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ESTIMATES FOR THE HIGHER ORDER RIESZ TRANSFORMS RELATED TO SCHRÖDINGER TYPE OPERATORS

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Abstract. We consider the Schrödinger type operator $\mathcal{L}_k = (-\Delta)^k + V^k$ on $\mathbb{R}^n (n \geq 2k+1)$, where $k = 1, 2$ and the nonnegative potential V belongs to the reverse Hölder class RH_s with $n/2 < s < n$. In this paper, we establish the (L^p, L^q)-boundedness of the higher order Riesz transform $T_{\alpha,\beta} = V^{2\alpha}\nabla^2\mathcal{L}_2^{-\beta} (0 \leq \alpha \leq 1/2 < \beta \leq 1, \beta - \alpha \geq 1/2)$ and its adjoint operator $T^*_{\alpha,\beta}$ respectively. We show that $T_{\alpha,\beta}$ is bounded from Hardy type space $H^1_{\mathcal{L}_2}(\mathbb{R}^n)$ into $L^{p_2}(\mathbb{R}^n)$ and $T^*_{\alpha,\beta}$ is bounded from $L^{p_1}(\mathbb{R}^n)$ into BMO type space $BMO_{\mathcal{L}_1}(\mathbb{R}^n)$ when $\beta - \alpha > 1/2$, where $p_1 = \frac{n}{4(\beta-\alpha)-2}$, $p_2 = \frac{n}{n-4(\beta-\alpha)+2}$. Moreover, we prove that $T_{\alpha,\beta}$ is bounded from $BMO_{\mathcal{L}_1}(\mathbb{R}^n)$ to itself when $\beta - \alpha = 1/2$.

1. Introduction and results

For $s > 1$, a nonnegative locally L^s-integrable function V is said to belong to the reverse Hölder class RH_s if there exists a constant $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(y)^s dy\right)^{1/s} \leq \frac{C}{|B|}\int_B V(y)dy$$

holds for every ball $B \subset \mathbb{R}^n$. It is obvious that V is a doubling measure if $V \in RH_s$ with $s > 1$.

Given a potential $V \in RH_s$ with $s > n/2$, we define the auxiliary function (see [7])

$$\rho(x) = \sup\left\{r > 0 : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)dy \leq 1\right\}, \quad x \in \mathbb{R}^n,$$

where $B(x,r)$ denotes the ball centered at x with radius r. It is well known that $0 < \rho(x) < \infty$ for any $x \in \mathbb{R}^n$.

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We consider the Schrödinger type operator

$$\mathcal{L}_k = (-\Delta)^k + V^k, \quad n \geq 2k+1, \; k = 1, 2.$$

When V is a nonnegative polynomial, Zhong [10] showed the L^p boundedness of the operators $V^{2-j/2}\nabla^j \mathcal{L}_2^{-1}$, where $j = 0, 1, 2, 3, 4$. For the potential V which belongs to $RH_s, n/2 < s < n$, and there exists a constant C such that $V(x) \leq C\rho(x)^{-2}$, Sugano in [8] established estimates of the fundamental solutions for $\mathcal{L}_2$ and showed the L^p boundedness of the operators $V^{2-j/2}\nabla^j \mathcal{L}_2^{-1}$, where $j = 0, 1, 2, 3$. When $V \in RH_s$ with $s > n/2$, Wang in [9] obtained the (L^p, L^q)-boundedness of the operator $V^{2\alpha}\mathcal{L}_2^{-\beta}$ for $0 < \alpha \leq \beta \leq 1$.

Let V belong to RH_s with $n/2 < s < n$. We concentrate on the higher order Riesz transform

$$T_{\alpha,\beta} = V^{2\alpha}\nabla^2 \mathcal{L}_2^{-\beta}, \; 0 \leq \alpha \leq 1/2 \leq \beta \leq 1, \; \beta - \alpha \geq 1/2,$$

and its adjoint operator $T^*_{\alpha,\beta}$. Obviously, if $(\alpha, \beta) = (0, \frac{1}{2})$, $T_{\alpha,\beta}$ just is the transform $\mathcal{R} = \nabla^2 \mathcal{L}_2^{-\frac{1}{2}}$. Liu and Dong in [5] investigated the L^p and weak $(1,1)$ estimates of $\mathcal{R}$; Liu et al. in [6] obtained the L^p boundedness of the commutators of $\mathcal{R}$.

We first establish the following (L^p, L^q)-boundedness.

Theorem 1.1. *Suppose $V \in RH_s$ with $n/2 < s < n$. Let $0 \leq \alpha \leq 1/2 < \beta \leq 1, \beta - \alpha \geq \frac{1}{2}$, and $\frac{1}{p_\alpha} = \frac{2\alpha+2}{s} - \frac{2}{n}$.*

(i) *If $p'_\alpha < p < \frac{n}{4(\beta-\alpha)-2}$ and $\frac{1}{q} = \frac{1}{p} - \frac{4(\beta-\alpha)-2}{n}$, then*

$$\|T^*_{\alpha,\beta}(f)\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)};$$

(ii) *If $1 < p < \frac{1}{\frac{1}{p_\alpha} + \frac{4(\beta-\alpha)-2}{n}}$ and $\frac{1}{q} = \frac{1}{p} - \frac{4(\beta-\alpha)-2}{n}$, then*

$$\|T_{\alpha,\beta}(f)\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}.$$

Let us recall the concept of Hardy spaces related to Schrödinger type operators.

The Schrödinger type operators $\mathcal{L}_k = (-\Delta)^k + V^k$ $(k = 1, 2)$ generate C_0 semigroups $\{e^{-t\mathcal{L}_k}\}_{t>0}$. The maximal function with respect to the semigroup $\{e^{-t\mathcal{L}_k}\}_{t>0}$ are given by

$$M^{\mathcal{L}_k} f(x) = \sup_{t>0} |e^{-t\mathcal{L}_k} f(x)|.$$

By [1, 3], a function $f \in L^1(\mathbb{R}^n)$ is said to be in $H^1_{\mathcal{L}_k}(\mathbb{R}^n)(k = 1, 2)$ if the semigroup maximal function $M^{\mathcal{L}_k} f$ belongs to $L^1(\mathbb{R}^n)$. The norm of such a function is defined by

$$\|f\|_{H^1_{\mathcal{L}_k}(\mathbb{R}^n)} = \|M^{\mathcal{L}_k} f\|_{L^1(\mathbb{R}^n)}.$$

It was showed that $H^1_{\mathcal{L}_2}(\mathbb{R}^n) = H^1_{\mathcal{L}_1}(\mathbb{R}^n)$ with equivalent norms (see Theorem 1.1 in [1]).

We consider the boundedness of $T_{\alpha,\beta}$ at the endpoint $p = 1$, and get the following result.

Theorem 1.2. *Suppose $V \in RH_s$ with $n/2 < s < n$. Let $0 \le \alpha \le 1/2 < \beta \le 1, \beta - \alpha \ge 1/2$. Then*

$$\|T_{\alpha,\beta}(f)\|_{L^{p_2}(\mathbb{R}^n)} \le C\|f\|_{H^1_{\mathcal{L}_1}(\mathbb{R}^n)},$$

where $p_2 = \frac{n}{n-4(\beta-\alpha)+2}$.

The dual space of $H^1_{\mathcal{L}_1}(\mathbb{R}^n)$ is the BMO type space $BMO_{\mathcal{L}_1}(\mathbb{R}^n)$ (see [2]). Let f be a locally integrable function on $\mathbb{R}^n$ and $B = B(x,r)$. Set $f_B = \frac{1}{|B|}\int_B f(y)dy$ and $f(B,V) = f_B$ if $r < \rho(x)$; $f(B,V) = 0$ if $r \ge \rho(x)$. We say $f \in BMO_{\mathcal{L}_1}(\mathbb{R}^n)$ if

$$\|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)} = \sup_B \frac{1}{|B|}\int_B |f(y) - f(B,V)|dy < \infty.$$

It follows from [2] that $\|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)}$ is actually a norm which makes $BMO_{\mathcal{L}_1}(\mathbb{R}^n)$ a Banach space. Since $H^1(\mathbb{R}^n) \subset H^1_{\mathcal{L}_1}(\mathbb{R}^n)$, we conclude by duality that $BMO_{\mathcal{L}_1}(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$.

By Theorem 1.2 and duality, we get:

Corollary 1.3. *Given $V \in RH_s$ with $n/2 < s < n$. Let $0 \le \alpha \le 1/2 < \beta \le 1$. If $\beta - \alpha > 1/2$, we have*

$$\|T^*_{\alpha,\beta}(f)\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)} \le C\|f\|_{L^{p_1}(\mathbb{R}^n)},$$

where $p_1 = \frac{n}{4(\beta-\alpha)-2}$.

In case $\beta - \alpha = 1/2$, we have:

Theorem 1.4. *Given $V \in RH_s$ with $n/2 < s < n$, and $0 \le \alpha \le 1/2 < \beta \le 1$. If $\beta - \alpha = 1/2$, we have*

$$\|T^*_{\alpha,\beta}(f)\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)} \le C\|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)}.$$

2. Preliminaries

Throughout this section, we always assume $V \in RH_s$ with $n/2 < s < n$. Let us first recall some important properties concerning the auxiliary function.

Lemma 2.1 ([4])**.** *There exist constants C and $l_0 > 0$ such that*

$$\frac{1}{r^{n-2}}\int_{B(x,r)} V(y)dy \le C\left(1 + \frac{r}{\rho(x)}\right)^{l_0}.$$

Lemma 2.2 ([7])**.** *For $0 < r < R < \infty$, we have*

$$\frac{1}{r^{n-2}}\int_{B(x,r)} V(y)dy \le C\left(\frac{R}{r}\right)^{n/s-2}\frac{1}{R^{n-2}}\int_{B(x,R)} V(y)dy.$$

Lemma 2.3 ([7]). *There exist constants C and $k_0 \geq 1$ such that*

$$C^{-1}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-k_0} \leq \frac{\rho(y)}{\rho(x)} \leq C\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{1+k_0}}$$

for all $x, y \in \mathbb{R}^n$.

A ball $B(x,\rho(x))$ is called critical. Assume that $Q = B(x_0,\rho(x_0))$. For $x \in Q$, Lemma 2.3 tell us that $\rho(x) \sim \rho(y)$, if $|x-y| < C\rho(x)$.

Let $\mathcal{W}_\beta = \nabla^2\mathcal{L}_2^{-\beta}$, and let K_β, K_β^* be the kernels of $\mathcal{W}_\beta$ and $\mathcal{W}_\beta^*$, respectively. Then $K_\beta^*(x,z) = K_\beta(z,x)$ and we have the following estimates.

Lemma 2.4. *Suppose* $1/2 < \beta \leq 1$.

(i) *For every positive integer N there exists a constant C_N such that*

$$\begin{aligned} |K_\beta^*(x,z)| \leq & \frac{C_N}{(1+\frac{|x-z|}{\rho(x)})^N}\frac{1}{|x-z|^{n-4\beta}} \\ & \times\left(\int_{B(z,|x-z|/4)}\frac{V(\xi)^2}{|\xi-z|^{n-2}}d\xi+\frac{1}{|x-z|^2}\right). \end{aligned}$$

Moreover, the inequality above also holds with $\rho(x)$ replaced by $\rho(z)$.

(ii) *For every positive integer N and some $\delta > 0$ there exists a constant C_N such that*

$$\begin{aligned} & |K_\beta^*(x,z) - K_\beta^*(y,z)| \\ \leq & \frac{C_N}{(1+\frac{|x-z|}{\rho(x)})^N}\frac{|x-y|^\delta}{|x-z|^{n-4\beta+\delta}}\left(\int_{B(z,|x-z|/4)}\frac{V(\xi)^2}{|\xi-z|^{n-2}}d\xi+\frac{1}{|x-z|^2}\right), \end{aligned}$$

whenever $|x-y| < \frac{1}{16}|x-z|$. Moreover, the inequality above also holds with $\rho(x)$ replaced by $\rho(z)$.

Proof. Let $\Gamma_{\mathcal{L}_2}(x,y,\lambda)$ be the fundamental solution of $\mathcal{L}_2+\lambda$, where $\lambda \geq 0$. When $\lambda = 0$, it follows from Theorem 2 in [8] that for any positive integer N there exists a positive constant C_N such that

$$0 \leq \Gamma_{\mathcal{L}_2}(x,y,0) \leq \frac{C_N}{\left(1+\frac{|x-y|}{\rho(x)}\right)^N}\frac{1}{|x-y|^{n-4}}.$$

When $\lambda > 0$, from [5] we have

$$0 \leq \Gamma_{\mathcal{L}_2}(x,y,\lambda) \leq \frac{C_N}{(1+\lambda^{\frac{1}{2}}|x-y|^2)^N\left(1+\frac{|x-y|}{\rho(x)}\right)^N}\frac{1}{|x-y|^{n-4}}.$$

By the functional calculus, we may write, for any $1/2 < \beta < 1$,

$$\mathcal{L}_2^{-\beta} = \frac{\sin\pi\beta}{\pi}\int_0^\infty \lambda^{-\beta}(\mathcal{L}_2+\lambda)^{-1}d\lambda.$$

Let $f \in C_0^\infty$. It follows from $(\mathcal{L}_2+\lambda)^{-1}f(x)=\int_{\mathbb{R}^n}\Gamma_{\mathcal{L}_2}(x,z,\lambda)f(z)dz$ that

$$\mathcal{W}_\beta(f)(x)=\nabla^2\mathcal{L}_2^{-\beta}(f)(x)=\int_{\mathbb{R}^n}K_\beta(x,z)f(z)dz.$$

Then

$$\mathcal{W}_\beta^*(f)(x)=\int_{\mathbb{R}^n}K_\beta^*(x,z)f(z)dz,$$

where

$$K_\beta^*(x,z)=\frac{\sin\pi\beta}{\pi}\int_0^\infty \lambda^{-\beta}\nabla_z^2\Gamma_{\mathcal{L}_2}(z,x,\lambda)d\lambda$$

for $1/2<\beta<1$, and

$$K_\beta^*(x,z)=\nabla_z^2\Gamma_{\mathcal{L}_2}(z,x,0)$$

for $\beta=1$.

Fix $x_0,z_0\in\mathbb{R}^n$. Let $u(z)=\Gamma_{\mathcal{L}_2}(z,x_0,\lambda)$ and $R=\frac{|x_0-z_0|}{4}$. By the proof of Lemma 9 in [5] we know

$$\nabla^2u(z_0)\le\frac{C}{(1+\lambda^{\frac12}R^2)^N\big(1+\frac{R}{\rho(x_0)}\big)^N}\left\{\frac{1}{R^{n-4}}\int_{B(z_0,R)}\frac{V(\xi)^2d\xi}{|\xi-z_0|^{n-2}}+\frac{1}{R^{n-2}}\right\}.$$

Then, for $\beta=1$ we have

$$\begin{aligned}|K_\beta^*(x_0,z_0)|&=|\nabla_z^2\Gamma_{\mathcal{L}_2}(z_0,x_0,0)|\\&\le\frac{C}{\big(1+\frac{R}{\rho(x_0)}\big)^N}\left\{\frac{1}{R^{n-4}}\int_{B(z_0,R)}\frac{V(\xi)^2d\xi}{|\xi-z_0|^{n-2}}+\frac{1}{R^{n-2}}\right\}.\end{aligned}$$

Note that

$$\int_0^\infty\frac{\lambda^{-\beta}}{\big(1+\lambda^{\frac12}R^2\big)^N}d\lambda\le CR^{4\beta-4}.$$

So, for $1/2<\beta<1$, we get

$$|K_\beta^*(x_0,z_0)|\le\frac{C_N}{\big(1+\frac{R}{\rho(x_0)}\big)^N}\frac{1}{R^{n-4\beta}}\left\{\int_{B(z_0,|u-z_0|<R)}\frac{V(\xi)^2d\xi}{|u-z_0|^{n-2}}+\frac{1}{R^2}\right\}.$$

(ii) Fix $x,z\in\mathbb{R}^n$. Let $R=\frac{|x-z|}{8}$ and $\delta=4-2n/s$. By the functional calculus we have

$$|K_\beta^*(x,z)-K_\beta^*(y,z)|\le C\int_0^\infty\lambda^{-\beta}|\nabla_z^2\Gamma_{\mathcal{L}_2}(z,x,\lambda)-\nabla_z^2\Gamma_{\mathcal{L}_2}(z,y,\lambda)|d\lambda.$$

From the proof of Lemma 3.2 in [6] we get

$$\begin{aligned}&|\nabla_z^2\Gamma_{\mathcal{L}_2}(z,x,\lambda)-\nabla_z^2\Gamma_{\mathcal{L}_2}(z,y,\lambda)|\\&\le\frac{CR^{-\delta}|x-y|^\delta}{(1+\lambda^{\frac12}R^2)^N\big(1+\frac{R}{\rho(x)}\big)^N}\left\{\frac{1}{R^{n-4}}\int_{B(z,5R)}\frac{V(\xi)^2d\xi}{|\xi-z|^{n-2}}+\frac{1}{R^{n-2}}\right\}.\end{aligned}$$

Note that

$$\int_0^\infty \frac{\lambda^{-\beta}}{\left(1+\lambda^{\frac{1}{2}}R^2\right)^N}d\lambda \le CR^{4\beta-4}.$$

Then

$$|K_\beta^*(x,z)-K_\beta^*(y,z)| \le \frac{C|x-y|^\delta}{\left(1+\frac{R}{\rho(x)}\right)}\frac{1}{R^{n-4\beta+\delta}}\left\{\int_{B(z,5R)}\frac{V(\xi)^2d\xi}{|u-z|^{n-2}}+\frac{1}{R^2}\right\}.$$

This concludes the proof of Lemma 2.4. □

Let $\gamma \ge 1, f \in L^\gamma_{loc}(\mathbb{R}^n)$. For $0 \le \sigma < n/\gamma$, the fractional Hardy-Littlewood maximal function $M_{\sigma,\gamma}$ is defined by

$$M_{\sigma,\gamma}(f)(x) = \sup\left(\frac{1}{|B|^{1-\frac{\sigma\gamma}{n}}}\int_B |f(y)|^\gamma dy\right)^{1/\gamma}.$$

Lemma 2.5. *Suppose* $1 \le \gamma < p < \frac{n}{\sigma}$, *and* $\frac{1}{q}=\frac{1}{p}-\frac{\sigma}{n}$. *Then*

$$\|M_{\sigma,\gamma}f\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}.$$

At the end of this section we give a characterization of function space $BMO_{\mathcal{L}_1}(\mathbb{R}^n)$.

Lemma 2.6 ([2])**.** *Let* $1 \le p < \infty, B = B(x,r)$. *If* $f \in BMO_{\mathcal{L}_1}(\mathbb{R}^n)$, *then*

$$\sup_B\left(\frac{1}{|B|}\int_B |f(y)-f(B,V)|^p dy\right)^{1/p} \le C\|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)}.$$

A function $f \in BMO_{\mathcal{L}_1}(\mathbb{R}^n)$ *if and only if there exists a suitable constant* c_B *depending on* B *and satisfying* $c_B = 0$ *whenever* $r \ge \rho(x)$ *such that*

$$\sup_B\left(\frac{1}{|B|}\int_B |f(y)-c_B|^p dy\right)^{1/p} < \infty$$

and

$$\|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)} \le C\sup_B\left(\frac{1}{|B|}\int_B |f(y)-c_B|^p dy\right)^{1/p}.$$

3. Proof of main results

First we prove the following lemma.

Lemma 3.1. *Suppose* $V \in RH_s$ *with* $n/2 < s < n$. *Let* $0 \le \alpha \le 1/2 < \beta \le 1, \beta-\alpha \ge \frac{1}{2}$. *Then there exists a constant* C *such that*

$$|T^*_{\alpha,\beta}f(x)| \le CM_{\gamma,p'_\alpha}(f)(x)$$

for all $f \in C_0^\infty(\mathbb{R}^n)$, *where* $\frac{1}{p'_\alpha}=1-\frac{1}{p_\alpha}$, $\frac{1}{p_\alpha}=\frac{2\alpha+2}{s}-\frac{2}{n}$, *and* $\gamma = 4(\beta-\alpha)-2$.

Proof. Let $r=\rho(x), C_j=\{z:\ 2^{j-1}r<|z-x|\le 2^jr\}$. We choose t such that $1/t=2/s-2/n$. Then $1/t+1/p'_\alpha+(2\alpha)/s=1$. By Hölder inequality,

$$\begin{aligned}|T^*_{\alpha,\beta}(f)(x)| \le& \sum_{j=-\infty}^{+\infty}\int_{C_j}|K^*_\beta(x,z)|V(z)^{2\alpha}|f(z)|dz\\ \le& C\sum_{j=-\infty}^{+\infty}(2^jr)^n\left(\frac{1}{(2^jr)^n}\int_{C_j}|K^*_\beta(x,z)|^tdz\right)^{1/t}\\ &\times\left(\frac{1}{(2^jr)^n}\int_{B(x,2^jr)}V(z)^sdz\right)^{2\alpha/s}\left(\frac{1}{(2^jr)^n}\int_{B(x,2^jr)}|f(z)|^{p'_\alpha}dz\right)^{1/p'_\alpha}.\end{aligned}$$

Due to $V\in RH_s$, we have

$$\begin{aligned}\left(\frac{1}{(2^jr)^n}\int_{B(x,2^jr)}V(z)^sdz\right)^{2\alpha/s} \le& C\left(\frac{1}{(2^jr)^n}\int_{B(x,2^jr)}V(z)dz\right)^{2\alpha}\\ \le& C\ (2^jr)^{-4\alpha}\left(\frac{(2^jr)^2}{(2^jr)^n}\int_{B(x,2^jr)}V(z)dz\right)^{2\alpha}.\end{aligned}$$

Let $\mathcal{I}_2(f)(x)=\int_{\mathbb{R}^n}\frac{f(y)dy}{|x-y|^{n-2}}$. By (i) of Lemma 2.4, Minkowski inequality and Hardy-Littlewood-Sobolev inequality, we obtain

$$\begin{aligned}&(2^jr)^n\left(\frac{1}{(2^jr)^n}\int_{C_j}|K^*_\beta(x,z)|^tdz\right)^{1/t}\\ \le& \frac{C}{(1+2^j)^N}\frac{1}{(2^jr)^{\frac{n}{t}-4\beta}}\left(\left(\int_{C_j}\left(\mathcal{I}_2(V^2\chi_{B(x,2^{j+1}r)})(z)\right)^tdz\right)^{1/t}+(2^jr)^{\frac{n}{t}-2}\right)\\ \le& \frac{C}{(1+2^j)^N}\frac{1}{(2^jr)^{\frac{n}{t}-4\beta}}\left((2^jr)^{2n/s}\left(\frac{1}{(2^jr)^n}\int_{B(x,2^{j+1}r)}V(z)^sdz\right)^{2/s}+(2^jr)^{\frac{n}{t}-2}\right)\\ \le& \frac{C}{(1+2^j)^N}\frac{1}{(2^jr)^{2-4\beta}}\left(\left(\frac{(2^jr)^2}{(2^jr)^n}\int_{B(x,2^jr)}V(z)dz\right)^2+1\right).\end{aligned}$$

For case $j\ge 1$, by Lemma 2.1 we have

$$\frac{(2^jr)^2}{(2^jr)^n}\int_{B(x,2^jr)}V(z)dz\le C2^{jl_0}.$$

For the case $j\le 0$, by Lemma 2.2 we get

$$\frac{(2^jr)^2}{(2^jr)^n}\int_{B(x,2^jr)}V(z)dz\le C2^{j(2-n/s)}\frac{1}{r^{n-2}}\int_{B(x,r)}V(z)dz\le C2^{j(2-n/s)}.$$

Then, taking $N \geq 2l_0(\alpha+1)$ we get

$$\begin{aligned}|T^*_{\alpha,\beta}(f)(x)| &\leq C\left(\sum_{j=1}^{\infty}\frac{1}{(2^j)^{N-2l_0(\alpha+1)}}+\sum_{j=-\infty}^{0}(2^j)^{2\alpha(2-\frac{n}{s})}\right)\\ &\qquad\times\frac{1}{(2^jr)^{2-4(\beta-\alpha)}}\left(\frac{1}{(2^jr)^n}\int_{B(x,2^jr)}|f(z)|^{p'_\alpha}dz\right)^{1/p'_\alpha}\\ &\leq C\left(\frac{1}{(2^jr)^{n-(4(\beta-\alpha)-2)p'_\alpha)}}\int_{B(x,2^jr)}|f(z)|^{p'_\alpha}dz\right)^{1/p'_\alpha}\\ &\leq CM_{\gamma,p'_\alpha}(f)(x).\end{aligned}$$

□

Proof of Theorem 1.1. By Lemma 2.5 and Lemma 3.1 we know

$$\|T^*_{\alpha,\beta}(f)\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$$

holds if $p'_\alpha < p < \frac{n}{4(\beta-\alpha)-2}$ and $\frac{1}{q} = \frac{1}{p} - \frac{4(\beta-\alpha)-2}{n}$.

By duality, we get

$$\|T_{\alpha,\beta}(f)\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$$

for $\frac{n}{n-4(\beta-\alpha)+2} < q < p_\alpha$ and $\frac{1}{q} = \frac{1}{p} - \frac{4(\beta-\alpha)-2}{n}$. These conditions are equivalent to

$$1 < p < \frac{1}{\frac{1}{p_\alpha}+\frac{4(\beta-\alpha)-2}{n}} \quad\text{and}\quad \frac{1}{q} = \frac{1}{p} - \frac{4(\beta-\alpha)-2}{n}.$$

Theorem 1.1 is proved. □

Proof of Theorem 1.2. Since $H^1_{\mathcal{L}_2}(\mathbb{R}^n) = H^1_{\mathcal{L}_1}(\mathbb{R}^n)$ and their norms are equivalent, we only need to show that the operator $T_{\alpha,\beta}$ maps the Hardy space $H^1_{\mathcal{L}_1}(\mathbb{R}^n)$ continuously into $L^{p_2}(\mathbb{R}^n)$, where $p_2 = \frac{n}{n-4(\beta-\alpha)+2}$. Firstly, we review the concept of $(1,q)_\rho$-atom.

Let $1 < q \leq \infty$. A measurable function a is called a $(1,q)_\rho$-atom associated with the ball $B(x,r)$ if $r < \rho(x)$ and the following conditions hold:

(i) supp $a \subset B(x,r)$ for some $x \in \mathbb{R}^n$ and $r > 0$,

(ii) $\|a\|_{L^q(\mathbb{R}^n)} \leq |B(x,r)|^{1/q-1}$,

(iii) if $r < \rho(x)/4$, then $\int_{B(x,r)} a(x)dx = 0$.

It follows from [3] that the Hardy space $H^1_{\mathcal{L}_1}(\mathbb{R}^n)$ admits the following atomic decomposition:

Lemma 3.2. *$f \in H^1_{\mathcal{L}_1}(\mathbb{R}^n)$ if and only if f can be written as $f = \sum_j \lambda_j a_j$, where a_j are $(1,q)_\rho$-atoms and $\sum_j |\lambda_j| < \infty$. Moreover*

$$\|f\|_{H^1_{\mathcal{L}_1}} \sim \inf\left\{\sum_j |\lambda_j|\right\},$$

where the infimum is taken over all atomic decompositions of f into $(1,q)_\rho$-atoms.

Since $0 \le \alpha \le 1/2 < \beta \le 1, \beta-\alpha \ge 1/2$, we can choose q_1 and q_2 such that

$$1 < q_1 < \frac{1}{\frac{1}{p_\alpha} + \frac{4(\beta-\alpha)-2}{n}}$$

and

$$\frac{1}{q_2} = \frac{1}{q_1} - \frac{4(\beta-\alpha)-2}{n}.$$

By Lemma 3.2 we only need to prove $\|T_{\alpha,\beta}(a)\|_{L^{p_2}(\mathbb{R}^n)} \le C$ holds for any $(1,q_1)_\rho$-atom, where the constant $C>0$ is independent of a.

Assume that $\operatorname{supp} a \subset B(x_0,r), r<\rho(x_0)$. Then

$$\begin{aligned}\|T_{\alpha,\beta}(a)\|_{L^{p_2}(\mathbb{R}^n)} &\le \|\chi_{16B}T_{\alpha,\beta}(a)\|_{L^{p_2}(\mathbb{R}^n)} + \|\chi_{(16B)^c}T_{\alpha,\beta}(a)\|_{L^{p_2}(\mathbb{R}^n)}\\ &= I_1 + I_2.\end{aligned}$$

By Hölder inequality, Theorem 1.1 and $\frac{1}{p_2} = 1 - \frac{4(\beta-\alpha)-2}{n}$, we have

$$\begin{aligned}I_1 &= \|\chi_{16B}T_{\alpha,\beta}(a)\|_{L^{p_2}(\mathbb{R}^n)}\\ &\le C\ |16B|^{\frac{1}{p_2}-\frac{1}{q_2}} \left(\int_{\mathbb{R}^n} |T_{\alpha,\beta}a(x)|^{q_2}dx\right)^{1/q_2}\\ &\le C\ |16B|^{\frac{1}{p_2}-\frac{1}{q_2}} \left(\int_B |a(x)|^{q_1}dx\right)^{1/q_1}\\ &\le C\ \ |16B|^{\frac{1}{p_2}-\frac{1}{q_2}}|B|^{\frac{1}{q_1}-1} \le C.\end{aligned}$$

We divided into two cases for the estimate of $I_2 : r \ge \rho(x_0)/4$ and $r < \rho(x_0)/4$.

Case I: $r \ge \rho(x_0)/4$. In this case, we have $r \sim \rho(x_0)$. For $z \in B(x_0,r)$, we have $\rho(z) \sim \rho(x_0)$. By Lemma 2.4,

$$\begin{aligned}&\int_B \big|K_\beta(x,z)a(z)\big|dz\\ &\le C\int_B \frac{|a(z)|dz}{\left(1+\frac{|x-z|}{\rho(z)}\right)^N |x-z|^{n-4\beta+2}}\\ &\quad + \int_B \frac{|a(z)|}{\left(1+\frac{|x-z|}{\rho(z)}\right)^N |x-z|^{n-4\beta}} \int_{B(x,|x-z|/4)} \frac{V(\xi)^2}{|\xi-x|^{n-2}}d\xi dz.\end{aligned}$$

For $z \in B$, $x \in C_k = \{x : 2^k r < |x-x_0| \le 2^{k+1}r\}$, $k \ge 5$, we have

$$\begin{aligned}&\int_B \big|K_\beta(x,z)a(z)\big|dz\\ &\le C\frac{1}{(1+2^k)^N(2^kr)^{n-4\beta+2}}\int_B |a(z)|dz\end{aligned}$$

$$+\frac{1}{(1+2^k)^N(2^kr)^{n-4\beta}}\mathcal{I}_2(V^2\chi_{B(x_0,2^{k+1}r)})(x)\int_B|a(z)|dz.$$

Then

$$\begin{aligned}
I_2 &= \left(\int_{(16B)^c}\left|T_{\alpha,\beta}a(x)\right|^{p_2}dx\right)^{1/p_2}\\
&\leq \left(\sum_{k\geq 5}\int_{C_k}V(x)^{2\alpha p_2}\left(\int_B\left|K_\beta(x,z)a(z)\right|dz\right)^{p_2}dx\right)^{1/p_2}\\
&\leq C\left(\sum_{k\geq 5}\frac{(2^kr)^{(4\beta-n-2)p_2+n}}{(1+2^k)^{Np_2}}\frac{1}{|2^kB|}\int_{2^kB}V(x)^{2\alpha p_2}dx\right)^{1/p_2}\int_B|a(z)|dz\\
&\quad+\left(\sum_{k\geq 5}\frac{(2^kr)^{(4\beta-n)p_2+n}}{(1+2^k)^{Np_2}}\frac{1}{|2^kB|}\int_{2^kB}V(x)^{2\alpha p_2}(\mathcal{I}_2(V^2\chi_{2^kB})(x))^{p_2}dx\right)^{1/p_2}\\
&\quad\times\int_B|a(z)|dz\\
&= I_{21}+I_{22}.
\end{aligned}$$

Notice

$$p_2=\frac{n}{n-4(\beta-\alpha)+2}<\frac{n}{4\alpha}<\frac{s}{2\alpha}.$$

By Hölder inequality, $V\in RH_s$ and Lemma 2.1 we have

$$\frac{1}{|2^kB|}\int_{2^kB}V(x)^{2\alpha p_2}dx\leq C(2^kr)^{-4\alpha p_2}\left(1+\frac{2^kr}{\rho(x_0)}\right)^{2l_0\alpha p_2}. \tag{1}$$

Since a is a $(1,q_1)_\rho$-atom, we have

$$\int_B|a(y)|dy\leq 1. \tag{2}$$

Due to $4(\beta-\alpha)-n-2+\frac{n}{p_2}=0$, by (1) and (2) we obtain

$$I_{21}\leq C\left(\sum_{k\geq 5}\frac{(2^k)^{(4(\beta-\alpha)-n-2)p_2+n}}{(1+2^k)^{Np_2-2l_0\alpha p_2}}\right)^{1/p_2}\leq C\left(\sum_{k\geq 5}\frac{1}{(2^k)^{Np_2-2l_0\alpha p_2}}\right)^{1/p_2}.$$

Taking N large enough such that $N>2l_0\alpha$, we get $I_{21}\leq C$.

It is easy to check that $\frac{1}{p_2}>\frac{1}{p_\alpha}=\frac{2\alpha}{s}+\frac{1}{t}$, $\frac{1}{t}=\frac{2}{s}-\frac{2}{n}$. By Hölder inequality, Hardy-Littlewood-Sobolev inequality and $V\in RH_s$, we obtain

$$\begin{aligned}
&\frac{1}{|2^kB|}\int_{2^kB}V(x)^{2\alpha p_2}(\mathcal{I}_2(V^2\chi_{2^kB})(x))^{p_2}dx\\
&\leq C\left(\frac{1}{|2^kB|}\int_{2^kB}V(x)^sdx\right)^{2p_2\alpha/s}\left(\frac{1}{|2^kB|}\int_{2^kB}(\mathcal{I}_2(V^2\chi_{2^kB})(x))^tdx\right)^{p_2/t}
\end{aligned}$$

$$\leq C\left(\frac{1}{|2^kB|}\int_{2^kB}V(x)dx\right)^{2p_2\alpha}\left(\frac{1}{|2^kB|}\int_{2^kB}V(x)^sdx\right)^{2p_2/s}|2^kB|^{p_2(\frac{2}{s}-\frac{1}{t})}$$
$$\leq C\left(\frac{1}{|2^kB|}\int_{2^kB}V(x)dx\right)^{2p_2(\alpha+1)}(2^kr)^{2p_2}.$$

Then

$$\frac{1}{|2^kB|}\int_{2^kB}V(x)^{2\alpha p_2}(\mathcal{I}_2(V^2\chi_{2^kB})(x))^{p_2}dx \leq C(2^kr)^{-2p_2(2\alpha+1)}\left(1+\frac{2^kr}{\rho(x_0)}\right)^{2p_2l_0(\alpha+1)}. \tag{3}$$

Note that $(4\beta-n)p_2+n-2p_2(2\alpha+1)=p_2(4(\beta-\alpha)-n-2)+n=0$. Then, taking $N>2l_0(\alpha+1)$, we get

$$I_{22}\leq C\left(\sum_{k\geq 5}\frac{1}{(1+2^k)^{Np_2-2p_2l_0(\alpha+1)}}\right)^{1/p_2}\leq C.$$

Case II: $r<\rho(x_0)/4$. Let $z\in B$, $x\in C_k$, $k\geq 5$. By (ii) of Lemma 2.4, for some $\delta>0$, we have

$$\begin{aligned}
&|K_\beta(x,z)-K_\beta(x,x_0)|\\
&\leq \frac{C_N}{\left(1+\frac{|x-z|}{\rho(x_0)}\right)^N}\frac{|z-x_0|^\delta}{|x-z|^{n-4\beta+\delta+2}}\\
&\quad+\frac{C_N}{\left(1+\frac{|x-z|}{\rho(x_0)}\right)^N}\frac{|z-x_0|^\delta}{|x-z|^{n-4\beta+\delta}}\int_{B(z,|x-z|/4)}\frac{V(\xi)^2}{|\xi-x|^{n-2}}d\xi\\
&\leq C\frac{1}{\left(1+\frac{2^kr}{\rho(x_0)}\right)^N}\frac{r^\delta}{(2^kr)^{n+\delta-4\beta+2}}\\
&\quad+\frac{1}{\left(1+\frac{2^kr}{\rho(x_0)}\right)^N}\frac{r^\delta}{(2^kr)^{n+\delta-4\beta}}\mathcal{I}_2(V^2\chi_{2^{k+1}B})(x).
\end{aligned}$$

By the vanishing condition of a and (2),

$$\begin{aligned}
I_2&=\left(\int_{(16B)^c}\left|T_{\alpha,\beta}a(x)\right|^{p_2}dx\right)^{1/p_2}\\
&\leq C\left(\sum_{k\geq 5}\int_{C_k}V(x)^{2\alpha p_2}\left(\int_B\left|(K_\beta(x,z)-K_\beta(x,x_0))a(z)\right|dz\right)^{p_2}dx\right)^{1/p_2}\\
&\leq CI'_{21}+I'_{22},
\end{aligned}$$

where

$$I'_{21} = \left(\sum_{k\geq 5} \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{Np_2}} \frac{r^{\delta p_2}}{(2^k r)^{(n+\delta-4\beta+2)p_2}} \int_{2^k B} V(x)^{2\alpha p_2} dx \right)^{1/p_2}$$

and

$$I'_{22} = \left(\sum_{k\geq 5} \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{Np_2}} \frac{r^{\delta p_2}}{(2^k r)^{(n+\delta-4\beta)p_2}} \int_{2^k B} V(x)^{2\alpha p_2} (\mathcal{I}_2(V^2\chi_{2^k B})(x))^{p_2} dx \right)^{1/p_2}.$$

Since $4(\beta-\alpha)-n-2+\frac{n}{p_2}=0$, by (1) we obtain

$$\begin{aligned} I'_{21} &\leq C\left(\sum_{k\geq 5} \frac{1}{2^{k\delta p_2}} \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{Np_2-2l_0\alpha p_2}} \right)^{1/p_2} \\ &\leq C\left(\sum_{k\geq 5} \frac{1}{2^{k\delta p_2}} \right)^{1/p_2} \leq C. \end{aligned}$$

By (3) and $n-(n-4\beta)p_2+2p_2(2\alpha+1)=0$, we get

$$\begin{aligned} I'_{22} &\leq C\left(\sum_{k\geq 5} \frac{1}{2^{k\delta p_2}} \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{Np_2-2l_0(\alpha+1)p_2}} \right)^{1/p_2} \\ &\leq C\left(\sum_{k\geq 5} \frac{1}{2^{k\delta p_2}} \right)^{1/p_2} \leq C. \end{aligned}$$

This completes the proof of Theorem 1.2. □

Proof of Theorem 1.4. If $\beta-\alpha=1/2$, it follows from Theorem 1.1 that

$$\|T^*_{\alpha,\beta} f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$$

for $p'_\alpha < p < \infty$.

Let $f \in BMO_{\mathcal{L}_1}(\mathbb{R}^n)$, $B=B(x_0,r)$, and let $B^*=B(x_0,2r)$. Write

$$f = f\chi_{B^*} + f\chi_{(B^*)^c} = f_1 + f_2.$$

Consider the case $r \geq \rho(x_0)$. Owing to the fact that $T^*_{\alpha,\beta}$ is bounded on $L^p(\mathbb{R}^n)$, we have

$$\begin{aligned} \frac{1}{|B|}\int_B |T^*_{\alpha,\beta}(f_1)(x)|dx &\leq C\left(\frac{1}{|B|}\int_B |T^*_{\alpha,\beta}(f_1)(x)|^p dx\right)^{1/p} \\ &\leq C\left(\frac{1}{|B^*|}\int_{B^*} |f(x)|^p dx\right)^{1/p}. \end{aligned}$$

By Lemma 2.6 we get

$$\frac{1}{|B|}\int_B |T^*_{\alpha,\beta}(f_1)(x)|dx \le C\|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)}.$$

For any $x \in B(x_0, r)$, by Lemma 2.3, we have $\rho(x) \sim \rho(x_0)$. Then

$$\begin{aligned}
&|T^*_{\alpha,\beta}(f_2)(x)| \\
&\le \int_{(B^*)^c} |K^*_\beta(x,z)| V(z)^{2\alpha} |f(z)| dz \\
&\le C\int_{(B^*)^c} \frac{V(z)^{2\alpha}}{\left(1+\frac{|x-z|}{\rho(x_0)}\right)^N} \frac{|f(z)|}{|x-z|^{n-4\beta+2}} dz \\
&\quad + \int_{(B^*)^c} \frac{V(z)^{2\alpha}}{\left(1+\frac{|x-z|}{\rho(x_0)}\right)^N} \frac{|f(z)|}{|x-z|^{n-4\beta}} \int_{B(z,|x-z|/4)} \frac{V(\xi)^2}{|\xi-z|^{n-2}} d\xi dz \\
&\le C\sum_{k=2}^{\infty} \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^N} \frac{1}{(2^k r)^{n-4\beta+2}} \int_{2^k B} V(z)^{2\alpha}|f(z)|dz \\
&\quad + C\sum_{k=2}^{\infty} \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^N} \frac{1}{(2^k r)^{n-4\beta}} \int_{2^k B} V(z)^{2\alpha}|f(z)|\mathcal{I}_2(V^2\chi_{2^{k+1}B})(z)dz \\
&= J_1 + J_2.
\end{aligned}$$

Note that $\frac{1}{p'_\alpha} + \frac{2\alpha}{s} + \frac{1}{t} = 1$, and $\frac{1}{t} = \frac{2}{s} - \frac{2}{n}$, by Hölder inequality and Hardy-Littlewood-Sobolev inequality we get

$$\begin{aligned}
&\frac{1}{(2^k r)^n}\int_{2^k B} V(z)^{2\alpha}|f(z)|\mathcal{I}_2(V^2\chi_{2^{k+1}B})(z)dz \\
&\le C\left(\frac{1}{(2^k r)^n}\int_{2^k B} V(z)^s dz\right)^{2\alpha/s} \left(\frac{1}{(2^k r)^n}\int_{2^k B} |f(z)|^{p'_\alpha} dz\right)^{1/p'_\alpha} \\
&\quad \times \left(\frac{1}{(2^k r)^n}\int_{2^k B} |\mathcal{I}_2(V^2\chi_{2^{k+1}B})(z)|^t dz\right)^{1/t} \\
&\le C\left(\frac{1}{(2^k r)^n}\int_{2^k B} V(z) dz\right)^{2\alpha} \|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)} \\
&\quad \times \left(\frac{1}{(2^{k+1} r)^n}\int_{2^{k+1} B} V(z)^s dz\right)^{2/s} (2^k r)^{\frac{2n}{s}-\frac{n}{t}} \\
&\le C(2^k r)^{-4(\alpha+1)}\left(1+\frac{2^k r}{\rho(x_0)}\right)^{2l_0(\alpha+1)} (2^k r)^{\frac{2n}{s}-\frac{n}{t}} \|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)}.
\end{aligned}$$

Because $4\beta - 4(\alpha+1) + \frac{2n}{s} - \frac{n}{t} = 0$, then, taking $N > l_0(\alpha+1)$ we get

$$J_2 \le C\sum_{k=2}^{\infty} \frac{1}{(2^k)^{N-2l_0(\alpha+1)}} \|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)}$$

$$\leq C\|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)}.$$

In the same way, by Hölder inequality and $\beta - \alpha = 1/2$, we get

$$\begin{aligned}
J_1 &= \sum_{k=2}^{\infty} \frac{(2^k r)^{-n+4\beta-2}}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^N} \int_{2^k B} V(z)^{2\alpha}|f(z)|dz \\
&\leq C \sum_{k=2}^{\infty} \frac{(2^k r)^{4\beta-2}}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^N} \left(\frac{1}{(2^k r)^n}\int_{2^k B} V(z)^s dz\right)^{2\alpha/s} \\
&\quad \times \left(\frac{1}{(2^k r)^n}\int_{2^k B} |f(z)|^{p'_\alpha} dz\right)^{1/p'_\alpha} \\
&\leq C \sum_{k=2}^{\infty} \frac{(2^k r)^{4\beta-2}}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^N} \left(\frac{1}{(2^k r)^n}\int_{2^k B} V(z) dz\right)^{2\alpha} \|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)} \\
&\leq C \sum_{k=2}^{\infty} \frac{(2^k r)^{4\beta-2-4\alpha}}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{N-2l_0\alpha}} \|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)} \\
&\leq C \sum_{k=2}^{\infty} \frac{1}{(2^k)^{N-2l_0\alpha}} \|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)} \leq C\|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)}.
\end{aligned}$$

Thus, for any $x \in B(x_0, r)$, we get

$$|T^*_{\alpha,\beta}(f_2)(x)| \leq C\|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)}.$$

Consequently,

$$\frac{1}{|B|}\int_B |T^*_{\alpha,\beta}(f_2)(x)|dx \leq C\|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)}.$$

Let us consider the case $r < \rho(x_0)$. We set $B^\sharp = B(x_0, 2\rho(x_0))$ and write

$$f = f\chi_{B^\sharp} + f\chi_{(B^\sharp)^c} = f_1^\sharp + f_2^\sharp.$$

Similar to the estimates for $|T^*_{\alpha,\beta} f_2(x)|$, we have

$$|T^*_{\alpha,\beta}(f_2^\sharp)(x)| \leq C\|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)}.$$

Then

$$\frac{1}{|B|}\int_B |T^*_{\alpha,\beta}(f_2^\sharp)(x) - (T^*_{\alpha,\beta}(f_2^\sharp))_B|dx \leq C\|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)}.$$

For any $x \in B(x_0, r)$, let $B_{x,k} = B(x, 2^{2-k}\rho(x_0))$. It is obvious that

$$|f(B_{x,k}, V)| = 0$$

for $k = 0, 1, 2$. Notice

$$|f(B_{x,3}, V) - f(B_{x,2}, V)| = |f(B_{x,3}, V)|$$

$$\leq C\frac{1}{|B(x,\rho(x_0))|}\int_{B(x,\rho(x_0))}|f(z)|dz$$
$$\leq C\|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)}.$$

So, for $k=3,4,\dots$, we have

$$|f(B_{x,k},V)-f(B_{x,k-1},V)|\leq C\|f\|_{BMO(\mathbb{R}^n)}\leq C\|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)}.$$

Then, for $k=3,4,\dots$, we get

$$|f(B_{x,k},V)|\leq Ck\|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)}.$$

Hence, for any $p>1$ and $k=0,1,2,\dots$, we have

$$\begin{aligned}&\left(\frac{1}{|B_{x,k}|}\int_{B_{x,k}}|f(z)|^p dz\right)^{1/p}\\&\leq C\left(\frac{1}{|B_{x,k}|}\int_{B_{x,k}}|f(z)-f(B_{x,k},V)|^p dz\right)^{1/p}+|f(B_{x,k},V)|\\&\leq C(k+1)\|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)}.\end{aligned}$$

For any $x\in B$,

$$\begin{aligned}&|T^*_{\alpha,\beta}(f_1^\sharp)(x)|\\&\leq C\int_{B^\sharp}|K^*_\beta(x,z)|V(z)^{2\alpha}|f(z)|dz\\&\leq C\sum_{k=0}^\infty\int_{B_{x,k}\setminus B_{x,k+1}}\frac{V(z)^{2\alpha}}{\left(1+\frac{|x-z|}{\rho(x_0)}\right)^N}\frac{|f(z)|}{|x-z|^{n-4\beta+2}}dz\\&\quad+\sum_{k=0}^\infty\int_{B_{x,k}\setminus B_{x,k+1}}\frac{V(z)^{2\alpha}}{\left(1+\frac{|x-z|}{\rho(x_0)}\right)^N}\frac{|f(z)|}{|x-z|^{n-4\beta}}\int_{B(z,|x-z|/4)}\frac{V(\xi)^2}{|\xi-z|^{n-2}}d\xi dz\\&\leq C\sum_{k=0}^\infty\frac{(2^{2-k}\rho(x_0))^{4\beta-2}}{|B_{x,k}|}\int_{B_{x,k}}V(z)^{2\alpha}|f(z)|dz\\&\quad+\sum_{k=0}^\infty\frac{(2^{2-k}\rho(x_0))^{4\beta}}{|B_{x,k}|}\int_{B_{x,k}}V(z)^{2\alpha}|f(z)|\mathcal{I}_2(V^2\chi_{B_{x,k}})(z)dz\\&=K_1+K_2.\end{aligned}$$

Note that $2\alpha<s$, by Hölder inequality and $\beta-\alpha=1/2$, we get

$$\begin{aligned}K_1&\leq C\sum_{k=0}^\infty(2^{2-k}\rho(x_0))^{4\alpha}\frac{1}{|B_{x,k}|}\int_{B_{x,k}}V(y)^{2\alpha}|f(y)|dy\\&\leq C\sum_{k=0}^\infty(2^{2-k}\rho(x_0))^{4\alpha}\left(\frac{1}{|B_{x,k}|}\int_{B_{x,k}}V(y)^s dy\right)^{2\alpha/s}\end{aligned}$$

$$\times \left(\frac{1}{|B_{x,k}|} \int_{B_{x,k}} |f(y)|^{(\frac{s}{2\alpha})'} dy \right)^{1/(\frac{s}{2\alpha})'}$$

$$\leq C \, \|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)} \sum_{k=0}^{\infty} (k+1) \left(\frac{1}{(2^{2-k}\rho(x_0))^{n-2}} \int_{B_{x,k}} V(y) dy \right)^{2\alpha}.$$

It follows from Lemma 2.2 that

$$\frac{1}{(2^{2-k}\rho(x_0))^{n-2}} \int_{B_{x,k}} V(y) dy \leq C 2^{-k\delta}$$

for $k = 2, 3, \ldots$, where $\delta = 2 - n/s > 0$. Then $K_1 \leq C\|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)}$.

Pay attention to $\frac{1}{p'_\alpha} + \frac{2\alpha}{s} + \frac{1}{t} = 1, \frac{1}{t} = \frac{2}{s} - \frac{2}{n}$. By Hölder inequality and Hardy-Littlewood-Sobolev inequality we get

$$\frac{1}{|B_{x,k}|} \int_{B_{x,k}} V(z)^{2\alpha} |f(z)| \mathcal{I}_2(V^2 \chi_{B_{x,k}})(z) dz$$

$$\leq C \, \left(\frac{1}{|B_{x,k}|} \int_{B_{x,k}} V(z)^s dz \right)^{2\alpha/s} \left(\frac{1}{|B_{x,k}|} \int_{B_{x,k}} |f(z)|^{p'_\alpha} dz \right)^{1/p'_\alpha}$$

$$\times \left(\frac{1}{|B_{x,k}|} \int_{B_{x,k}} |\mathcal{I}_2(V^2 \chi_{B_{x,k}})(z)|^t dz \right)^{1/t}$$

$$\leq C \, (k+1) \|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)} \left(\frac{1}{|B_{x,k}|} \int_{B_{x,k}} V(z) dz \right)^{2\alpha}$$

$$\times \left(\frac{1}{|B_{x,k}|} \int_{B_{x,k}} V(z)^s dz \right)^{2/s} |B_{x,k}|^{\frac{2}{s} - \frac{1}{t}}$$

$$\leq C \, (k+1) \|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)} (2^{2-k}\rho(x_0))^{-4\alpha-2}$$

$$\times \left(\frac{1}{(2^{2-k}\rho(x_0))^{n-2}} \int_{B_{x,k}} V(z) dz \right)^{2\alpha+2}.$$

Then

$$K_2 \leq C\|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)} \sum_{k=0}^{\infty} (k+1) 2^{-k\delta(2\alpha+2)} \leq C\|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)}.$$

Combining the estimates for K_1 and K_2, we have proved the inequality

$$|T^*_{\alpha,\beta}(f_1^\sharp)(x)| \leq C\|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)}$$

holds for any $x \in B(x_0, r), r < \rho(x_0)$. Thence

$$\frac{1}{|B|} \int_B |T^*_{\alpha,\beta}(f_1^\sharp)(x) - (T^*_{\alpha,\beta}(f_1^\sharp))_B| dx \leq C\|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 1.4. □

References

[1] J. Cao, Y. Liu, and D. Yang, *Hardy spaces $H^1_{\mathrm{L}}(\mathbb{R}^n)$ associated to Schrödinger type operators $(-\Delta)^2 + V^2$*, Houston J. Math. **36** (2010), no. 4, 1067–1095.

[2] J. Dziubański, G. Garrigós, T. Martínez, J. L. Torrea, and J. Zienkiewicz, *BMO spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality*, Math. Z. **249** (2005), no. 2, 329–356. https://doi.org/10.1007/s00209-004-0701-9

[3] J. Dziubański and J. Zienkiewicz, *Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality*, Rev. Mat. Iberoamericana **15** (1999), no. 2, 279–296. https://doi.org/10.4171/RMI/257

[4] ———, *H^p spaces associated with Schrödinger operators with potentials from reverse Hölder classes*, Colloq. Math. **98** (2003), no. 1, 5–38. https://doi.org/10.4064/cm98-1-2

[5] Y. Liu and J. Dong, *Some estimates of higher order Riesz transform related to Schrödinger type operators*, Potential Anal. **32** (2010), no. 1, 41–55. https://doi.org/10.1007/s11118-009-9143-7

[6] Y. Liu, J. Zhang, J. Sheng, and L. Wang, *Some estimates for commutators of Riesz transform associated with Schrödinger type operators*, Czechoslovak Math. J. **66(141)** (2016), no. 1, 169–191. https://doi.org/10.1007/s10587-016-0248-z

[7] Z. W. Shen, *L^p estimates for Schrödinger operators with certain potentials*, Ann. Inst. Fourier (Grenoble) **45** (1995), no. 2, 513–546.

[8] S. Sugano, *L^p estimates for some Schrödinger type operators and a Calderón-Zygmund operator of Schrödinger type*, Tokyo J. Math. **30** (2007), no. 1, 179–197. https://doi.org/10.3836/tjm/1184963655

[9] Y. Wang, *Estimates for Riesz transforms associated with Schrödinger type operators*, Bull. Korean Math. Soc. **56** (2019), no. 5, 1117–1127. https://doi.org/10.4134/BKMS.b180836

[10] J. Zhong, *Harmonic analysis for some Schroedinger type operators*, ProQuest LLC, Ann Arbor, MI, 1993.

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