linear span of E. For any multiprojective space let ν denote its Segre embedding. Let $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a multiprojective space. Let $\pi_i : Y \to \mathbb{P}^{n_i}$ be the projection of Y onto its i-th factor. Set $Y_i := \prod_{j\neq i} \mathbb{P}^{n_j}$ and let $\eta_i : Y \to Y_i$ be the projection. Thus for any $p = (p_1, \dots, p_k) \in Y$, $\pi_i(p) = p_i$ is the i-th component of p, while $\eta_i(p) = (p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_k)$ deletes the i-th component of p.

For any $i \in \{1, \dots, k\}$ let $\epsilon_i \in \mathbb{N}^k$ (resp. $\hat{\epsilon}_i$) be the multiindex $(a_1, \dots, a_k) \in \mathbb{P}^k$ with $a_i = 1$ and $a_h = 0$ for all $h \neq i$ (resp. $a_i = 0$ and $a_h = 1$ for all $h \neq i$). Thus $\mathcal{O}_Y(\epsilon_i)$ and $\mathcal{O}_Y(\hat{\epsilon}_i)$ are line bundles on Y and $\mathcal{O}_Y(\epsilon_i) \otimes \mathcal{O}_Y(\hat{\epsilon}_i) \cong \mathcal{O}_Y(1, \dots, 1)$.

If needed we usually call $\mathbb{P}^r$ the projectivization of the space of tensors with prescribed format we are working, i.e., the projective space in which the given Segre sits. For instance, if the given Segre is $\nu(Y)$ we take $r = -1+\prod_{i=1}^{k}(n_i+1)$. For any $q \in \mathbb{P}^r$ let $r_Y(q)$ or $r_{\nu(Y)}(q)$ denote the tensor rank of q. For any finite set $A \subset Y$ the minimal multiprojective subspace of Y containing A is the multiprojective space $\prod_{i=1}^{k}\langle\pi_i(A)\rangle \subseteq Y$. For any positive integer t let $\mathcal{S}(Y, q, t)$ denote the set of all $S \subset Y$ such that $q \in \langle\nu(S)\rangle$, $\#S = t$ and S irredundantly spans q. The set $\mathcal{S}(Y,q) := \mathcal{S}(Y,q,r_{\nu(Y)}(q))$ is the set of all tensor decompositions of q with minimal length. By concision given any $A \in \mathcal{S}(Y,q)$ the minimal multiprojective subspace of Y containing A is the minimal multiprojective subspace $Y' \subseteq Y$ such that $q \in \langle \nu(Y')\rangle$ ([9, Proposition 3.1.3.1]).

Remark 2.1. Take $S \subset Y$ such that $e(S) > 0$ and $\#S \le 3$. Since ν is an embedding, we have $\#S = 3$, $e(S) = 1$ and (by the structure of linear subspaces contained in a Segre variety) there is $i \in \{1,\dots,k\}$ such that $\#\pi_h(S) = 1$ for all $h \ne 1$, $\pi_{i|S}$ is injective and $\pi_i(S)$ is contained in a line.

Lemma 2.2. *Fix a multiprojective space Y and any finite set $Z \subset Y$ with $z := \#Z \ge 3$ and concise for Y. Set $e(Z) := z - 1 - \dim\langle \nu(Z)\rangle$. We have $e(Z) \le z-2$ and equality holds if and only if $Y = \mathbb{P}^1$.*

Proof. Since ν is an embedding, $\nu(Z)$ is a set of $z \ge 2$ points of $\mathbb{P}^N$ and hence $\dim\langle\nu(Z)\rangle \ge 1$. The Grassmann's formula gives $e(Z) \le z-2$ and that equality holds if and only if $\nu(Z)$ is formed by collinear points. Since the Segre $\nu(Y)$ is cut out by quadrics and $z \le 3$, we get $\langle\nu(Z)\rangle \subseteq \nu(Y)$. Since the lines of a Segre variety are Segre varieties, the concision assumption gives $Y = \mathbb{P}^1$.

The converse is trivial, because $h^0(\mathcal{O}_{\mathbb{P}^1}(1)) = 2$. □

The following construction was implicitly used in the proof of [3, Theorem 3.8].

Definition. Fix a multiprojective space $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, $n_h > 0$ for all $h \ne i$, and $i \in \{1,\dots,k\}$ (we allow the case $n_i = 0$ so that $\mathbb{P}^{n_i}$ may be a single point). Fix an integer m_i such that $n_i \le m_i \le n_i+1$; if $n_i = 0$ assume $m_i = 1$. Let $W \supseteq Y$ be a multiprojective space with $\mathbb{P}^{n_j}$ as its j-th factor for all $j \ne i$ and with $\mathbb{P}^{m_i}$ as its i-th factor. Thus $W = Y$ if $m_i = n_i$ and $\dim W = \dim Y + 1$ if $m_i = n_i + 1$. If $W \ne Y$ we identify Y with a multiprojective subspace of W identifying its factor $\mathbb{P}^{n_i}$ with a hyperplane $M_i \subset \mathbb{P}^{m_i}$. Fix a finite set $E \subset Y$ (we allow the case $E = \emptyset$) and $o = (o_1,\dots,o_k) \in Y \setminus E$. Set $E_i := \pi_i(E) \subset \mathbb{P}^{n_i}$. Fix any $u_i \in \mathbb{P}^{m_i} \setminus (E_i \cup \{o_i\})$ and any $v_i \in \langle\{o_i, v_i\}\rangle$ with $v_i \notin E_i$. Set $u = (u_1,\dots,u_k)$ and $v := (v_1,\dots,v_k)$ with $u_h = v_h = o_h$ for all $h \ne i$. Set $F := E \cup \{o\}$ and $G := E \cup \{u,v\}$. We say that G is an *elementary increasing* of F with respect to o and the i-th factor. Note that $\#G = \#E+2$, $\#F = \#E+1$ and $\langle\nu(F)\rangle \subseteq \langle\nu(G)\rangle$. If $n_i > 0$ we have $w(Y) = w(W)$, while if $n_i = 0$ we have $w(W) = w(Y)+1$. Thus an elementary increasing may increase the width, but only by 1 and only if $n_i = 0$.

Remark 2.3. Let $U \subset Y$ be a finite set, $W \supseteq Y$ any multiprojective space and $V \subset W$ any set obtained from U making an elementary increasing. For any finite set $G \subset W$ either $w(V \cup G) = w(U \cup G)$ or $w(V \cup G) = w(U \cup G)+1$, but the latter may occur only if $w(V) = w(U)+1$. Even when $w(V) = w(U)+1$ it is quite easy to see for which G we have $w(V \cup G) = w(U \cup G)+1$.

Proof of Proposition 1.1. Set $s := \#S$. If $e = 1$, then we apply [10, Corollary 14]. Assume $e > 1$ and take $U \subset S$ such that $\#U = e-1$ and $S \setminus U$ is a circuit. Let Y' be the minimal multiprojective space containing $S \setminus U$. By [10, Corollary 14] we have $w(S\setminus U) \le (s-e+1)-2$. Since $h^1(\mathcal{I}_{S\setminus U}(1,\dots,1)) = h^1(\mathcal{I}_S(1,\dots,1)) - \#U$, $\langle \nu(S\setminus U)\rangle = \langle S\rangle$. Thus $\nu(S) \subseteq \langle \nu(Y')\rangle$. Concision ([9, Proposition 3.1.3.1]) gives $S \subset Y'$. Thus $w(S) = w(S\setminus U)$. □

3. The examples

Example 3.1. Fix an integer $k \ge 2$ and integers $n_1, n_2 \in \{1,2\}$. We take $Y = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times (\mathbb{P}^1)^{k-2}$. Take $o = (o_1,\dots,o_k) \in Y$ and $p = (p_1,\dots,p_k) \in Y$ such that $p_i \ne o_i$ for all i. Take $u = (u_1,\dots,u_k) \in Y$, $v = (v_1,\dots,v_k) \in Y$, $w = (w_1,\dots,w_k) \in Y$ and $z = (z_1,\dots,z_k) \in Y$ such that $u_i = v_i = o_i$ for all $i \ne 1$, $w_i = z_i = p_i$ for all $i \ne 2$, $\#\{u_1,v_1,o_1,p_1\} = \#\{o_2,p_2,w_2,z_2\} = 4$. If $n_1 = 2$ (resp. $n_2 = 2$) we also require that $\langle\{u_1,v_1,o_1\}\rangle \subset \mathbb{P}^2$ is a line not containing p_1 (resp. $\langle\{w_2,z_2,p_2\}\rangle \subset \mathbb{P}^2$ is a line not containing o_2). Set $S := \{o,p,u,v,w,z\}$. By construction $\#S = 6$, S is concise for Y, and $e(S) = 2$. It is easy to check that $e(S') = 1$ (but S' is not a circuit) for any $S' \subset S$ such that $\#S' = 5$. The family of these sets S has dimension $n_1 + n_2 + 2$. If $k > 2$ instead of taking the first two factors of Y we may take two arbitrary (but distinct) factors and obtain another family of sets S not projectively equivalent to the one constructed using the first two factors. A small modification of the construction works even if $o_i = p_i$ for some $i \in \{1,2\}$, but in that case we are forced to take $n_i = 1$.

Example 3.2. Fix integers $n \in \{1,2,3\}$ and $k \ge 1$. Set $Y := \mathbb{P}^n \times (\mathbb{P}^1)^{k-1}$. If $k > 1$ fix any $o_i, p_i \in \mathbb{P}^1$, $2 \le i \le k$, such that $o_i \ne p_i$ for all i. Fix lines $L \subseteq \mathbb{P}^n$ and $D \subseteq \mathbb{P}^n$. If $n = 2$ assume $L \ne D$. If $n = 3$ assume $L \cap D = \emptyset$. Fix 3 distinct points $o_1, u_1, v_1 \subset L$ and 3 distinct points w_1, p_1, z_1 of D. If $n = 1$ assume $\#\{o_1,p_1,u_1,v_1,w_1,z_1\} = 6$. If $n = 2$ assume $L \cap D \notin \{o_1,p_1,u_1,v_1,w_1,z_1\}$. Set $o := (o_1,o_2,\dots,o_k)$, $u := (u_1,o_2,\dots,o_k)$, $v := (v_1,o_2,\dots,o_k)$, $p := (p_1,p_2,\dots,p_k)$, $w := (w_1,p_2,\dots,p_k)$, $z := (z_1,p_2,\dots,p_k)$, $A := \{o,u,v\}$, $B := \{p,w,z\}$, and $S := A \cup B$. The decomposition $S = A \cup B$ immediately gives that S is equally dependent. If $k = 1$ we have $e(S) = 5 - n$. Now assume $k > 1$. Since neither $\nu(A)$ nor $\nu(B)$ are linearly independent and $A \cap B = \emptyset$, we have $e(S) \ge 2$. Take $D \in |\mathcal{I}_p(\epsilon_2)|$. By construction we have $S \cap D = B$. Thus the residual exact sequence of D gives the exact sequence

$$0 \to \mathcal{I}_A(\hat{\epsilon}_2) \to \mathcal{I}_S(1,\dots,1) \to \mathcal{I}_{B,D}(1,\dots,1) \to 0. \tag{1}$$

It is easy to check that $h^1(\mathcal{I}_A(\hat{\epsilon}_2)) = 1$ and that $h^1(D, \mathcal{I}_{B,D}(1,\dots,1)) = 1$. Thus (1) gives $e(S) \le 2$. Thus $e(S) = 2$. A small modification of the construction works even if $o_1 = p_1$, but in this case we take $n < 3$.

Example 3.3. Assume $k > 1$. Fix $n \in \{1,2,3\}$ and an integer $s \ge 6$ and set $Y := \mathbb{P}^n \times (\mathbb{P}^1)^{k-1}$. We mimic the proof of Example 3.2 taking 3 points on L and $s-3$ points on $Y \setminus L$. We get $S \subset Y$ concise for Y and such that $\#S = s$, $e(S) = s-4$ and $e(S') < e(S)$ for all $S' \subset S$, $S' \ne S$. We get examples similar to Example 3.1 taking instead of two points a fixed set S' of points and get a set with $\#S'+2$ points.

Example 3.4. Take $Y=\mathbb{P}^2$. Fix a line $L\subset\mathbb{P}^2$, any $E\subset L$ such that $\#E=3$ and a general $G\subset\mathbb{P}^2\setminus L$ such that $\#G=2$. Set $S:=E\cup G$. We have $e(S)=2$ and for any $p\in E$, the set $S\setminus\{p\}$ is a circuit. However, E shows that S is not uniformly dependent.

4. $4\le \#S\le 5$

In this paper we often use two results from [1] which give a complete classification of circuits S with $\#S\le 5$ ([1, Theorem 1.1 and Proposition 5.2]). In this section we extend them to the case of equally dependent subsets $S\subset Y$ with $e(S)\ge 2$. Sometimes we will use them later, but the key point is that the cases with arbitrarily large width and fixed $s:=\#S$ occur exactly when $s\ge 6$. We always call $Y=\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$ the minimal Segre variety containing S.

Fix a set $S\subset Y$ such that $\#S\le 5$, $e(S')<e(S)$ for all $S'\subset S$, $S'\ne S$, and $e(S)\ge 2$. We put the last assumption because we described all circuits (i.e., the case $e(S)=1$) in [1, Proposition 5.2] (case $\#S=4$) and [1, Theorem 1.1] (case $\#S=5$).

Now the two new observations for the case $e(S)\ge 2$. We always assume that S is concise for Y.

Remark 4.1. Assume $\#S=4$ and $e(S)\ge 2$. By Lemma 2.2 we have $e(S)=2$ and $Y=\mathbb{P}^1$. Any union F of 4 distinct points of $\mathbb{P}^1$ has $e(F)=2$ and it is equally dependent. For the existence of this case we need $\#K\ne 2$.

Remark 4.2. Assume $\#S=5$. If $e(S)\ge 3$, then $e(S)=3$, $Y=\mathbb{P}^1$ and S is an arbitrary subset of $\mathbb{P}^1$ with cardinality 5 (Lemma 2.2). Assume $e(S)=2$. Thus for all $o\in S$ we have $e(S\setminus\{o\})=1$. Let $S_o\subseteq S\setminus\{o\}$ the minimal subset with $e(S_o)=1$. Each S_o is a circuit. Let $Y[o]\subsetneq Y$ be the minimal multiprojective subspace containing o. The plane $\langle\nu(S)\rangle$ contains at least 5 points of $\nu(Y)$. Since $\nu(Y)$ is cut out by quadrics either $\langle\nu(S)\rangle\subseteq\nu(Y)$ (and hence $Y=\mathbb{P}^2$ by the assumption that Y is the minimal multiprojective space containing S) or $\langle\nu(S)\rangle\cap\nu(Y)$ is a conic. In the latter case the conic may be smooth or reducible, but not a double line. In this case $Y=\mathbb{P}^1\times\mathbb{P}^1$. To show that this case occurs we take an element $C\in|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)|$ and take a union S of 5 points of C, with no restriction if C is irreducible, with the restriction that no component of C contains 4 or 5 points of S if C is reducible. To get examples with C irreducible we need $\#K\ge 4$, but even if $\#K\in\{2,3\}$ there are examples contained in a reducible C.

In the last part of this section we classify the quintuples (W,Y,q,A,B), where W and Y are multiprojective spaces, $Y\subseteq W$, $q\in\langle\nu(Y)\rangle$, $r_{\nu(Y)}(q)=2$, $A\in\mathcal{S}(Y,q)$, $B\subset W$ and $B\in\mathcal{S}(W,q,3)$. We assume that q is concise for Y. By [9, Proposition 3.1.3.1] this assumption is equivalent to the conciseness of A for Y. We assume that B is concise for W, but we do not assume $W=Y$.

Since Y is concise for A and $\#A = 2$, we have $Y = (\mathbb{P}^1)^k$ for some $k > 0$. Since $r_{\nu(Y)}(q) \neq 1$, we have $k \geq 2$. Since W is concise for B and $\#B = 3$ we have $W = \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_s}$ for some $s \geq k$ and $m_i \in \{1, 2\}$ for all $i = 1, \dots, s$. We see the inclusion $Y \subseteq W$, fixing for $i = 1, \dots, k$ a one-dimensional linear subspace $L_i \subseteq \mathbb{P}^{m_i}$ and for $i = k+1, \dots, s$ fixing $o_i \in \mathbb{P}^{m_i}$.

We prove the following statement.

Proposition 4.3. *Fix $q \in \mathbb{P}^r$ with rank 2 and take a multiprojective space $Y = (\mathbb{P}^1)^k$ concise for q. Take a multiprojective space $W \supseteq Y$ and assume the existence of $B \in \mathcal{S}(W, q, 3)$. Fix $A \in \mathcal{S}(Y, q)$. Then one of the following cases occurs*:

(1) *$A \cap B \neq \emptyset$, B is obtained from A making an elementary increasing and either $W = Y$ or $W \cong \mathbb{P}^2 \times (\mathbb{P}^1)^{k-1}$ or $W \cong (\mathbb{P}^1)^{k+1}$;*
(2) *$A \cap B = \emptyset$; in this case either $W \cong \mathbb{P}^2 \times \mathbb{P}^1$ or $W \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $W \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.*

The multiprojective spaces W's listed in (2) of Proposition 4.3 are the ones with $k > 1$ in the list of [1, Theorem 1.1].

For more on the possibles B's in case (1), see Lemma 4.5. For the proof of Proposition 4.3 we set $S := A \cup B$. Our working multiprojective space is W and cohomology of ideal sheaves is with respect to W. Since $\nu(A)$ and $\nu(B)$ irredundantly spans q, we have $e(S) > 0$. Note that $k > 1$, because we assumed that the tensor q has tensor rank $\neq 1$.

Lemma 4.4. *If $A \cap B = \emptyset$, then S is irredundantly dependent and either $e(S) = 1$ or $e(S) = 2$, $Y = \mathbb{P}^1 \times \mathbb{P}^1$ and S is formed by 5 points of some $C \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)|$.*

Proof. Since $A \cap B = \emptyset$, we have $e(S) - 1 = \dim(\langle \nu(A) \rangle \cap \langle \nu(B) \rangle)$. Since $\nu(A)$ (resp. $\nu(B)$) irredundantly spans q, we have $\langle \nu(A \setminus \{a\}) \rangle \cap \langle \nu(B) \rangle \subset \langle \nu(A) \rangle \cap \langle \nu(B) \rangle$ (with strict inclusion) for all $a \in A$ and $\langle \nu(A) \rangle \cap \langle \nu(B \setminus \{b\}) \rangle \subset \langle \nu(A) \rangle \cap \langle \nu(B) \rangle$ (with strict inclusion) for all $b \in B$. Thus $e(S') < e(S)$ for all $S' \subset S$, $S' \neq S$, by the Grassmann's formula. Assume $e(S) \geq 2$. Since $k > 1$ we have $e(S) = 2$ (Lemma 2.2). Since $e(S) = 2$, Remark 4.2 gives $W = Y = \mathbb{P}^1 \times \mathbb{P}^1$ and that S is formed by 5 points of any smooth $C \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)|$. For the existence of this case we need $\#K \geq 4$. □

Lemma 4.5. *If $A \cap B \neq \emptyset$, then B is obtained from A making an elementary increasing of A with respect to the point $A \setminus A \cap B$ and one of the coordinates. In this case for any $Y = (\mathbb{P}^1)^k$ concise for q the concise W for B is either Y or isomorphic to $\mathbb{P}^2 \times (\mathbb{P}^1)^{k-1}$ in which we may prescribe which of the k factors of W has dimension 2. For any rank 2 point $q \in \langle \nu(Y) \rangle$, any $A \in \mathcal{S}(Y, q)$, any point $a \in A$ and any $i \in \{1, \dots, k\}$ we get a 2-dimensional family of such sets B's with $W = Y$ and a 3-dimensional family of such B's with* $\dim W = \dim Y + 1$.

Proof. Assume $A \cap B \neq \emptyset$. Since $\nu(A)$ and $\nu(B)$ irredundantly span q, A is not contained in B. Thus $A \cap B \neq A$. Assume $A \cap B = \{o\}$ with $A = \{o, p\}$. Thus $\#S = 4$. Since $q \neq \nu(o)$, and $q \in \langle \nu(B) \rangle$, we get $\langle \nu(B) \rangle \supset \langle \nu(A) \rangle$ and in particular $\nu(p) \subset \langle \nu(B) \rangle$.

First assume that S is equally dependent. Since S is equally dependent and $s \geq k \geq 2$, by Remark 4.1 and [1, Proposition 5.2] we get $W = Y = \mathbb{P}^1 \times \mathbb{P}^1$ and the list of all possible S's. In this list $\nu(p) \notin \langle \nu(S \setminus \{p\} \rangle$, a contradiction.

Now assume that S is not equally dependent. The proof of Lemma 4.4 gives that $e(S') = e(S)$ only if $S' = S \setminus \{o\}$. Since $\#S' = 3$, there is $i \in \{1, \dots, s\}$ such that $\#\pi_h(S') = 1$ for all $h \neq i$. We see that B is obtained from A keeping o and making an elementary increasing with respect to p to get two other points of B. □

5. $\nu(A)$ or $\nu(B)$ linearly dependent

Recall that $\#S = 6$, Y is concise for S and we fixed a partition $S = A \cup B$ such that $\#A = \#B = 3$. In this section we assume that at least one among $\nu(A)$ and $\nu(B)$ is linearly dependent, while in the next sections we will always assume that both $\nu(A)$ and $\nu(B)$ are linearly independent. Just to fix the notation we assume $e(A) > 0$. Thus $\nu(A)$ is the union of 3 collinear points and there is $i \in \{1, \dots, k\}$ such that $\#\pi_h(A) = 1$ for all $h \neq i$ and $\pi_i(A)$ is formed by the points spanning a line (Remark 2.1). With no loss of generality we may assume $i = 1$.

Remark 5.1. Assume also $e(B) > 0$. We want to prove that we are in one of the cases described in Example 3.1 or 3.2, up to a permutation of the factors of Y (assuming obviously $k > 1$). By Remark 2.1 there is $j \in \{1, \dots, k\}$ such that $\#\pi_h(B) = 1$ for all $h \neq j$ and $\pi_j(B)$ is formed by 3 collinear points.

(a) Assume $i \neq j$. Up to a permutation of the factors of Y we may assume $i = 1$ and $j = 2$. Fix $o = (o_1, \dots, a_k) \in A$ and $p = (p_1, \dots, p_k) \in B$. Set $\{u_1, o_1, v_1\} := \pi_1(A)$ and $\{w_2, z_2, o_2, p_2\} := \pi_2(B)$. Since $\#\pi_i(A) = 1$ for all $i > 1$, $\pi_i(a) = o_i$ for all $a \in A$ and all $i > 1$. Since $\#\pi_i(B) = 1$ for all $i \neq 1$, $\pi_i(b) = p_i$ for all $b \in B$ and all $i \neq 1$. Thus we are as in Example 3.1.

(b) Now assume $i = j$. Up to a permutation of the factors of Y we may assume $i = 1$. In this case we are in the set-up of Example 3.2.

Remark 5.2. Now assume $e(B) = 0$. Since $A \subset S$, $A \neq S$ and $e(A) > 0$, we have $e(S) \geq 2$. Take $i \in \{1, \dots, k\}$ as in part (a) and set $\{o_i\} := \pi_i(A)$. By assumption $\langle \nu(B) \rangle$ is a plane and either $\langle \nu(B) \rangle \cap \langle \nu(A) \rangle = \emptyset$ (i.e., $e(S) = 2$) or $\langle \nu(B) \rangle \cap \langle \nu(A) \rangle$ is a point (call it q') (i.e., $e(S) = 3$) or $\langle \nu(B) \rangle \supset \langle \nu(A) \rangle$ (i.e., $e(S) = 4$). In the latter case we have $Y = \mathbb{P}^1$ (Lemma 2.2). Take any $A_1 \subset A$ such that $\#A_1 = 2$ and set $S_1 := A_1 \cup B$. We have $e(S_1) = e(S) - 1$ and $e(S') < e(S_1)$ for any $S' \subset S_1$ with $S' \neq S_1$. The set S_1 is very particular, because it contains a subset A_1 such that $\#A_1 = 2$ and $\#\pi_i(A) = 1$ for $k - 1$ integers $i \in \{1, \dots, k\}$, say for all $i \neq 1$.

(a) Assume $e(S) = 3$ and hence $e(S_1) = 2$. We may apply Remark 4.2 to this very particular S_1. Either $Y = \mathbb{P}^2$ or $Y = \mathbb{P}^1 \times \mathbb{P}^1$. The case $Y = \mathbb{P}^2$ may obviously occur (take 6 points, 3 of them on a line). To get examples with $Y = \mathbb{P}^1 \times \mathbb{P}^1$ we need $S \subset C \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)|$, because $e(S) = 3$. The existence of A gives C reducible say $C = L \cup D$ with $L \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)|$ and $D \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1)|$ with $D \supset A$. Since $h^1(\mathcal{I}_B(1,1)) = 0$, we see that $\#(B \cap L) = 2$, $\#(B \cap D) = 1$ and $B \cap D \cap L = \emptyset$.

(b) Now assume $e(S) = 2$. Thus $e(S_1)$ is a circuit and we may use the list in [1, Theorem 1.1]. Hence $k \le 3$, $k = 3$ implies $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, while $k = 2$ implies $n_1 + n_2 \in \{2, 3\}$. Obviously the case $k = 1$, $Y = \mathbb{P}^3$ occurs (6 points of $\mathbb{P}^3$ with the only restriction that 3 of them are collinear).

(b1) Assume $Y = \mathbb{P}^2 \times \mathbb{P}^1$. We are in the set-up of [1, Example 5.7], case $C = T_1 \cup L_1$ with L_1 a line and $\#(L_1 \cap S_1) = 2$. This case obviously occurs (as explained in [1, last 8 lines of Example 5.7]). To get S just add another point of L_1.

(b2) Assume $Y = \mathbb{P}^1 \times \mathbb{P}^1$. Here we may take as S_1 (resp. S) the union of 2 (resp. 3) points of any $D \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1)|$ and 3 sufficiently general points of Y.

(b3) Assume $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. It does not occur here (it occurs when $e(A) = e(B) = 0$ and $r_Y(q) = 3$), because $\#(L \cap C) \le 1$ for every integral curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with multidegree $(1,1,1)$ and each curve $L \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ such that $\nu(L)$ is a line and we may apply [1, part (c) of Lemma 5.8].

6. $r_Y(q) = 1$

We recall that q is a general element of $\langle \nu(A) \rangle \cap \langle \nu(B) \rangle$ and that in Sections 6, 7, and 8 we assume $e(A) = e(B) = 0$ and $k > 1$. In this section we assume $r_Y(q) = 1$. Take $o \in Y$ such that $\nu(o) = q$ and write $o = (o_1, \dots, o_k)$. Set $A' = A \cup \{o\}$ and $B' := B \cup \{o\}$.

(a) Assume $o \in A$. Since $\nu(o)$ is general in $\langle \nu(A) \rangle \cap \langle \nu(B) \rangle$ and A has finitely many points, we have $\langle \nu(A) \rangle \cap \langle \nu(B) \rangle = \{\nu(o)\}$. The Grassmann's formula gives $\dim \langle \nu(S) \rangle = 4$, i.e., $e(S) = 1$. Since $A \cap B = \emptyset$, we have $o \notin B$. Thus $\nu(B \cup \{o\})$ is linearly dependent. Since $B \cup \{o\}$ is strictly contained in S, $e(S) = 1$ and S is assumed to be equally dependent, we get a contradiction. In the same way we prove that $\#B' = 4$.

(b) By step (a) we have $\#A' = \#B' = 4$. Write $o = (o_1, \dots, o_k)$. The sets $\nu(A')$ and $\nu(B')$ are linearly dependent. Assume for the moment the existence of A'' strictly contained in A' such that $e(A'') = e(A')$. We have $\#A'' = 3$, $e(A'') = 1$ and there is $i \in \{1, \dots, k\}$ such that $\#\pi_h(A'') = 1$ for all $h \ne 1$. Since $e(A) = 0$, $o \in A''$. Set $\{b\} := A \setminus A \cap A'$. We see that A is obtained from $\{o, b\}$ making an elementary increasing with respect to o and the i-th factor. But then $\nu(o)$ is spanned by $\nu(A \cap A'')$, contradicting the generality of $q \in \langle \nu(A) \rangle \cap \langle \nu(B) \rangle$ and that S is equally dependent. In the same way we handle the case in which there is B'' strictly contained in A' such that $\nu(A'')$ is dependent.

(c) By steps (a) and (b) we may assume that $\nu(A')$ and $\nu(B')$ are circuits. Let $Y' = \prod_{i=1}^{s} \mathbb{P}_i^m$ (resp. $Y'' = \prod_{i=1}^{c} \mathbb{P}^{t_i}$) be the minimal multiprojective subspace of Y containing A' (resp. B'). By [1, Proposition 5.2] either $s = 1$ and $m_1 = 2$ or $s = 2$ and $m_1 = m_1 = 2$, either $c = 2$ and $t_1 = 2$ or $c = 2$ and $t_1 = t_2 = 1$.

(c1) Assume $s = c = 2$. Up to a permutation of the factors we may assume $\#\pi_h(A') = 1$ for all $h > 1$. Call $1 \le i < j \le k$ the two indices such that $\#\pi_h(B') = 1$ for all $h \notin \{i, j\}$. Note that $\pi_h(S) = \pi_h(o)$ if $h \notin \{1, 2, i, j\}$.

Claim 1. $k = j$.

Proof of Claim 1. Assume $k > j$. Since $k > j \ge 2$, we have $\pi_k(A) = \pi_k(o) = \pi_k(B)$. Thus the pair (Y, S) is not concise. □

Claim 2. $k \le 4$ and $Y = (\mathbb{P}^1)^4$ if $k = 4$.

Proof of Claim 2. By Claim 1 we have $k \le 4$. Assume $k = 4$, i.e., assume $i = 3$ and $j = 4$. Assume $Y \ne (\mathbb{P}^1)^4$, i.e., assume $n_h \ge 2$ for some h, say for $h = 1$. Fix $a \in A$. Since $h^0(\mathcal{O}_Y(\epsilon_1)) = n_1 + 1 \ge 3$, there is $H \in |\mathcal{O}_Y(\epsilon_1)|$ containing o and at least one point of B. By concision S is not contained in H. Since A and B irredundantly span q, [2, Lemma 5.1] or [4, Lemmas 2.4 and 2.5] give $h^1(\mathcal{I}_{S\setminus S\cap H}(0, 1, 1, 1)) > 0$. Since $\#\pi_1(B') = 1$, we have $B \subset H$. Thus $\#(S \setminus S \cap H) \le 2$. Since $\mathcal{O}_Y(\epsilon_1)$ is globally generated, we get $\#(S \setminus S \cap H) = 2$, i.e., $S \setminus S \cap H = A \setminus \{a\}$. Since $\mathcal{O}_{Y_1}(1, 1, 1)$ is very ample, we get $\#\eta_1(A \setminus \{a\}) = 1$. Taking another $a' \in A$ instead of a, we get $\#\eta_1(A) = 1$, i.e., A does not depend on the second factor of Y. Since $\nu(A)$ irredundantly spans $\nu(o)$, we get $\#\pi_1(A') = 1$, a contradiction. □

(c2) Assume $s = 2$ and $c = 1$ (the case $s = 2$ and $c = 1$) being similar. We may assume $\pi_h(A') = 1$ for all $h > 2$. Call i the only index such that $\#\pi_i(B') > 1$. As in step (c1) we get $k \le \#\{1, 2, 3\} \le 3$.

(c3) Assume $s = c = 1$. As is step (c1) and (c2) we get $k \le 2$.

7. $r_Y(q) = 2$

In this section we assume $r_Y(q) = 2$. We fix $E \in \mathcal{S}(Y, q)$. Set $M := \langle\nu(A)\rangle \cap \langle\nu(E)\rangle$. Call Y' (resp. Y'') the minimal multiprojective subspace of Y containing $E \cup A$ (resp. $E \cup B$)

Lemma 7.1. *If $w(Y) \ge 4$, then either $\nu(A)$ and $\nu(B)$ irredundantly span q.*

Proof. Assume for instance that $\nu(A)$ does not span irredundantly q. Since $r_Y(q) = 2$, there is $A' \subset A$ such that $\#A' = 2$ and $A' \in \mathcal{S}(Y, q)$. Since $A \cap B = \emptyset$, $A' \cap B = \emptyset$. Since $w(S) > 2$, [5, Proposition 2.3] gives that B irredundantly spans q. Let $W \subseteq Y$ be the minimal multiprojective space containing $A' \cup B$. Since $q \in \langle\nu(A')\rangle \cap \langle\nu(B)\rangle$ and $A' \cap B = \emptyset$, $e(A' \cup B) > 0$. Since S is equally dependent, $e(S) = e(A' \cup B) + 1$ and $\langle \nu(S) \rangle = \langle \nu(A' \cup B) \rangle$. Since $A' \cap B = \emptyset$, Proposition 4.3 gives $w(W) \le 3$. Set $\{a\} := A \setminus A'$. Since $\langle \nu(S) \rangle = \langle \nu(A' \cup B) \rangle$, $a \in \langle \nu(W) \rangle$. Concision for rank 1 tensors implies $\langle \nu(W) \rangle \cap \nu(Y) = \nu(W)$. Thus $a \in W$. Hence $W = Y$, contradicting the assumption $w(Y) \ge 4$. □

Remark 7.2. By Lemma 7.1 from now on in this section we assume that each set $\nu(A)$ and $\nu(B)$ irredundantly spans q.

Lemma 7.3. *Take a circuit $F \subset Y := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ concise for Y and with $\#F = 5$. Write $F = U \cup G$ with $\#U = 2$ and $\#G = 3$. Then Y is concise for U.*

Proof. By [1, Lemma 5.8] F is contained in an integral curve $C \subset Y$ of tridegree $(1, 1, 1)$. Each map $\pi_{i|C} : C \to \mathbb{P}^1$ is an isomorphism. Thus each $\pi_{i|U}$ is injective. □

Lemma 7.4. *$E \cap A \neq \emptyset$ (resp. $E \cap B \neq \emptyset$) if and only if either $w(S) \le 3$ or A (resp. B) is obtained form E making an elementary increasing.*

Proof. It is sufficient to prove the lemma for the set A. The "if" part follows from the definition of elementary increasing, because $\#E > 1$.

Assume $E \cap A \neq \emptyset$. Since $\nu(A)$ irredundantly spans q (Remark 7.2), we have E is not contained in A. Write $E \cap A = \{a\}$, $E = \{a, b\}$ and $A = \{a, u, v\}$. We need to prove that there is i such that $\pi_h(a) = \pi_h(u) = \pi_h(v)$ for all $h \neq i$, while $\pi_i(\{a, u, v\})$ spans a line.

(a) First assume that $E \cup A$ is not equally dependent. Since $\#(E \cup A) = 4$, we have $e(E \cup A) = 1$ and there is $F \subset E \cup A$ such that $\#F = 3$ and $e(F) = 1$. By Remark 2.1 there is i such that $\#\pi_h(F) = 1$ for all $h \neq i$ and $\pi_i(F)$ is formed by 3 collinear points. Since $\nu(E)$ and $\nu(A)$ irredundantly span q (Remark 7.2 and the assumption $E \in \mathcal{S}(Y, q)$), it is easy to check that $(E \cup A) \setminus F = \{a\}$. Thus A is obtained from E applying an elementary increasing with respect to b and the i-th factor of the multiprojective space.

(b) Now assume that $E \cup A$ is equally dependent. Since $\#(E \cup A) = 4$, [1, Proposition 5.2] says that $w(E \cup A) \le 2$ and that $\mathbb{P}^1 \times \mathbb{P}^1$ is the minimal multiprojective space containing $E \cup A$. Since $E \in \mathcal{S}(Y, q)$ and $r_Y(q) > 1$, $Y' \cong \mathbb{P}^1 \times \mathbb{P}^1$ is the minimal multiprojective space containing E.

(b1) Assume $E \cap B \neq \emptyset$ and $E \cup B$ is not equally dependent. By step (a) applied to B we get that B is obtained from E making a positive elementary increasing. Thus either $w(B) = 2$ or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is the minimal multiprojective space containing B (last sentence of Example 3.1) and it contains A, too, since it contains E. Thus $w(S) \le 3$.

(b2) Assume $E \cap B \neq \emptyset$ and $E \cup B$ equally dependent. Thus $Y'' \cong \mathbb{P}^1 \times \mathbb{P}^1$ and Y'' is the minimal multiprojective subspace containing E. Hence $Y'' = Y'$ and $Y = \mathbb{P}^1 \times \mathbb{P}^1$.

(b3) Assume $E \cap B = \emptyset$. We get $w(Y'') \le 3$ by Proposition 4.3 and (since $W \supseteq Y'$) we get $Y = W$. □

Lemma 7.5. *Assume $E \cap A \neq \emptyset$ and $E \cap B \neq \emptyset$. Then either $w(S) \leq 2$ or S is as in one of Examples 3.1 and 3.2.*

Proof. Assume $w(S) > 2$. By Lemma 7.4 A and B are obtained from E making an elementary increasing. Since $A \cap B = \emptyset$, we have $\#A \cap E = \#B \cap E = 1$ and $E \subset S$. By the definition of elementary increasing it is obvious that S is as in one of Examples 3.1 and 3.2 (Example 3.2 occurs if and only if we are doing the elementary increasings giving A and B from E with respect to the same factor of the multiprojective space). □

Lemma 7.6. *Assume $E \cap A = \emptyset$ (resp. $E \cap B = \emptyset$). Then $E \cup A$ (resp. $E \cup B$) is equally dependent.*

Proof. It is sufficient to prove the lemma for $E \cup A$. The assumption is equivalent to $\dim M = e(E \cup A) - 1$. Fix $a \in A$. Since $q \notin \langle \nu(A \setminus \{a\}) \rangle$, $\langle \nu(A \setminus \{a\}) \rangle \cap \langle \nu(E) \rangle$ is strictly contained in M. The Grassmann's formula gives $e((E \cup A) \setminus \{a\}) < e(E \cup A)$. Take $b \in E$. Since $q \notin \langle \nu(E \setminus \{b\}) \rangle$, we have $\langle \nu(E \setminus \{b\}) \rangle \cap \langle \nu(A) \rangle$ is strictly contained in M. Thus $E \cup A$ is equally dependent. □

Lemma 7.7. *Assume $E \cap A = E \cap B = \emptyset$. Then $w(S) \leq 3$ and $Y \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ if $w(S) = 3$.*

Proof. By Proposition 4.3 and Lemmas 7.3 and 7.6 we have $w(Y') \leq 3$, $w(Y'') \leq 3$ and if one of them, say $w(Y')$, is 3, then $Y' \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is the minimal multiprojective space containing E. Hence $w(Y'') = 3$ and $Y' = Y''$, i.e., $Y \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Now assume $w(Y') = w(Y'') = 2$. In this case both Y' and Y'' have the same number of factors as the minimal multiprojective space containing E and exactly the same non-trivial factor, i.e., if $E = \{u, v\}$ with $u = (u_1, \dots, u_k)$, $v = (v_1, \dots, v_k)$ and $u_i = v_i$ for all $i > 2$, then $\#\pi_i(Y') = \#\pi_i(Y'') = 1$ for all $i > 2$. Since $\pi_i(Y') = \{u_i\} = \pi_i(Y'')$ for all $i > 2$, we get $w(Y) = 2$. □

Lemma 7.8. *Either S is as in Examples 3.1 and 3.2 or $w(S) \leq 4$ with $Y = (\mathbb{P}^1)^4$ if $w(S) = 4$.*

Proof. By the previous lemmas we may assume that exactly one among $E \cap A$ and $E \cap B$, say the first one, is empty. Thus B is obtained from E making a positive elementary increasing, while $w(Y') \leq 3$ and $Y' \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ if $w(Y') = 3$. First assume $w(Y') = 3$ and $Y' \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. By Lemma 7.3 Y' is the minimal multiprojective space containing E. Hence $w(E \cup B) \leq 4$ and $Y'' = (\mathbb{P}^1)^4$ with $Y \supset Y'$ if $w(Y'') = 4$ (last part of Example 3.1). We get $w(Y) \leq 4$ and $Y \cong (\mathbb{P}^1)^4$ if S is not as in Examples 3.1 and 3.2. Now assume $w(Y') = 2$. Thus $w(E) = 2$. We get that either $w(Y'') = 2$ or $Y'' \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with $\#\pi_3(A) = 1$. Hence $w(Y) \leq 3$. □

8. $r_Y(q) = 3$

The point $q \in \mathbb{P}^N$ has tensor rank 3 and hence $\nu(A)$ and $\nu(B)$ are tensor decompositions of it with the minimal number of terms. By concision ([9, Proposition 3.1.3.1]) Y is the minimal multiprojective space containing A and the minimal multiprojective space containing B. Hence $1 \le n_i \le 2$ for all i. Y is as in the cases of [5, Theorem 7.1] coming from the cases $\#S = 6$, i.e., we exclude case (6) of that list. In all cases (1), (2), (3), (4), (5) of that list we have $w(Y) \le 4$ and $w(Y) = 4$ if and only if $Y \cong (\mathbb{P}^1)^4$. The sets $\mathcal{S}(Y, q)$ to which A and B belong are described in the same paper. The possible concise Y's are listed in [5, Theorem 7.1], but we stress that from the point of view of tensor ranks among the sets S described in one of the examples of [5] there is some structure. If we start with S with $e(S) = 1$ and arising in this section and any decomposition $S = A \cup B$ with $\#A = \#B = 3$, the assumption $e(S) = 1$ and $e(A) = e(B) = 0$ gives that $\langle \nu(A) \rangle \cap \langle \nu(B) \rangle$ is a single point by the Grassmann's formula. Call q this point. If we assume $r_X(q) = 3$, then in [5] there is a description of all $S \in \mathcal{S}(Y, q)$. Changing the decomposition $S = A \cup B$ change q and hence all sets associated to S using the point q. Thus if $e(S) = 1$ and there is a partition $S = A \cup B$ of S such that the point $\langle \nu(A) \rangle \cap \langle \nu(B) \rangle$ has tensor rank 3, then to S and the partition $S = A \cup B$ we may associate a family $\mathcal{S}(Y, q)$ of circuits associated to q.

End of the proof of Theorem 1.2. In the last 4 sections we considered all possible cases coming from a fixed partition of $A \cup B$. We summarized the case $r_Y(q) = 2$ in the statement of Lemma 7.8. □

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