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# SUBPERMUTABLE SUBGROUPS OF SKEW LINEAR GROUPS AND UNIT GROUPS OF REAL GROUP ALGEBRAS

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Abstract. Let $D$ be a division ring and $n > 1$ be an integer. In this paper, it is shown that if $D \neq \mathbb{F}_3$, then every subpermutable subgroup of the general skew linear group $\mathrm{GL}_n(D)$ is normal. By applying this result, we show that every subpermutable subgroup of the unit group $(\mathbb{R}G)^*$ of the real group algebras $\mathbb{R}G$ of finite groups $G$ is normal in $(\mathbb{R}G)^*$.

## 1. Introduction and main results

Let $G$ be a group. A subgroup $Q$ of $G$ is called *permutable* in $G$, write $Q <_p G$, if $PQ = QP$ for every subgroup $P$ of $G$. A permutable subgroup is also called *quasinormal* in some papers or books. In this paper, following [23], we prefer to use “a permutable subgroup” rather than “a quasinormal subgroup”. Also in [23], $Q$ is called *subpermutable* in $G$ if there exists a permutable series of subgroups of $G$

$$Q = Q_n <_p Q_{n-1} <_p \cdots <_p Q_1 <_p Q_0 = G.$$

Clearly, notions of permutable and subpermutable subgroups extend naturally of normal and subnormal ones respectively. There are some examples of groups having permutable subgroups that are even not subnormal (see [23]). There are many papers on permutable subgroups of certain groups (e.g., see [1–5, 9, 11, 14, 15, 21, 23, 24]). For a survey on permutable subgroups, we refer to [9, 16, 25]. It seems to be interesting to find classes of groups whose (sub)permutable subgroups are (sub)normal. It is well known that in the case when $G$ is finite, then every permutable subgroup of $G$ is subnormal [19]. An another example, [22, Corollary C2] showed that a simple group contains no proper non-trivial permutable subgroups. The nicest result on this topic is maybe from [23]. In fact, it is shown in [23] that if $G$ is finitely generated, then every subpermutable

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subgroup of $G$ is subnormal. Our groups in this paper are unit groups of algebras and these algebras are matrix rings over division rings and real group algebras of finite groups.

There are many reasons why we focus on the class of matrix rings over division rings and real group algebras. First of all, the unit groups of these algebras are "big" enough to work on. For example, if $\mathbb{R}H$ is the real group algebra of a finite group $H$, then the unit group $(\mathbb{R}H)^*$ contains strictly the set of trivial units $\{ah \mid a \in \mathbb{R}^* \text{ and } h \in H\}$. The second reason is from the fact that the classification of skew linear groups is a classic and interesting topic. Recently, the first author showed some results on permutable subgroups of $\mathrm{GL}_n(D)$ and this paper can be considered as a continuation work of [6]. An another reason is from techniques. In details, if $\mathbb{R}H$ is the group algebra of a finite group $H$ over the field of real numbers $\mathbb{R}$, then by Wedderburn-Artin's Theorem, $\mathbb{R}H$ is isomorphic to a direct product of finitely many matrix rings over division rings. We may use results on subpermutable subgroups of general skew linear groups to apply for the group algebras.

The following theorem is our first main result:

**Theorem 1.1.** *Let $D$ be a division ring, $n \geq 2$ an integer and $Q$ a non-central subgroup of $\mathrm{GL}_n(D)$. The following statements are equivalent unless $\mathrm{GL}_n(D) = \mathrm{GL}_2(\mathbb{F}_3)$:*

(1) *$Q$ contains $\mathrm{SL}_n(D)$;*
(2) *$Q$ is normal in $\mathrm{GL}_n(D)$;*
(3) *$Q$ is subnormal in $\mathrm{GL}_n(D)$;*
(4) *$Q$ is permutable in $\mathrm{GL}_n(D)$;*
(5) *$Q$ is subpermutable in $\mathrm{GL}_n(D)$.*

Our assumption requires $n > 1$ since it was shown that there exists a division ring $D$ such that $D^*$ contains a non-subnormal subpermutable subgroup (see [6]). We also except the case $\mathrm{GL}_2(\mathbb{F}_3)$ where $\mathbb{F}_q$ is denoted by the field of $q$ elements. For convenience, we also present results on $\mathrm{GL}_2(\mathbb{F}_3)$ whose proof is simple here.

Denote by $Q_8$ the quaternion group. It is shown that $Q_8$ is isomorphic to the subgroup $\langle i, j\rangle$ of $\mathbb{H}^*$ generated by $\{i, j\}$, where $\mathbb{H}$ is the division ring of real quaternions.

**Proposition 1.2.** $\mathrm{GL}_2(\mathbb{F}_3)$ *contains a normal subgroup $Q$ which is isomorphic to the quaternion group $Q_8$ such that*:

(1) *$Q \lhd \mathrm{GL}_2(\mathbb{F}_3)$;*
(2) *$\langle i\rangle \lhd Q_8$ and;*
(3) *The embedding of $\langle i\rangle$ into $\mathrm{GL}_2(\mathbb{F}_3)$ is not normal.*

The following corollary is implied directly from Theorem 1.1 and Proposition 1.2.

**Corollary 1.3.** *Every subpermutable subgroup of $\mathrm{GL}_n(D)$ is subnormal in $\mathrm{GL}_n(D)$.*

The second aim of this paper is to show that if $G$ is a finite group, then every subpermutable subgroups of $(\mathbb{R}G)^*$ is normal. To do this, we need to work on division rings over $\mathbb{R}$. It is well known that there are equivalent finite-dimensional division rings over $\mathbb{R}$ which are $\mathbb{R}$, the field of complex numbers $\mathbb{C}$ and the real quaternion division ring $\mathbb{H}$. There is nothing difficult for $\mathbb{R}$ and $\mathbb{C}$. With the real quaternion division ring $\mathbb{H}$, we have the following statement which may be considered as a generalization of [8, Example].

**Theorem 1.4.** *Let $\mathbb{H}$ be the real quaternion division ring. Put*

$$\mathbb{H}_1 = \{a + bi + cj + dk \in \mathbb{H} \mid a^2 + b^2 + c^2 + d^2 = 1\}.$$

*For a non-central subgroup $Q$ of $\mathbb{H}^*$, the following assertions are equivalent*:

(1) *$Q$ contains $\mathbb{H}_1$;*
(2) *$Q$ is normal in $\mathbb{H}^*$;*
(3) *$Q$ is subnormal in $\mathbb{H}^*$;*
(4) *$Q$ is permutable in $\mathbb{H}^*$;*
(5) *$Q$ is subpermutable in $\mathbb{H}^*$.*

By applying results above, we have the next main result on the real group algebra of a finite group.

**Theorem 1.5.** *Let $G$ be a finite group and $\mathbb{R}$ the field of real numbers. Then every subpermutable subgroup of the unit group $(\mathbb{R}G)^*$ is normal in $(\mathbb{R}G)^*$.*

## 2. Proofs

We begin this section with the following result.

**Lemma 2.1.** *Let $D$ be a division ring, $n \geq 2$ an integer and $N$ a non-central subgroup of $\mathrm{GL}_n(D)$. The following statements are equivalent unless $\mathrm{GL}_n(D) = \mathrm{GL}_2(\mathbb{F}_2)$ or $\mathrm{GL}_n(D) = \mathrm{GL}_2(\mathbb{F}_3)$:*

(1) *$N$ contains $\mathrm{SL}_n(D)$.*
(2) *$N$ is normal in $\mathrm{GL}_n(D)$.*
(3) *$N$ is subnormal in $\mathrm{GL}_n(D)$.*

*Proof.* See [17, Theorem 4]. □

Next, we recall useful properties of the permutability.

**Lemma 2.2.** *Let $G$ be a group. Then*

(1) *For any subgroup $K$ of $G$, if $H$ is a permutable subgroup of $G$, then so is $H \cap K$ of $K$.*
(2) *If $H$ and $K$ are both permutable in $G$, then so is $HK$ in $G$.*
(3) *For a normal subgroup $N$ of $G$, if $H$ is a permutable subgroup of $G$ containing $N$, then so is $H/N$ of $G/N$.*
(4) *For a homomorphism $\varphi$ from $G$, if $H$ is permutable in $G$, then so is $\varphi(H)$ in $\varphi(G)$.*

*Proof.* The proofs are elementary. □

**Lemma 2.3.** *Let $G$ be any group and $H$ its permutable subgroup. Then, either*

(1) *$G$ is radical over $H$, that is, for every elements $x \in G$, there exists a positive integer $n_x$ such that $x^{n_x} \in H$, or*
(2) *$H$ is subnormal in $G$.*

*Proof.* See [6, Lemma 2.4]. □

Let $D$ be a division ring, $n$ an integer greater than 1 and $\mathrm{SL}_n(D)$ the special linear group of degree $n$ over $D$. The following result describes all permutable subgroups of $\mathrm{SL}_n(D)$ in case when $\mathrm{PSL}_n(D)$ is simple. Recall that if $C$ is the center of $\mathrm{SL}_n(D)$, then $\mathrm{PSL}_n(D)$ is the quotient group $\mathrm{SL}_n(D)/C$.

**Lemma 2.4.** *Let $D$ be a division ring and an integer $n \geq 2$. Suppose that either $n \geq 3$ or $n = 2$ but $D$ contains at least four elements. Then every proper permutable subgroup of $\mathrm{SL}_n(D)$ is central.*

*Proof.* Let $Q$ be a non-central proper permutable subgroup of $\mathrm{SL}_n(D)$ and $C$ the center of $\mathrm{SL}_n(D)$. Put $H = QC$. It is obvious that $H$ is permutable in $\mathrm{SL}_n(D)$ and hence so is $H/C$ in $\mathrm{PSL}_n(D) = \mathrm{SL}_n(D)/C$ (Lemma 2.2). Therefore, $H/C$ is either trivial or equal to $\mathrm{PSL}_n(D)$ by [23, Corollary C2]. If $H/C = \mathrm{PSL}_n(D)$, then $H = \mathrm{SL}_n(D)$. Notice that $Q$ is normal in $H$: for each $a \in Q$ and $x = bc \in H$ where $b \in Q$ and $c \in C$,

$$x^{-1}ax = c^{-1}b^{-1}abc = b^{-1}ab \in Q.$$

Hence $Q \trianglelefteq H \trianglelefteq \mathrm{GL}_n(D)$. As Lemma 2.1, $\mathrm{SL}_n(D) \leq Q$ and hence $Q = \mathrm{SL}_n(D)$, a contradiction. Thus, $H/C$ must be trivial, which implies that $Q$ is central. □

Now we are ready to show the first aim of paper.

*Proof of Theorem 1.1.* The implications (1) ⇒ (2) ⇒ (4) ⇒ (5) and (2) ⇒ (3) are trivial. We will show the implications (3) ⇒ (2) and (5) ⇒ (1).

Assume that $Q$ is a non-central subpermutable subgroup of $\mathrm{GL}_n(D)$. We begin with the first case when $n = 2$ and $D = \mathbb{F}_2$. One has $\mathrm{GL}_2(\mathbb{F}_2)$ is isomorphic to $S_3$, the symmetric group of degree 3. Moreover, $S_3$ has 4 proper non-central subgroups which are $A_3$, $\langle(1\ 2)\rangle$, $\langle(1\ 3)\rangle$ and $\langle(2\ 3)\rangle$. By checking with these 4 subgroups, we conclude that only $A_3$ is subpermutable in $S_3$. Since $A_3 \cong \mathrm{SL}_2(\mathbb{F}_2)$, the proof is complete in this case.

Now we consider the case when $\mathrm{GL}_n(D)$ is different from $\mathrm{GL}_2(\mathbb{F}_2)$ and $\mathrm{GL}_2(\mathbb{F}_3)$. By Lemma 2.1, (3) ⇒ (2). We next prove the implication (5) ⇒ (1). According to [7, §21, Theorem 2], $\mathrm{PSL}_n(D)$ is simple. Let

$$Q = Q_r <_p Q_{r-1} <_p \cdots <_p Q_0 = \mathrm{GL}_n(D)$$

be a permutable series of subgroups of $Q$ in $\mathrm{GL}_n(D)$, that is, $Q_i$ is permutable in $Q_{i-1}$ for each $i \in \{1, \dots, r\}$. Notice that $Q_i$ is non-central for every $0 \leq i \leq r$ since $Q$ is its subgroup and non-central. We shall prove that $Q_i$ contains $\mathrm{SL}_n(D)$ by induction on $0 \leq i \leq r$. It is clear that $\mathrm{SL}_n(D) \subseteq Q_0 = \mathrm{GL}_n(D)$. Suppose that $\mathrm{SL}_n(D) \subseteq Q_i$ for $i \geq 0$. Then, $Q_i$ is normal in $\mathrm{GL}_n(D)$. Using

Lemma 2.2, we obtain that $Q_{i+1} \cap \mathrm{SL}_n(D)$ is permutable in $\mathrm{SL}_n(D)$ since $Q_{i+1}$ is permutable in $Q_i$. We claim that $Q_{i+1} \cap \mathrm{SL}_n(D)$ is non-central. By Lemma 2.3, either $Q_{i+1}$ is subnormal in $Q_i$ or for every element $x \in Q_i$, there exists $n_x \in \mathbb{N}$ such that $x^{n_x} \in Q_{i+1}$. If the first one occurs, then $Q_{i+1}$ is subnormal in $\mathrm{GL}_n(D)$, which implies, by Lemma 2.1, that $\mathrm{SL}_n(D) \subseteq Q_{i+1}$. Therefore, $Q_{i+1} \cap \mathrm{SL}_n(D)$ is non-central. Now we assume that for every $x \in Q_i$, there exists a positive integer $n_x$ such that $x^{n_x} \in Q_{i+1}$. Let $P$ be the prime subfield of $D$.

*Case 1. The case when char*$(P) = 0$. Denote by $e_{ij}$ the matrix in which the $(i,j)$-entry is 1 and all the others are 0. Put $x = I_n + e_{1n}$. As a corollary, $x^{n_x} = I_n + n_x e_{1n} \in Q_{i+1}$. It implies that $x^{n_x} \in Q_{i+1} \cap \mathrm{SL}_n(D)$. Hence, in this subcase, $Q_{i+1} \cap \mathrm{SL}_n(D)$ is non-central.

*Case 2. The case when char*$(P) = p > 0$. For every $\alpha \in D, 1 \leq i,j \leq n$, let $m$ be the smallest positive integer such that $a_{ij}(\alpha)^m \in Q_{i+1}$ with $a_{ij}(\alpha) = I_n + \alpha e_{ij}$. If the greatest common divisor of $m$ and $p$ is 1, then there exist $k, \ell \in \mathbb{Z}$ such that $mk + p\ell = 1$. Therefore,

$$a_{ij}(\alpha) = a_{ij}((mk + p\ell)\alpha) = a_{ij}(mk\alpha) = a_{ij}(m\alpha)^k \in Q_{i+1},$$

which implies that $Q_{i+1} \cap \mathrm{SL}_n(D)$ is non-central. If the greatest common divisor of $m$ and $p$ is $p$, then $a_{ij}(\alpha)^p = a_{ij}(p\alpha) = I_n$. As minimality of $m$, $m = p$. Hence, $\langle a_{ij}(\alpha)\rangle$ is of order $p$. If $a_{ij}(\alpha) \in Q_{i+1}$, then obviously $a_{ij}(\alpha) Q_{i+1} a_{ij}(\alpha)^{-1} \subseteq Q_{i+1}$. If $a_{ij}(\alpha) \notin Q_{i+1}$, then $Q_{i+1} \cap \langle a_{ij}(\alpha)\rangle = 1$. It implies that $[Q_{i+1}\langle a_{ij}(\alpha)\rangle : Q_{i+1}] = p$. By [9, Theorem 1], $Q_{i+1} \trianglelefteq Q_{i+1}\langle a_{ij}(\alpha)\rangle$. Again, we have $a_{ij}(\alpha) Q_{i+1} a_{ij}(\alpha)^{-1} \subseteq Q_{i+1}$. Since $\mathrm{SL}_n(D) = \langle a_{ij}(\alpha) = I_n + \alpha e_{ij} \mid 1 \leq i,j \leq n, \alpha \in D\rangle$, $a(\alpha) Q_{i+1} (a(\alpha))^{-1} \subseteq Q_{i+1}$ for every $a \in \mathrm{SL}_n(D)$. According to [18, Theorem 3.1], $\mathrm{SL}_n(D) \subseteq Q_{i+1}$.

Two cases lead us the fact that $Q_{i+1} \cap \mathrm{SL}_n(D)$ is non-central. The claim is shown. These facts combining with Lemma 2.4 imply that $Q_{i+1} \cap \mathrm{SL}_n(D)$ coincides with $\mathrm{SL}_n(D)$, or equivalently, $\mathrm{SL}_n(D) \subseteq Q_{i+1}$. By inductive hypothesis, we conclude that $\mathrm{SL}_n(D) \subseteq Q_i$ for all $i \geq 0$. Thus, $Q$ must contain $\mathrm{SL}_n(D)$. ☐

*Proof of Proposition 1.2.* A representation of $Q_8$ in $\mathrm{GL}_2(\mathbb{F}_3)$ given by

$$1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad i \mapsto \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad j \mapsto \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad k \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

determines a subgroup $Q$ of $\mathrm{GL}_2(\mathbb{F}_3)$ which is isomorphic to $Q_8$. It is easy to verify that $Q$ is normal in $\mathrm{GL}_2(\mathbb{F}_3)$. Furthermore, the group generated by $i$ with four elements $\{1, -1, i, -i\}$ is of index 2 and hence normal in $Q_8$. However, the embedding of $\langle i\rangle$ into $\mathrm{GL}_2(\mathbb{F}_3)$, namely

$$H = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \right\},$$

is not normal in $\mathrm{GL}_2(\mathbb{F}_3)$ since

$$\begin{bmatrix}-1 & -1\\ 1 & 0\end{bmatrix}^{-1}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}-1 & -1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}1 & -1\\ -1 & -1\end{bmatrix}\notin H.$$ □

*Remark* 2.5. The group $H$ is subnormal in $\mathrm{GL}_2(\mathbb{F}_3)$, that is $H \triangleleft Q \triangleleft \mathrm{GL}_2(\mathbb{F}_3)$, and hence subpermutable in $\mathrm{GL}_2(\mathbb{F}_3)$. But $H$ is not normal in $\mathrm{GL}_2(\mathbb{F}_3)$.

*Proof of Corollary 1.3.* This corollary is implied directly from Theorem 1.1 and Proposition 1.2. □

Before showing the next theorem, we recall some simple facts on the real quaternion division ring $\mathbb{H}$. Let $\alpha = a+bi+cj+dk \in \mathbb{H}$. Put $\overline{\alpha} = a-bi-cj-dk$. Then, $\alpha\overline{\alpha} = \overline{\alpha}\alpha = a^2+b^2+c^2+d^2$. Put $\mathrm{N}(\alpha) = \sqrt{\alpha\overline{\alpha}} = \sqrt{a^2+b^2+c^2+d^2}$. $\mathrm{N}(\alpha)$ is called the *norm* of $\alpha$. If $\alpha \neq 0$, equivalently, $\mathrm{N}(\alpha)^2 \neq 0$, then $\alpha^{-1} = \frac{1}{\mathrm{N}(\alpha)^2}\overline{\alpha}$. Now let $\mathbb{H}_1$ be the set of all elements of $\mathbb{H}$ whose norms are 1, that is,

$$\mathbb{H}_1 := \{a+bi+cj+dk \in \mathbb{H} \mid a^2+b^2+c^2+d^2 = 1\}.$$

It is easy to check that for every $\alpha, \beta \in \mathbb{H}$, $\mathrm{N}(\alpha\beta) = \mathrm{N}(\beta\alpha)$, $\mathrm{N}(\alpha\beta) = \mathrm{N}(\alpha)\mathrm{N}(\beta)$ and $\mathrm{N}(\alpha^{-1}) = \mathrm{N}(\alpha)^{-1}$ with $\alpha \neq 0$, which implies that $\mathbb{H}_1$ is a non-central normal subgroup of $\mathbb{H}^*$.

*Proof of Theorem 1.4.* The technique we use in this theorem is a slight modification of one of Theorem 1.1. The implications (2) ⇒ (4) ⇒ (5) and (2) ⇒ (3) are trivial.

*The implication* (1) ⇒ (2): Assume that $Q$ contains $\mathbb{H}_1$. For every $\alpha, \beta \in \mathbb{H}^*$, one has

$$\mathrm{N}(\alpha\beta\alpha^{-1}\beta^{-1}) = \mathrm{N}(\alpha)\mathrm{N}(\beta)\mathrm{N}(\alpha)^{-1}\mathrm{N}(\beta)^{-1} = 1.$$

Hence, $\alpha\beta\alpha^{-1}\beta^{-1} \in \mathbb{H}_1$, which implies $\mathbb{H}_1$ contains the commutator subgroup $[\mathbb{H}^*, \mathbb{H}^*]$, so does $Q$. Therefore, $Q$ is normal in $\mathbb{H}^*$.

*The implication* (2) ⇒ (1): This implication is from [8, Claim, Page 163].

*The implication* (3) ⇒ (2): This implication is from [8, Example, Page 162].

*The implication* (5) ⇒ (2): Let $Q$ be a non-central subpermutable subgroup of $\mathbb{H}^*$. Let

$$Q = Q_r <_p Q_{r-1} <_p \cdots <_p Q_0 = \mathbb{H}^*.$$

be a permutable series of subgroups of $Q$ in $\mathbb{H}^*$. Firstly, notice that all of $Q_i$ is non-central since they contains $Q$ as the non-central subgroup. We show that $Q_i$ is normal in $\mathbb{H}^*$ by induction on $i$ with $0 \leq i \leq r$. Indeed, it is trivial that $Q_0 \trianglelefteq \mathbb{H}^*$. Suppose that $Q_i \trianglelefteq \mathbb{H}^*$ for some integer $i \geq 0$. It remains to prove that $Q_{i+1}$ is also a normal subgroup of $\mathbb{H}^*$. According to Lemma 2.3, either $Q_{i+1}$ is subnormal subgroup of $Q_i$ or $Q_i$ is radical over $Q_{i+1}$. If $Q_{i+1}$ is subnormal in $Q_i$, then $Q_{i+1}$ is subnormal in $\mathbb{H}^*$, which implies that $Q_{i+1} \trianglelefteq \mathbb{H}^*$ by [8, Example]. Now we assume that $Q_i$ is radical over $Q_{i+1}$, that is, for every $x \in Q_i$, there exists a positive integer $n_x$ such that $x^{n_x} \in Q_{i+1}$. Put $N = Q_{i+1} \cap \mathbb{H}_1$. We claim that $N = \mathbb{H}_1$. Indeed, set $M = NZ(\mathbb{H}_1)$. By Lemma 2.2, $M/Z(\mathbb{H}_1)$ is permutable in $\mathbb{H}_1/Z(\mathbb{H}_1)$. According to [13, (5.8)], we have $\mathbb{H}_1/Z(\mathbb{H}_1) \simeq \mathrm{SO}(3)$ where SO(3) is the special orthogonal group. It implies that $\mathbb{H}_1/Z(\mathbb{H}_1)$ is simple since SO(3) is the simple group (see [22]). By [23, Corollary C2], either $M/Z(\mathbb{H}_1) = \overline{1}$ or $M/Z(\mathbb{H}_1) = \mathbb{H}_1/Z(\mathbb{H}_1)$. If the first case holds, then $M = Z(\mathbb{H}_1)$, which implies that $N$ is contained in $Z(\mathbb{H}_1)$. Observe that $Z(\mathbb{H}_1) = \{1, -1\}$, so $N \subseteq \{-1, 1\}$. On the other hand, by the inductive hypothesis that $\mathbb{H}_1 \subseteq Q_i$ and $Q_i$ is radical over $Q_{i+1}$, for every $x \in \mathbb{H}_1$, there exists a positive integer $n_x \in$ such that $x^{n_x} \in Q_{i+1}$. It implies that $x^{n_x} \in N$. Hence, $x^{2n_x} = 1$. This fact holds wherever $x$ ranges over $\mathbb{H}_1$, so $\mathbb{H}_1$ is torsion. By [10, Theorem 8], $\mathbb{H}_1$ is central, a contradiction. Thus, the second case holds, that is, $M/Z(\mathbb{H}_1) = \mathbb{H}_1/Z(\mathbb{H}_1)$, equivalently, $M = \mathbb{H}_1$. Furthermore, since $Z(\mathbb{H}_1) = \{-1, 1\}$, $[\mathbb{H}_1 : N] = [M : N] \leq 2$. Therefore, $N \trianglelefteq \mathbb{H}_1$. Consequently, $N$ is subnormal in $\mathbb{H}^*$. As [8, Example], $N$ is normal in $\mathbb{H}^*$ and again by [8, Claim, Page 163], $N = \mathbb{H}_1$. As a corollary, $\mathbb{H}_1 \subseteq Q_{i+1}$. By the hypothesis again, $Q_{i+1}$ is normal in $\mathbb{H}_1$. □

**Corollary 2.6.** *If $D$ is a field or the real quaternion division ring, then every subpermutable subgroup of $\mathrm{GL}_n(D)$ is normal for all positive integer $n$ unless $\mathrm{GL}_n(D) = \mathrm{GL}_2(\mathbb{F}_3)$.*

*Proof.* Let $N$ be a subpermutable subgroup of $\mathrm{GL}_n(D)$. If $N$ is central, then obviously $N$ is normal in $\mathrm{GL}_n(D)$. Now assume that $N$ is non-central. If $n > 1$, then the corollary is from Theorem 1.1; otherwise from Theorem 1.4. □

Recall that for a finite group $G$, one may define the real group algebra $\mathbb{R}G$ as follows: each element of $\mathbb{R}G$ has the form $\sum_{g\in G} a_g g$ where $a_g \in \mathbb{R}$ with the addition and multiplication defined as:

$$\left(\sum_{g\in G} a_g g\right) + \left(\sum_{g\in G} b_g g\right) = \sum_{g\in G}(a_g + b_g)g$$

and

$$\left(\sum_{g\in G} a_g g\right)\left(\sum_{g\in G} b_g g\right) = \sum_{g\in G}\left(\sum_{\substack{u,v\in G;\\ uv=g}} a_u b_v\right) g$$

respectively. Since $G$ is a basis of $\mathbb{R}G$ over $\mathbb{R}$, $\mathbb{R}G$ is a finite-dimensional vector space over $\mathbb{R}$.

*Proof of Theorem 1.5.* Assume that $G$ is a finite group and $Q$ is a subpermutable subgroup of the unit group $(\mathbb{R}G)^*$ of the group algebra of $G$ over $\mathbb{R}$. There is nothing to do if $Q$ is central, so we assume that $Q$ is non-central. According to [20, Chapter 7, §4, Lemma 4.2] or also known as Maschke's Theorem, $\mathbb{R}G$ is an artinian semisimple ring since $G$ is finite and $\mathrm{char}(\mathbb{R}) = 0$.

Write $\mathbb{R}G = \bigoplus_{k=1}^{r} n_k V_k$ where $V_1, \ldots, V_r$ are representatives of the isomorphic classes of simple $\mathbb{R}G$-modules. Then,

$$\mathbb{R}G \cong \mathrm{End}_{\mathbb{R}G}\left({}_{\mathbb{R}G}\mathbb{R}G\right) \cong \prod_{k=1}^{r} \mathrm{End}_{\mathbb{R}G}\left(n_k V_k\right) \cong \prod_{k=1}^{r} \mathrm{M}_{n_k}\left(\mathrm{End}_{\mathbb{R}G}(V_k)\right).$$

Indeed, the number $r$ is uniquely determined, as are the pairs

$$(n_k, \mathrm{End}_{\mathbb{R}G}(V_k))_{k=1}^{r}.$$

The above fact is known as the famous Wedderburn-Artin Theorem. For convenience, set $D_k = \mathrm{End}_{\mathbb{R}G}(V_k)$ for each $k \in \{1, \ldots, r\}$. By our setting, it is easy to verify that $D_k$ is a division ring containing $\mathbb{R}$ as a subfield. The isomorphism above can be written, in short, as following:

$$\mathbb{R}G \cong \mathrm{M}_{n_1}(D_1) \times \cdots \times \mathrm{M}_{n_r}(D_r).$$

Consequently,

$$(\mathbb{R}G)^* \cong \mathrm{GL}_{n_1}(D_1) \times \cdots \times \mathrm{GL}_{n_r}(D_r).$$

Moreover, since $\mathbb{R}G$ is finite dimensional over $\mathbb{R}$, so is every $D_k$. It is well known by Frobenius's Theorem (or see [12, Theorem 13.12]), either $D_k \cong \mathbb{R}$, $D_k \cong \mathbb{C}$ or $D_k \cong \mathbb{H}$. Now for every $1 \le k \le r$, put

$$\begin{array}{rccc} \pi_k : & (\mathbb{R}G)^* & \to & \mathrm{GL}_{n_k}(D_k), \\ & (A_1, \ldots, A_k, \ldots, A_r) & \mapsto & A_k, \end{array}$$

the $k$-th projection of $(\mathbb{R}G)^*$ onto $\mathrm{GL}_n(D_k)$. On other hand, since $Q$ is subpermutable in $(\mathbb{R}G)^*$, we obtain a permutable series

$$Q = Q_s <_p \cdots <_p Q_1 <_p Q_0 = (\mathbb{R}G)^*.$$

Then, one may simply check that for each $0 \le \ell \le s$, $Q_\ell \cong Q_{\ell,1} \times \cdots \times Q_{\ell,r}$ where $Q_{\ell,k} = \pi_k(Q_\ell)$. By Lemma 2.2, it is implied that for each $0 \le k \le r$, $Q_{\ell+1,k}$ is permutable in $Q_{\ell,k}$ for every $0 \le \ell \le s-1$. Hence, $Q_{s,k}$ is subpermutable in $Q_{0,k} = \mathrm{GL}_{n_k}(D_k)$. According to Corollary 2.6, $Q_{s,k}$ is normal in $\mathrm{GL}_{n_k}(D_k)$ for every $1 \le k \le r$. Thus, $\pi_1(Q_s) \times \cdots \times \pi_r(Q_s)$ is normal in $\mathrm{GL}_{n_1}(D_1) \times \cdots \times \mathrm{GL}_{n_r}(D_r)$. As a corollary, $Q$ is normal in $(\mathbb{R}G)^*$. □

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## References

[1] A. Ballester-Bolinches, L. A. Kurdachenko, J. Otal, and T. Pedraza, *Local characterizations of infinite groups whose ascendant subgroups are permutable*, Forum Math. **22** (2010), no. 1, 187–201. https://doi.org/10.1515/FORUM.2010.010

[2] G. Busetto and F. Napolitani, *On some questions concerning permutable subgroups of finitely generated groups and projectivities of perfect groups*, Rend. Sem. Mat. Univ. Padova **87** (1992), 303–305.

[3] M. R. Celentani and A. Leone, *On quasinormal subgroups of infinite groups*, Ricerche Mat. **47** (1998), no. 1, 205–211.

[4] J. Cossey and S. E. Stonehewer, *Cyclic permutable subgroups of finite groups*, J. Aust. Math. Soc. **71** (2001), no. 2, 169–176. https://doi.org/10.1017/S1446788700002810

[5] G. Cutolo and A. Leone, *A note on torsion-free locally cyclic quasinormal subgroups*, J. Algebra **424** (2015), 464–472. https://doi.org/10.1016/j.jalgebra.2014.09.023

[6] L. Q. Danh, M. H. Bien, and B. X. Hai, *Permutable subgroups in* $\mathrm{GL}_n(D)$ *and applications to locally finite group algebras*, submitted, 2019, arXiv:1912.09887.

[7] P. K. Draxl, *Skew fields*, London Mathematical Society Lecture Note Series, 81, Cambridge University Press, Cambridge, 1983. https://doi.org/10.1017/CBO9780511661907

[8] G. R. Greenfield, *A note on subnormal subgroups of division algebras*, Canadian J. Math. **30** (1978), no. 1, 161–163. https://doi.org/10.4153/CJM-1978-014-9

[9] A. Hassani and S. Sedghi, *On the quasi-normality and subnormality of subgroups of finite groups*, Vietnam J. Math. **27** (1999), no. 3, 253–258.

[10] I. N. Herstein, *Multiplicative commutators in division rings*, Israel J. Math. **31** (1978), no. 2, 180–188. https://doi.org/10.1007/BF02760549

[11] B. Hu, J. Huang, and A. Skiba, *On two open problems of the theory of permutable subgroups of finite groups*, Publ. Math. Debrecen **94** (2019), no. 3-4, 477–491. https://doi.org/10.5486/pmd.2019.8473

[12] T. Y. Lam, *A first course in noncommutative rings*, Graduate Texts in Mathematics, 131, Springer-Verlag, New York, 1991. https://doi.org/10.1007/978-1-4684-0406-7

[13] ———, *Hamilton's quaternions*, in *Handbook of algebra, Vol. 3*, 429–454, Handb. Algebr., 3, Elsevier/North-Holland, Amsterdam, 2003. https://doi.org/10.1016/S1570-7954(03)80068-2

[14] J. C. Lennox, *A note on quasinormal subgroups of finitely generated groups*, J. London Math. Soc. (2) **24** (1981), no. 1, 127–128. https://doi.org/10.1112/jlms/s2-24.1.127

[15] ———, *On quasinormal subgroups of certain finitely generated groups*, Proc. Edinburgh Math. Soc. (2) **26** (1983), no. 1, 25–28. https://doi.org/10.1017/S0013091500028042

[16] A. Leone, *Quasinormal subgroups of infinite groups*, Ric. Mat. **63** (2014), no. 1, suppl., S201–S207. https://doi.org/10.1007/s11587-014-0195-7

[17] M. Mahdavi-Hezavehi and S. Akbari, *Some special subgroups of* $\mathrm{GL}_n(D)$, Algebra Colloq. **5** (1998), no. 4, 361–370.

[18] N. K. Ngoc, M. H. Bien, and B. X. Hai, *Free subgroups in almost subnormal subgroups of general skew linear groups*, St. Petersburg Math. J. **28** (2017), no. 5, 707–717; translated from Algebra i Analiz **28** (2016), no. 5, 220–235. https://doi.org/10.1090/spmj/1468

[19] O. Ore, *On the application of structure theory to groups*, Bull. Amer. Math. Soc. **44** (1938), no. 12, 801–806. https://doi.org/10.1090/S0002-9904-1938-06868-1

[20] D. S. Passman, *The algebraic structure of group rings*, Pure and Applied Mathematics, Wiley-Interscience, New York, 1977.

[21] H. H. G. Seneviratne, *On permutable subgroups of soluble minimax groups*, Arch. Math. (Basel) **44** (1985), no. 6, 481–484. https://doi.org/10.1007/BF01193986

[22] J. Stillwell, *Naive Lie Theory*, Undergraduate Texts in Mathematics, Springer, New York, 2008. https://doi.org/10.1007/978-0-387-78214-0

[23] S. E. Stonehewer, *Permutable subgroups of infinite groups*, Math. Z. **125** (1972), 1–16. https://doi.org/10.1007/BF01111111

[24] ———, *Permutable subgroups of some finite permutation groups*, Proc. London Math. Soc. (3) **28** (1974), 222–236. https://doi.org/10.1112/plms/s3-28.2.222

[25] ———, *Old, recent and new results on quasinormal subgroups*, Irish Math. Soc. Bull. No. 56 (2005), 125–133.

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