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DING INJECTIVE MODULES OVER FROBENIUS EXTENSIONS

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Abstract. In this paper, we study Ding injective modules over Frobenius extensions. Let $R \subset A$ be a separable Frobenius extension of rings and M any left A-module, it is proved that M is a Ding injective left A-module if and only if M is a Ding injective left R-module if and only if $A \otimes_R M$ $(\mathrm{Hom}_R(A, M))$ is a Ding injective left A-module.

1. Introduction

In this paper, all rings are associative rings with identity and all modules are unitary. In 1995, Enochs and Jenda introduced Gorenstein projective modules over any associative rings, and at the same time, Gorenstein injective modules are dually defined in [1]. Mao and Ding introduced Gorenstein FP-injective modules in 2008 in [6] (Gillespie called Gorenstein FP-injective modules Ding injective modules in [2]). In 2013, Yang, Liu and Liang [13] further studied some properties of Ding injective modules.

The theory of Frobenius extensions was developed by Kasch [4] in 1954 as a generalization of Frobenius algebra, and was further studied by Nakayama and Tsuzuku [8, 9] in 1960 and Morita [7] in 1965. In 2018, Ren [10] studied Gorenstein injective (projective) modules over Frobenius extensions. It was proved that if $R \subset A$ is a Frobenius extension and M is any left A-module, then M is Gorenstein injective (projective) left A-module if and only if the underlying left R-module ${}_RM$ is Gorenstein injective (projective) if and only if $A \otimes_R M$ and $\mathrm{Hom}_R(A, M)$ are Gorenstein injective (projective) left A-modules.

Inspired by above conclusions, we intend to study some properties of Ding injective modules along Frobenius extensions of rings.

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2. FP-injective modules over Frobenius extensions

We refer to [3, Definition 1.1 and Theorem 1.2] for the definition of Frobenius extensions.

Definition. An extension of rings $R \subset A$ is a Frobenius extension, which provided that one of the following equivalent conditions holds:

(1) The functors $\mathrm{T} = A \otimes_R -$ and $\mathrm{H} = \mathrm{Hom}_R(A, -)$ are naturally equivalent.

(2) ${}_RA$ is finite generated projective, and ${}_AA_R \cong ({}_RA_A)^* = \mathrm{Hom}_R({}_RA_A, R)$.

(3) A_R is finite generated projective, and ${}_RA_A \cong ({}_AA_R)^* = \mathrm{Hom}_R({}_AA_R, R)$.

(4) There exist an R-R-homomorphism $\tau : A \to R$ and the elements x_i and y_i in A, such that for any $a \in A$, one has $\sum_i x_i\tau(y_ia) = a$ and $\sum_i \tau(ax_i)y_i = a$.

Example 2.1. (1)([11, Lemma 3.1]) Let R be an arbitrary ring and $A = R[x]/(x^2)$ be the quotient of the polynomial ring, where x is a variable which is supposed to commute with all the elements of R. Then the $R \subset A$ is a Frobenius extension.

(2) For any finite group G, the integral group ring extension $\mathbb{Z} \subset \mathbb{Z}G$ is a Frobenius extension.

Recall that a left R-module M is called FP-injective in [12] if $\mathrm{Ext}^1_R(N, M) = 0$ for all finitely presented left R-modules N.

Lemma 2.2 ([5, Lemma 3.18])**.** *Let R and S be rings. Suppose ${}_SN_R$ is an S-R-bimodule, N_R is a flat right R-module, and left S-module ${}_SN$ is finitely generated projective. If M is a finitely presented left R-module, then ${}_SN \otimes_R M$ is a finitely presented left S-module.*

Lemma 2.3. *Let $R \subset A$ be a Frobenius extension of rings. The following hold*:

(1) *If ${}_AM$ is a finitely presented left A-module, then ${}_RM$ is finitely presented as a left R-module.*

(2) *If ${}_RM$ is a finitely presented left R-module, then $A \otimes_R M$ is finitely presented as a left A-module.*

Proof. (1) Since the extension of rings $R \subset A$ is a Frobenius extension, ${}_RA$ is a finitely generated projective left R-module. It is clear that A_A is a regular module, so A_A is a flat right A-module. According to Lemma 2.2, $A \otimes_A M$ is a finitely presented left R-module. Note that $A \otimes_A M \cong {}_RM$, hence ${}_RM$ is finitely presented as a left R-module.

(2) By the definition of Frobenius extension, we get that A_R is a projective right R-module, so it is flat. Note that ${}_AA$ is finitely generated projective as a left A-module, then it follows from Lemma 2.2 that $A \otimes_R M$ is a finitely presented left A-module. □

Proposition 2.4. *Let $R \subset A$ be a Frobenius extension of rings and M a left A-module. If ${}_AM$ is an FP-injective left A-module, then the underlying left R-module ${}_RM$ is also FP-injective.*

Proof. Let N be a finitely presented left R-module. A_R is a finitely generated and projective right R-module, then $A \otimes_R N$ is a finitely presented left A-module by Lemma 2.3. Since the left A-module ${}_AM$ is FP-injective by assumption, we have $\mathrm{Ext}^1_A(A \otimes_R N, M) = 0$. By the isomorphisms

$$\mathrm{Hom}_A(A \otimes_R N, M) \cong \mathrm{Hom}_R(N, \mathrm{Hom}_A(A, M)) \cong \mathrm{Hom}_R(N, M),$$

we have

$$\mathrm{Ext}^1_A(A \otimes_R N, M) \cong \mathrm{Ext}^1_R(N, M) = 0.$$

So M is an FP-injective left R-module. □

Proposition 2.5. *Let $R \subset A$ be a Frobenius extension of rings and M a left R-module. If M is an FP-injective left R-module, then $A \otimes_R M$* $(\mathrm{Hom}_R(A, M))$ *is an FP-injective left A-module.*

Proof. Let M be an FP-injective left R-module, and N be arbitrary finitely presented left A-module. We need to claim that $\mathrm{Ext}^1_A(N, A \otimes_R M) = 0$. Since N is a finitely presented left A-module, ${}_RN$ is a finitely presented left R-module by Lemma 2.3, and then $\mathrm{Ext}^1_R(N, M) = 0$. By the isomorphisms

$$\begin{aligned}\mathrm{Hom}_A(N, A \otimes_R M) &\cong \mathrm{Hom}_A(N, \mathrm{Hom}_R(A, M)) \\ &\cong \mathrm{Hom}_R(A \otimes_A N, M) \cong \mathrm{Hom}_R(N, M),\end{aligned}$$

we have

$$\mathrm{Ext}^1_A(N, A \otimes_R M) \cong \mathrm{Ext}^1_R(N, M) = 0.$$

So $A \otimes_R M$ is an FP-injective left A-module. By the definition of Frobenius extensions, we know that $A \otimes_R M \cong \mathrm{Hom}_R(A, M)$. Hence, $\mathrm{Hom}_R(A, M)$ is also FP-injective as a left A-module. □

We refer to [11, Definition 2.8] for the definition of separable Frobenius extensions.

Definition. An extension of rings $R \subset A$ is called a separable Frobenius extension, if hold:

(1) The extension of rings $R \subset A$ is a Frobenius extension;

(2) $R \subset A$ is a separable extension. That is, the multiplication map $\psi : A \otimes_R A \to A (a \otimes_R b \to ab)$ is a split epimorphism of A-bimodules.

Example 2.6. (1) ([11, Example 2.10]) Let G be any finite group. Then $\mathbb{Z} \subset \mathbb{Z}G$ is a separable Frobenius extension.

(2) ([3, Example 2.7]) Let F be a field and set $A = M_4(F)$. Let R be the subalgebra of A with F-basis consisting of the idempotents and matrix units $e_1 = e_{11} + e_{44}$, $e_2 = e_{22} + e_{33}$, e_{21}, e_{31}, e_{41}, e_{42}, e_{43}. Then $R \subset A$ is a separable Frobenius extension.

Proposition 2.7. *Let $R \subset A$ be a separable Frobenius extension of rings and M a left A-module. If $A \otimes_R M$* $(\mathrm{Hom}_R(A, M))$ *is an FP-injective left A-module, then M is an FP-injective left A-module.*

Proof. Let $R \subset A$ be a separable Frobenius extension of rings, there is a split epimorphism $A \otimes_R M \to M (a \otimes_R m \to am)$ of left A-module by [11, Lemma 2.9], and then left A-module ${}_A M$ is a direct summand of the left A-module $A \otimes_R M$. Since $A \otimes_R M$ is an FP-injective left A-module, we have $\mathrm{Ext}^1_A(N, A \otimes_R M) = 0$ for all finitely presented left A-modules N. Then $\mathrm{Ext}^1_A(N, M) = 0$, thus M is an FP-injective left A-module. □

3. Ding injective modules over Frobenius extensions

In the section, we set out the definition and basic properties, which is used in the sequel, of Ding injective modules (i.e., Gorenstein FP-injective modules) in [2, 6, 13], and then prove the main result in the paper.

Definition ([6, Definition 2.1]). An R-module M is called Gorenstein FP-injective if there exists an exact sequence of injective R-modules

$$\mathbb{I} = \cdots \to I_1 \to I_0 \to I_{-1} \to I_{-2} \to \cdots$$

with $M = \ker(I_0 \to I_{-1})$ and which remains exact after applying $\mathrm{Hom}_R(E, -)$ for any FP-injective R-module E.

Gorenstein FP-injective modules were renamed by Gillespie as Ding injective modules in [2]. In the paper, we prefer to use the name Ding injective modules.

Lemma 3.1 ([6, Lemma 2.3]). *If E is an FP-injective module, and N is a Ding injective R-module, then $\mathrm{Ext}^i_R(E, N) = 0$ for each $i \geq 1$.*

Lemma 3.2 ([13, Theorem 2.11]). *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-module. If B and C are Ding injective, then A is a Ding injective R-module if and only if $\mathrm{Ext}^1_R(E, A) = 0$ for any FP-injective R-module E.*

Lemma 3.3 ([13, Corollary 2.9]). *The class of Ding injective modules is closed under direct summand.*

Proposition 3.4. *Let $R \subset A$ be a Frobenius extension of rings and M a left A-module. If M is a Ding injective left A-module, then the underlying left R-module ${}_R M$ is also Ding injective.*

Proof. Let M be a Ding injective left A-module. Then there exists an exact sequence of injective left A-modules

$$\mathbb{I} = \cdots \to I_1 \to I_0 \to I_{-1} \to \cdots$$

with $M = \ker(I_0 \to I_{-1})$ and which remains exact after applying $\mathrm{Hom}_A(E, \mathbb{I})$ for any FP-injective left A-module E. Note that each I_i is an injective left R-module. Then by restricting $\mathbb{I}$ one get an exact sequence of injective left R-modules.

Let Q be an FP-injective left R-module. We get $A \otimes_R Q$ is an FP-injective left A-module by Proposition 2.5. Then $\mathrm{Hom}_A(A \otimes_R Q, \mathbb{I})$ is exact. Moreover, there are isomorphisms

$$\mathrm{Hom}_R(Q, \mathbb{I}) \cong \mathrm{Hom}_R(Q, \mathrm{Hom}_A(A, \mathbb{I})) \cong \mathrm{Hom}_A(A \otimes_R Q, \mathbb{I}).$$

This implies that the sequence $\mathrm{Hom}_R(Q,\mathbb{I})$ is exact, and hence the left R-module M is Ding injective. □

Proposition 3.5. *Let $R \subset A$ be a Frobenius extension of rings and M a left A-module. Then M is a Ding injective left R-module if and only if $A \otimes_R M$ $(\mathrm{Hom}_R(A, M))$ is a Ding injective left A-module.*

Proof. ⇒) Let M be a Ding injective left R-module. Then there exists an exact sequence of injective left R-modules

$$\mathbb{I} = \cdots \to I_1 \to I_0 \to I_{-1} \to \cdots$$

with $M = \ker(I_0 \to I_{-1})$ and which remains exact after applying $\mathrm{Hom}_R(E,\mathbb{I})$ for any FP-injective left R-module E. Since I_i is an injective left R-module, it is easy to see that $A \otimes_R I_i$ is an injective left A-module. Then $A \otimes_R \mathbb{I}$ is an exact sequence of injective A-modules, and

$$A \otimes_R M = \ker(A \otimes_R I_0 \to A \otimes_R I_{-1}).$$

Let Q be an FP-injective left A-module. According to Proposition 2.4, Q is an FP-injective left R-module. There are isomorphisms

$$\mathrm{Hom}_A(Q, A \otimes_R \mathbb{I}) \cong \mathrm{Hom}_A(Q, \mathrm{Hom}_R(A,\mathbb{I}) \cong \mathrm{Hom}_R(A \otimes_A Q,\mathbb{I}) \cong \mathrm{Hom}_R(Q,\mathbb{I}),$$

and then $\mathrm{Hom}_A(Q, A \otimes_R \mathbb{I})$ is exact. Hence $A \otimes_R M$ is a Ding injective left A-module. By the definition of Frobenius extensions, we know that $A \otimes_R M \cong \mathrm{Hom}_R(A, M)$. Hence, $\mathrm{Hom}_R(A, M)$ is Ding injective as a left A-module.

⇐) Let $R \subset A$ be a Frobenius extension of rings and M a left A-module. It is easy to see that M is a left R-module. By Proposition 3.4, we get that $A \otimes_R M$ is Ding injective left R-module. Since left R-module M is a direct summand of left R-module $A \otimes_R M$, M is a Ding injective left R-module by Lemma 3.3. □

Theorem 3.6. *Let $R \subset A$ be a separable Frobenius extension of rings and M a left A-module. Then M is a Ding injective left A-module if and only if the $A \otimes_R M$ $(\mathrm{Hom}_R(A, M))$ is a Ding injective left A-module.*

Proof. ⇒) It is clear by Propositions 3.4 and 3.5.

⇐) Let Q be an FP-injective left A-module. It is easy to see that Q is an FP-injective left R-module by Proposition 2.4. Note that for the Frobenius extension of rings $R \subset A$ and any A-module M, the module M is a left R-module. By isomorphisms

$$\mathrm{Hom}_A(A \otimes_R Q, M) \cong \mathrm{Hom}_R(Q, \mathrm{Hom}_A(A, M)) \cong \mathrm{Hom}_R(Q, M),$$

we have

$$\mathrm{Ext}^i_A(A \otimes_R Q, M) \cong \mathrm{Ext}^i_R(Q, M),$$

with each $i \geq 1$.

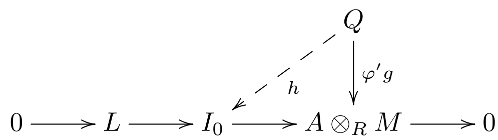
Assume that $A \otimes_R M$ is a Ding injective left A-module. Then M is a Ding injective left R-module by Proposition 3.5, and by Lemma 3.1, we get that $\mathrm{Ext}^i_R(Q, M) = 0$. Moreover, $\mathrm{Ext}^i_A(A \otimes_R Q, M) = 0$. Since left A-module ${}_AQ$

is a direct summand of left A-module $A \otimes_R Q$, the $\mathrm{Ext}^i_A(Q, M) = 0$ for each $i \geq 1$. There exists a short exact sequence of left A-modules

$$0 \to L \to I_0 \xrightarrow{f} A \otimes_R M \to 0,$$

with I_0 is injective and L is Ding injective.

According to [11, Lemma 2.9], $\varphi : A \otimes_R M \to M$ is a split epimorphism of A-module given by $\varphi(a \otimes_R m) = am$ for any $a \in A$ and $m \in M$, then there is an A-homomorphism $\varphi' : M \to A \otimes_R M$ such that $\varphi\varphi' = id_M$. Let Q be any FP-injective left R-module, and $g : Q \to M$ be any left R-homomorphism. Since L is a Ding injective left R-module by Proposition 3.4, for the left R-homomorphism $\varphi' g : Q \to A \otimes_R M$, there is an R-homomorphism $h : Q \to I_0$, such that $\varphi' g = fh$. That is, we have the following commutative diagram:



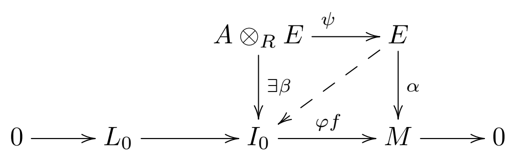
Now we have an A-epimorphism $\varphi f : I_0 \to M$. Consider the exact sequence of left A-modules

$$0 \to L_0 \to I_0 \xrightarrow{\varphi f} M \to 0,$$

which I_0 is injective, and $L_0 = \ker(\varphi f)$. Restricting the sequence, we note that it is $\mathrm{Hom}_R(Q, -)$-exact for any FP-injective left R-module Q, since for any R-homomorphism $g : Q \to M$, there exists an R-homomorphism $h : Q \to I_0$ such that $g = \varphi(\varphi' g) = \varphi(fh)$. Then, it follows from the exact sequence $\mathrm{Hom}_R(Q, I_0) \to \mathrm{Hom}_R(Q, M) \to \mathrm{Ext}^1_R(Q, L_0) \to 0$ that $\mathrm{Ext}^1_R(Q, L_0) = 0$. Moreover, the left R-module ${}_RM$ is Ding injective by Proposition 3.5, and I_0 is an injective left R-module, it follows from Lemma 3.2 that L_0 is a Ding injective left R-module.

Let E be any FP-injective left A-module. There is a split epimorphism $\psi : A \otimes_R E \to E$ of left A-module, and then there exists an A-homomorphism $\psi' : E \to A \otimes_R E$ such that $\psi\psi' = id_E$. Note that E is also FP-injective as a left R-module, then it follows from $\mathrm{Ext}^1_A(A \otimes_R E, L_0) \cong \mathrm{Ext}^1_R(E, L_0) = 0$ that the exact sequence $0 \to L_0 \to I_0 \xrightarrow{\varphi f} M \to 0$ remain exact after applying $\mathrm{Hom}_A(A \otimes_R E, -)$.

For any left A-homomorphism $\alpha : E \to M$, we consider the following diagram:



For $\alpha\psi : A\otimes_R E \to M$, there exists an A-map $\beta : A\otimes_R E \to I_0$ such that $\alpha\psi = (\varphi f)\beta$. And then, we have $\beta\psi' : E \to I_0$, such that $\alpha = (\psi\psi')\alpha = (\varphi f)(\beta\psi')$. This implies that the sequence $0 \to L_0 \to I_0 \xrightarrow{\varphi f} M \to 0$ is $\mathrm{Hom}_A(E, -)$-exact.

Note that L_0 is a Ding injective left R-module, and then $A \otimes_R L_0$ is a Ding injective left A-module by Proposition 3.5. Repeating the process we followed with M, we inductively construct an exact sequence of left A-modules

$$\cdots \to I_2 \to I_1 \to I_0 \to M \to 0,$$

with each I_i is injective and which is also exact after applying $\mathrm{Hom}_A(E, -)$ for any FP-injective left A-module E. This completes the proof. □

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