Bull. Korean Math. Soc. **58** (2021), No. 1, pp. 205–215
https://doi.org/10.4134/BKMS.b200202
pISSN: 1015-8634 / eISSN: 2234-3016

WEIGHTED NORM ESTIMATES FOR THE DYADIC PARAPRODUCT WITH VMO FUNCTION

Daewon Chung

Abstract. In [1], Beznosova proved that the bound on the norm of the dyadic paraproduct with $b \in \mathrm{BMO}$ in the weighted Lebesgue space $L^2(w)$ depends linearly on the A_2^d characteristic of the weight w and extrapolated the result to the $L^p(w)$ case. In this paper, we provide the weighted norm estimates of the dyadic paraproduct π_b with $b \in \mathrm{VMO}$ and reduce the dependence of the A_2^d characteristic to $1/2$ by using the property that for $b \in \mathrm{VMO}$ its mean oscillations are vanishing in certain cases. Using this result we also reduce the quadratic bound for the commutators of the Calderón-Zygmund operator $[b, T]$ to $3/2$.

1. Introduction

Let $\mathcal{D}$ denote the collection of dyadic intervals of the real line $\mathbb{R}$ and $\mathcal{D}(J)$ denote the dyadic subintervals of an interval J. For any interval $I \in \mathcal{D}$, there is a Haar function associated to I defined by

$$h_I(x) = \frac{1}{\sqrt{|I|}} \left(\mathbf{1}_{I_+}(x) - \mathbf{1}_{I_-}(x) \right),$$

where $|I|$ denotes the length of I, I_+ and I_- are the right and left halves respectively of I, and the characteristic function $\mathbf{1}_I(x) = 1$ if $x \in I$, zero otherwise. We say the positive almost everywhere and locally integrable function w, a weight, satisfies the A_p condition if:

$$[w]_{A_p^d} := \sup_{I \in \mathcal{D}} \langle w \rangle_I \langle w^{-1/(p-1)} \rangle_I^{p-1} < \infty, \tag{1.1}$$

where $\langle w \rangle_I$ stands for the integral average of a weight w over the interval I. It was known in the 1970s that the Maximal function, the Hilbert transform, the Calderón-Zygmund operator are bounded on $L^p(w)$ if and only if w satisfies the

Received March 3, 2020; Accepted August 21, 2020.
2010 *Mathematics Subject Classification.* Primary 42B20, 42B25; Secondary 47B38.
Key words and phrases. Weighted norm estimate, dyadic paraproduct, A_2-weights, Carleson sequence.
The author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Science, ICT & Future Planning(2015R1C1A1A02037331).

A_p condition respectively in [5, 8, 12]. A few decades later, we have obtained better understandings regarding the dependence of the weight characteristic, such as the Maximal function [2], the Hilbert transform [16], the dyadic paraproduct [1], Calderón-Zygmund operators [9] and their commutators [4]. One can find the detailed statements and their proofs in the indicated references but we also refer to [15] which kindly presented the most of this subject and progression. In this paper we are primarily interested in the result in [1] which is the now well-known fact that the dyadic paraproduct with b, a function in BMO obeys the linear dependence on the A_2 characteristic of the weight and also the dependence is optimal. Namely, for all $f \in L^2(w)$

$$\|\pi_b f\|_{L^2(w)} \leq C[w]_{A_2}\|b\|_{\mathrm{BMO}}\|f\|_{L^2(w)},$$

where $b \in L^1_{loc}(\mathbb{R})$, the dyadic paraproduct is defined as

$$\pi_b f := \sum_{I \in \mathcal{D}} \langle f \rangle_I b_I h_I.$$

Here $b_I := \langle b, h_I \rangle$ where $\langle \cdot, \cdot \rangle$ denotes the inner product on $L^2(\mathbb{R})$. However, in this paper, we can reduce the dependence of the weight characteristic by choosing the function b in VMO which was introduced by D. Sararon in [17]. More precisely, we show the following theorem.

Theorem 1.1 (Main result)**.** *Let $w \in A_2^d$ and $b \in \mathrm{VMO^d}$. Then there is a constant C, independent of the weight w, such that*

$$\|\pi_b\|_{L^2(w) \to L^2(w)} \leq C[w]_{A_2^d}^{\frac{1}{2}}\|b\|_{\mathrm{BMO^d}}.$$

This theorem, together with the sharp version of the Rubio De Francia's extrapolation theorem from [7], produces $L^p(w)$ bounds as follows.

Theorem 1.2. *Let $w \in A_p^d$ and $b \in \mathrm{VMO^d}$. Then the norm of the dyadic paraproduct operator π_b on the weighted $L^p(w)$ spaces satisfies*

$$\|\pi_b\|_{L^p(w) \to L^p(w)} \leq C(p)[w]_{A_p^d}^{\frac{1}{2}\max\left\{1, \frac{1}{p-1}\right\}}\|b\|_{\mathrm{BMO^d}}.$$

The functions in BMO are characterized by the boundedness of their mean oscillation over intervals. The functions in VMO on the circle $\mathbb{T}$ are those with the additional property that their mean oscillations over small intervals are small. The space VMO is a closed subspace of BMO and contains all uniformly continuous functions in BMO. Coifman and Weiss defined the space VMO on the real line [6] and they also proved that VMO($\mathbb{R}$) is the predual of the Hardy space $H^1(\mathbb{R})$. The precise definition of VMO($\mathbb{R}$) and several its alternative characterizations are given in the following section. Definitions and frequently used theorems are collected in Section 2. We give the proof of the main theorem in Section 3 and concluding remarks with an application of the main result to the weighted norm estimate of the commutator in Section 4.

2. Definition and useful lemmata

In this section, we will review some basic definitions, notations and some useful lemmas. Throughout the proofs a constant C will be a numerical constant that may change from line to line. Given a weight w and an interval I we also define the weighed Haar function associated to I as

$$h_I^w(x) = \frac{1}{\sqrt{w(I)}}\left(\sqrt{\frac{w(I_-)}{w(I_+)}}\mathbf{1}_{I_+}(x) - \sqrt{\frac{w(I_+)}{w(I_-)}}\mathbf{1}_{I_-}(x)\right).$$

The Haar systems $\{h_I\}_{I\in\mathcal{D}}$ and $\{h_I^w\}_{I\in\mathcal{D}}$ are orthonormal systems in L^2 and $L^2(w)$ respectively, where $L^2(w)$ is the collection of square integrable functions with respect to the measure wdx and it is a Hilbert space with the weighted inner product defined by $\langle f,g\rangle_w = \int fgwdx$. Then every function $f \in L^2(w)$ can be written as

$$f = \sum_{I\in\mathcal{D}} \langle f, h_I^w\rangle_w h_I^w\,,$$

where the sum converges a.e. in $L^2(w)$. Moreover, by the Bessel's inequality, we have

$$\sum_{I\in\mathcal{D}} |\langle f, h_I^w\rangle_w|^2 \le \|f\|^2_{L^2(w)}\,.$$

Also, the weighted and unweighted Haar functions are related linearly as follows.

Proposition 2.1. *For any weight w, there are numbers α_I^w and β_I^w such that*

$$h_I(x) = \alpha_I^w h_I^w(x) + \beta_I^w \frac{\mathbf{1}_I(x)}{\sqrt{|I|}},$$

where $|\alpha_I^w| \le \langle w\rangle^{1/2}$, $|\beta_I^w| \le \frac{|\Delta_I w|}{\langle w\rangle_I}$, and $\Delta_I w := \langle w\rangle_{I_+} - \langle w\rangle_{I_-}$.

We refer to [13] for the proof of Proposition 2.1. For the dyadic paraproduct to be defined on $L^p(\mathbb{R})$ the function b needs to be in $\mathrm{BMO^d}$, that is

$$\|b\|_{BMO^d} := \left(\sup_{I\in\mathcal{D}} \frac{1}{|I|}\int_I |b(x) - \langle b\rangle_I|^2 dx\right)^{1/2} < \infty\,.$$

We will use the alternative definition of the $\mathrm{BMO^d}$ norm of b which is

$$\|b\|^2_{BMO^d} = \sup_{J\in\mathcal{D}} \frac{1}{|J|}\sum_{I\in\mathcal{D}(J)} b_I^2\,.$$

As we mentioned in the beginning, the linear estimate of the dyadic paraproduct first presented by O. Beznosova in [1]. Here we state it as a theorem.

Theorem 2.2 ([1])**.** *Let $w \in A_2^d$ and $b \in \mathrm{BMO^d}$. Then the norm of the dyadic paraproduct operator π_b on the weighted $L^2(w)$ space is bounded by*

$$\|\pi_b\|_{L^2(w)\to L^2(w)} \le C[w]_{A_2^d}\|b\|_{\mathrm{BMO^d}}\,.$$

Theorem 2.2 plays an important role in the weighted norm estimates of the singular integral operators to obtain the its linear estimate in $L^2(w)$. In order to reduce the dependence of the weight characteristic, we are going to choose the function b in the space of $\mathrm{VMO}^{\mathrm{d}}$. First, we state the definition of $\mathrm{VMO}(\mathbb{R})$ as it appeared in [6].

Definition 1 ([6]). $\mathrm{VMO}(\mathbb{R})$ is the closure of $C_0^\infty(\mathbb{R})$ in the $\mathrm{BMO}(\mathbb{R})$ norm.

The alternative definitions in terms of oscillation conditions and Carleson measures on Haar coefficients as follows. One can find the proof and more detailed statements in [10]. Let us denote $B(0,2^R)$ be the ball in the real line centered origin with radius 2^R.

Definition 2. The space $\mathrm{VMO}^{\mathrm{d}}(\mathbb{R})$ is the set of all functions $b \in \mathrm{BMO}^{\mathrm{d}}(\mathbb{R})$ satisfying the conditions:

(1) $\displaystyle \lim_{N\to\infty} \sup_{\substack{J\in\mathcal{D}\\ |J|<2^{-N}}} \frac{1}{|J|} \sum_{I\in\mathcal{D}(J)} b_I^2 = 0;$

(2) $\displaystyle \lim_{M\to\infty} \sup_{\substack{J\in\mathcal{D}\\ |J|>2^{M}}} \frac{1}{|J|} \sum_{I\in\mathcal{D}(J)} b_I^2 = 0;$ and

(3) $\displaystyle \lim_{R\to\infty} \sup_{\substack{J\in\mathcal{D}\\ J\cap B(0,2^R)=\emptyset}} \frac{1}{|J|} \sum_{I\in\mathcal{D}(J)} b_I^2 = 0.$

If a function b belongs to the space $\mathrm{VMO}^{\mathrm{d}}$ then, for any positive number $\epsilon > 0$ there exist N and M such that

$$\sup\left\{ \frac{1}{|J|} \sum_{I\in\mathcal{D}(J)} b_I^2 \,\middle|\, |J| \le 2^{-N} \right\} < \epsilon$$

and

$$\sup\left\{ \frac{1}{|J|} \sum_{I\in\mathcal{D}(J)} b_I^2 \,\middle|\, |J| \ge 2^{M} \right\} < \epsilon.$$

Also, by the 3rd condition of Definition 2, there exists R such that

$$\sup\left\{ \frac{1}{|J|} \sum_{I\in\mathcal{D}(J)} b_I^2 \,\middle|\, J\cap B(0,2^R)=\emptyset \right\} < \epsilon\,.$$

Then, for any given $\epsilon > 0$, we can also define the collections of the dyadic intervals $\mathcal{B}$ and $\mathcal{G}$ as follows:

$$\mathcal{B} = \{I\in\mathcal{D} \,|\, 2^{-N} < |I| < 2^M \text{ and } I\cap B(0,2^R) \ne \emptyset\} \quad \text{and} \quad \mathcal{G} = \mathcal{D}\setminus\mathcal{B}\,.$$

Then $\mathcal{B}$ only contains a finite number of intervals, namely at most $2\big(2^{N+R+1}-1\big)$. The collection of the dyadic intervals $\mathcal{G}$ contains infinitely many intervals, but for any $J \in \mathcal{D}$, we have

$$\frac{1}{|J|} \sum_{I \in \mathcal{D}(J) \cap \mathcal{G}} b_I^2 < \epsilon \,. \tag{2.1}$$

We now introduce some useful theorems and lemmas which will be used frequently throughout the paper. The dyadic weighted maximal function M_w^d is defined as

$$M_w^d f(x) := \sup_{\substack{I \ni x \\ I \in \mathcal{D}}} \frac{1}{w(I)} \int_I |f(y)|\, w(y) dy.$$

A very important fact about the weighted maximal function is that the $L^p(w)$ norm of M_w^d only depends on $p' = p/(p-1)$ not on the weight w. This follows by Marcinkiewicz interpolation theorem, using the facts that M_w^d is bounded on $L^\infty(w)$ with constant 1 and it is weak-type $(1,1)$ also with constant 1. We use the notation $\langle f \rangle_{I,w}$ for the weighted average of f over I. It is worth to note here for later uses that

$$\langle |f| \rangle_{I,w} \leq \inf_{x \in I} M_w^d f(x) \tag{2.2}$$

for any $I \in \mathcal{D}$.

Theorem 2.3. *Let w be a locally integrable function such that $w > 0$ a.e. Then for all $1 < p < \infty$, M_w^d is bounded in $L^p(w)$. Moreover, for all $f \in L^p(w)$*

$$\|M_w^d f\|_{L^p(w)} \leq p' \|f\|_{L^p(w)} \,.$$

A positive sequence $\{\lambda_I\}_{I \in \mathcal{D}}$ is a w-Carleson sequence if there is a constant $C > 0$ such that for all dyadic intervals J

$$\sum_{I \in \mathcal{D}(J)} \lambda_I \leq C w(J). \tag{2.3}$$

The smallest constant $C > 0$ that satisfies the inequality (2.3) is called the intensity of the sequence. If $w = 1$ a.e. the sequence is called a Carleson sequence. One can find the relationship between unweighted and weighted Carleson sequences in the following lemma that was first presented in [1]. For example, $\{b_I^2\}_{I \in \mathcal{D}}$ for $b \in \mathrm{BMO^d}$ is a Carleson sequence with intensity $\|b\|^2_{\mathrm{BMO^d}}$ and, by the following lemma, $\{b_I^2/\langle w^{-1} \rangle_I\}_{I \in \mathcal{D}}$ is a w-Carleson sequence with intensity at most $4\|b\|^2_{\mathrm{BMO^d}}$.

Lemma 2.4 (Little Lemma, [1])**.** *Let w be a weight, such that w^{-1} is also a weight, and let $\{\alpha_I\}_{I \in \mathcal{D}}$ be a Carleson sequence with intensity B. Then $\{\alpha_I/\langle w^{-1} \rangle_I\}_{I \in \mathcal{D}}$ is a w-Carleson sequence with intensity at most $4B$, that is for all $J \in \mathcal{D}$,*

$$\frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} \frac{\alpha_I}{\langle w^{-1} \rangle_I} \leq 4B \langle w \rangle_J \,.$$

The weighted Carleson Lemma 2.5 will repeatedly appear in the proof of our main theorem. The original version of the lemma was first stated in [13]. The version we state here is called the Folklore lemma which was introduced and used in [14]. One can find the proof of the following version of lemma in [11].

Lemma 2.5 (Weighted Carleson Lemma). *Let w be a weight. Then $\{\lambda_I\}_{I\in\mathcal{D}}$ is a w-Carleson sequence with intensity $\mathcal{B}$ if and only if for all non-negative w-measurable functions F on the line,*

$$\sum_{I\in\mathcal{D}}(\inf_{x\in I} F(x))\lambda_I \leq \mathcal{B}\int_{\mathbb{R}} F(x)\,w(x)\,dx. \tag{2.4}$$

The following lemma is also necessary for us to prove our main theorem. One can find its proof in the indicated references. In fact, there is one more important lemma called α-Lemma for the sharp weighted estimate of the dyadic paraproduct in [1]. However, by taking advantage of the property of the $\mathrm{VMO^d}$ function, we can get our results without using α-Lemma.

Lemma 2.6 ([1]). *Let $w \in A_2^d$ be a weight. Then for all $J \in \mathcal{D}$*

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\frac{|\Delta_I w|^2}{\langle w\rangle_I}|I|\langle w^{-1}\rangle_I \leq C[w]_{A_2^d}\,.$$

3. Proof of the main result

In order to prove Theorem 1.1 it is enough to show that, for every $f \in L^2(w^{-1})$ and $g \in L^2(w)$,

$$|\langle \pi_b(fw^{-1}), gw\rangle| \leq C([w]_{A_2^d}, \|b\|_{\mathrm{BMO^d}})\|f\|_{L^2(w^{-1})}\|g\|_{L^2(w)}\,. \tag{3.1}$$

To obtain the inequality (3.1), we will split the left hand side of (3.1) by two parts as follows.

$$\begin{aligned}|\langle \pi_b(fw^{-1}), gw\rangle| &= \left|\sum_{I\in\mathcal{D}}\langle fw^{-1}\rangle_I b_I\langle gw, h_I\rangle\right|\\ &\leq \left|\sum_{I\in\mathcal{B}}\langle fw^{-1}\rangle_I b_I\langle gw, h_I\rangle\right| + \left|\sum_{I\in\mathcal{G}}\langle fw^{-1}\rangle_I b_I\langle gw, h_I\rangle\right|\\ &:= \Sigma_1 + \Sigma_2\,.\end{aligned}$$

For the summand Σ_1, applying the Cauchy-Schwarz inequality to the average of fw^{-1}, we get the following inequality

$$\Sigma_1 = \left|\sum_{I\in\mathcal{B}}\langle fw^{-1}\rangle_I b_I\langle gw, h_I\rangle\right| \leq \sum_{I\in\mathcal{B}}\langle |f|^2 w^{-1}\rangle_I^{1/2}\langle w^{-1}\rangle_I^{1/2}|b_I|\,|\langle gw, h_I\rangle|\,.$$

Replace h_I by $\alpha_I h_I^w + \beta_I \frac{\mathbf{1}_I}{\sqrt{|I|}}$, where $\alpha_I = \alpha_I^w$ and $\beta_I = \beta_I^w$ as described in Proposition 2.1 and get

$$\begin{aligned} &\sum_{I\in\mathcal{B}} \langle |f|^2 w^{-1}\rangle_I^{1/2} \langle w^{-1}\rangle_I^{1/2} |b_I|\, |\langle gw, h_I\rangle| \\ &= \sum_{I\in\mathcal{B}} \langle |f|^2 w^{-1}\rangle_I^{1/2} \langle w^{-1}\rangle_I^{1/2} |b_I| \left| \left\langle gw, \alpha_I h_I^w + \beta_I \frac{\mathbf{1}_I}{\sqrt{|I|}} \right\rangle \right| \\ &:= \Sigma_3 + \Sigma_4\,, \end{aligned} \tag{3.2}$$

where

$$\Sigma_3 = \sum_{I\in\mathcal{B}} \langle |f|^2 w^{-1}\rangle_I^{1/2} \langle w^{-1}\rangle_I^{1/2} |b_I|\, |\langle gw, \alpha_I h_I^w\rangle|$$

and

$$\Sigma_4 = \sum_{I\in\mathcal{B}} \langle |f|^2 w^{-1}\rangle_I^{1/2} \langle w^{-1}\rangle_I^{1/2} |b_I| \left| \left\langle gw, \beta_I \frac{\mathbf{1}_I}{\sqrt{|I|}} \right\rangle \right| .$$

Estimate for $\mathbf{\Sigma_3}$: Using the estimate $|\alpha_I| \le \langle w\rangle_I^{1/2}$ and $\langle w\rangle_I \langle w^{-1}\rangle_I \le [w]_{A_2^d}$ and applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \Sigma_3 &\le [w]_{A_2}^{1/2} \sum_{I\in\mathcal{B}} \langle |f|^2 w^{-1}\rangle_I^{1/2} |b_I|\, |\langle g, h_I^w\rangle_w| \\ &\le [w]_{A_2}^{1/2} \|f\|_{L^2(w^{-1})} \sum_{I\in\mathcal{B}} \frac{|b_I|}{\sqrt{|I|}} |\langle g, h_I^w\rangle_w| \\ &\le [w]_{A_2}^{1/2} \|f\|_{L^2(w^{-1})} \left(\sum_{I\in\mathcal{B}} \frac{|b_I|^2}{|I|} \right)^{1/2} \left(\sum_{I\in\mathcal{B}} |\langle g, h_I^w\rangle_w|^2 \right)^{1/2} \\ &\le [w]_{A_2}^{1/2} \|f\|_{L^2(w^{-1})} \|g\|_{L^2(w)} \left(\sup_{I\in\mathcal{B}} \frac{|K|}{|I|} \right)^{1/2} \left(\frac{1}{|K|} \sum_{I\in\mathcal{B}} |b_I|^2 \right)^{1/2} , \end{aligned}$$

where K denotes the smallest interval that contains all dyadic intervals in $\mathcal{B}$. Since the collection $\mathcal{B}$ only contains finite dyadic intervals, $\sup_{I\in\mathcal{B}} \frac{|K|}{|I|}$ exists as a finite number and only depends on the vanishing rate of the function b. Thus there exists a constant C such that

$$\Sigma_3 \le C[w]_{A_2}^{1/2} \|b\|_{\mathrm{BMO^d}} \|f\|_{L^2(w^{-1})} \|g\|_{L^2(w)}\,. \tag{3.3}$$

Estimate for $\mathbf{\Sigma_4}$: Using the estimate $|\beta_I| \le \frac{|\Delta_I w|}{\langle w\rangle_I}$ and similar argument with the estimate for Σ_3 we get

$$\begin{aligned} \Sigma_4 &\le \sum_{I\in\mathcal{B}} \langle |f|^2 w^{-1}\rangle_I^{1/2} \langle w^{-1}\rangle_I^{1/2} |b_I| \left| \left\langle gw, \beta_I \frac{\mathbf{1}_I}{\sqrt{|I|}} \right\rangle \right| \\ &\le \|f\|_{L^2(w^{-1})} \sum_{I\in\mathcal{B}} \langle w^{-1}\rangle_I^{1/2} |b_I| \langle gw\rangle_I \frac{|\Delta_I w|}{\langle w\rangle_I} \end{aligned}$$

$$
\begin{aligned}
&\le \|f\|_{L^2(w^{-1})}\sum_{I\in\mathcal{B}}\langle w^{-1}\rangle_I^{1/2}|b_I|\langle|g|^2w\rangle_I^{1/2}\langle w\rangle_I^{1/2}\frac{|\Delta_I w|}{\langle w\rangle_I}\\
&\le \|f\|_{L^2(w^{-1})}\|g\|_{L^2(w)}\sum_{I\in\mathcal{B}}\langle w^{-1}\rangle_I^{1/2}\frac{|b_I|}{\sqrt{|I|}}\frac{|\Delta_I w|}{\langle w\rangle_I^{1/2}}\\
&\le \|f\|_{L^2(w^{-1})}\|g\|_{L^2(w)}\|b\|_{\mathrm{BMO}}\left(|K|\sum_{I\in\mathcal{B}}\frac{1}{|I|}\right)^{1/2}\left(\frac{1}{|K|}\sum_{I\in\mathcal{B}}|I|\langle w^{-1}\rangle_I\frac{|\Delta_I w|^2}{\langle w\rangle_I}\right)^{1/2},
\end{aligned}
$$

where the last inequality uses the fact that $|b_I|/\sqrt{|I|}\le C\|b\|_{\mathrm{BMO^d}}$ and K denotes the smallest interval that contains all intervals in $\mathcal{B}$. Then, by Lemma 2.6, we get the following inequality

$$
\begin{aligned}
\Sigma_4 &\le C[w]_{A_2}^{1/2}\|f\|_{L^2(w^{-1})}\|g\|_{L^2(w)}\|b\|_{\mathrm{BMO^d}}\left(|K|\sum_{I\in\mathcal{B}}\frac{1}{|I|}\right)^{1/2}\\
&\le C[w]_{A_2}^{1/2}\|b\|_{\mathrm{BMO^d}}\|f\|_{L^2(w^{-1})}\|g\|_{L^2(w)}\,. \qquad (3.4)
\end{aligned}
$$

Combining the estimates (3.3) and (3.4) we obtain the estimate for the summand Σ_1:

$$
\Sigma_1 \le C[w]_{A_2}^{1/2}\|b\|_{\mathrm{BMO^d}}\|f\|_{L^2(w^{-1})}\|g\|_{L^2(w)}\,. \qquad (3.5)
$$

We now return to the remaining part. One can follow the Beznosova's linear estimate to estimate the summand Σ_2. However, to prove Theorem 1.1, we don't need to have the linear estimate because we can take an advantage of choosing ϵ. Before we estimate Σ_2 it is good to remind the property of the collection $\mathcal{G}$, (2.1), which is for any $I\in\mathcal{G}$,

$$
\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)\cap\mathcal{G}} b_I^2 < \epsilon\,.
$$

At the end of the estimate we are going to choose $\epsilon = \|b\|_{\mathrm{BMO^d}}/[w]_{A_2^d}$.

Estimate for $\boldsymbol{\Sigma_2}$: Similarly to Σ_1, replace h_I by $\alpha_I h_I^w + \beta_I\frac{\mathbf{1}_I}{\sqrt{|I|}}$ and split the summand into two parts:

$$
\begin{aligned}
\Sigma_2 &= \left|\sum_{I\in\mathcal{G}}\langle fw^{-1}\rangle_I b_I\langle gw, h_I\rangle\right|\\
&= \left|\sum_{I\in\mathcal{G}}\langle fw^{-1}\rangle_I b_I\left\langle gw, \alpha_I h_I^w + \beta_I\frac{\mathbf{1}_I}{\sqrt{|I|}}\right\rangle\right|\\
&\le \left|\sum_{I\in\mathcal{G}}\langle fw^{-1}\rangle_I b_I\left\langle gw, \alpha_I h_I^w\right\rangle\right| + \left|\sum_{I\in\mathcal{G}}\langle fw^{-1}\rangle_I b_I\left\langle gw, \beta_I\frac{\mathbf{1}_I}{\sqrt{|I|}}\right\rangle\right|\\
&\le \sum_{I\in\mathcal{G}}\langle |f|w^{-1}\rangle_I|b_I||\langle gw, h_I^w\rangle|\langle w\rangle_I^{1/2} + \sum_{I\in\mathcal{G}}\langle|f|w^{-1}\rangle_I|b_I|\sqrt{|I|}\langle|g|w\rangle_I\frac{|\Delta_I w|}{\langle w\rangle_I}
\end{aligned}
$$

$$:= \Sigma_5 + \Sigma_6 \,.$$

For the summand Σ_5, we use the Cauchy-Schwarz inequality and (2.2) as follows:

$$\begin{aligned}
\Sigma_5 &= \sum_{I\in\mathcal{G}} \langle |f|w^{-1}\rangle_I |b_I| |\langle gw, h_I^w\rangle| \langle w\rangle_I^{1/2} \\
&= \sum_{I\in\mathcal{G}} \frac{|b_I|}{\sqrt{\langle w\rangle_I}} \langle |f|\rangle_{I,w^{-1}} |\langle g, h_I^w\rangle_w| \langle w^{-1}\rangle_I \langle w\rangle_I \\
&\leq [w]_{A_2^d} \left(\sum_{I\in\mathcal{G}} \frac{|b_I|^2}{\langle w\rangle_I} \langle |f|\rangle_{I,w^{-1}}^2 \right)^{1/2} \left(\sum_{I\in\mathcal{G}} |\langle g, h_I^w\rangle_w|^2 \right)^{1/2} \\
&\leq [w]_{A_2^d} \|g\|_{L^2(w)} \left(\sum_{I\in\mathcal{G}} \frac{|b_I|^2}{\langle w\rangle_I} \left(\inf_{x\in I} M_{w^{-1}}^d f(x) \right)^2 \right)^{1/2} .
\end{aligned}$$

Let us define the sequence by

$$\gamma_I = \begin{cases} |b_I|^2/\langle w\rangle_I & \text{if } I \in \mathcal{G}, \\ 0 & \text{otherwise.} \end{cases}$$

Then γ_I is a w^{-1}-Carleson sequence with intensity at most ϵ. We now use Theorem 2.3 and Lemma 2.5 to get the following estimate:

$$\begin{aligned}
\Sigma_5 &\leq [w]_{A_2^d} \|g\|_{L^2(w)} \left(\sum_{I\in\mathcal{D}} \gamma_I \left(\inf_{x\in I} M_{w^{-1}}^d f(x) \right)^2 \right)^{1/2} \\
&\leq C\epsilon [w]_{A_2^d} \|g\|_{L^2(w)} \left(\int_R \left(M_{w^{-1}}^d f(x) \right)^2 w^{-1}(x) dx \right)^{1/2} \\
&\leq C\epsilon [w]_{A_2^d} \|g\|_{L^2(w)} \|f\|_{L^2(w^{-1})} \,.
\end{aligned}$$

The last summand Σ_6 can be estimated similarly to the summand Σ_5 as follows:

$$\begin{aligned}
\Sigma_6 &= \sum_{I\in\mathcal{G}} \langle |f|w^{-1}\rangle_I |b_I| \sqrt{|I|} \langle |g|w\rangle_I \frac{|\Delta_I w|}{\langle w\rangle_I} \\
&= \sum_{I\in\mathcal{G}} \langle |f|\rangle_{I,w^{-1}} |b_I| \sqrt{|I|} \langle |g|\rangle_{I,w} \frac{|\Delta_I w|}{\langle w\rangle_I} \langle w^{-1}\rangle_I \langle w\rangle_I \\
&\leq [w]_{A_2^d} \left(\sum_{I\in\mathcal{G}} \langle |f|\rangle_{I,w^{-1}}^2 \frac{|b_I|^2}{\langle w\rangle_I} \right)^{1/2} \left(\sum_{I\in\mathcal{G}} \langle |g|\rangle_{I,w}^2 \frac{|\Delta_I w|^2}{\langle w\rangle_I} |I| \right)^{1/2} \\
&\leq \epsilon [w]_{A_2^d} \|f\|_{L^2(w^{-1})} \left(\sum_{I\in\mathcal{D}} \frac{|\Delta_I w|^2}{\langle w\rangle_I} |I| \left(\inf_{x\in I} M_w^d g(x) \right)^2 \right)^{1/2} \\
&\leq \epsilon [w]_{A_2^d}^{3/2} \|f\|_{L^2(w^{-1})} \|g\|_{L^2(w)} .
\end{aligned}$$

Combining the estimates for Σ_5 and Σ_6, we get

$$\Sigma_2 \leq C[w]_{A_2^d}(1+[w]_{A_2^d}^{1/2})\epsilon\|f\|_{L^2(w^{-1})}\|g\|_{L^2(w)}\,. \tag{3.6}$$

We now choose ϵ is equal to $\|b\|_{\mathrm{BMO^d}}/[w]_{A_2^d}$, then we have

$$\begin{aligned}\Sigma_2 &\leq C(1+[w]_{A_2^d}^{1/2})\|f\|_{L^2(w^{-1})}\|g\|_{L^2(w)}\\ &\leq C[w]_{A_2^d}^{1/2}\|b\|_{\mathrm{BMO^d}}\|f\|_{L^2(w^{-1})}\|g\|_{L^2(w)}.\end{aligned} \tag{3.7}$$

This completes the proof.

4. Concluding remarks

In this section we will state an application of our main result to the commutator of the Hilbert transform. In [3] the author presents the sharp quadratic estimate for the commutator. In order to obtain the result one can decompose the commutator as follows:

$$[b,H](f) = (\pi_b(Hf) - H(\pi_b f)) + (\pi_b^*(Hf) - H(\pi_b^* f)) + (\pi_{Hf}(b) - H(\pi_f b))\,,$$

where π_b^* is the adjoint of the dyadic paraproduct. The author in [3] proved the linear estimates for the terms $H\pi_b$, $\pi_b^* H$ and $\pi_{H(\cdot)}(b) - H(\pi_{(\cdot)}b)$ using Bellman function techniques for $b \in \mathrm{BMO}$. However for the remaining two parts, namely $\pi_b H$ and $H\pi_b^*$, one was not able to establish the dependece less than quadratic due to the lack of localization property of those terms. However, by choosing b in $\mathrm{VMO^d}$, we can take an advantage of Theorem 1.1 and get the estimate for $\pi_b H$ and $H\pi_b^*$,

$$\|\pi_b H(f)\|_{L^2(w)} + \|H\pi_b^*(f)\|_{L^2(w)} \leq C\|b\|_{\mathrm{BMO^d}}[w]_{A_2^d}^{3/2}\|f\|_{L^2(w)}.$$

With this 3/2 estimate, the linear estimates for the other terms, and the extrapolation, we get, for $b \in \mathrm{VMO}$, $w \in A_p$, and $f \in L^p(w)$

$$\|[b,H]\|_{L^p(w)} \leq [w]_{A_p}^{\frac{3}{2}\max\{1,\frac{1}{p-1}\}}\|b\|_{\mathrm{BMO}}\|f\|_{L^p}(w)$$

for all $1 < p < \infty$. However, we don't believe this 3/2 estimate is optimal. We expect to have the linear estimate for the commutator with VMO function by more subtle calculations and it will be our next project.

References

[1] O. V. Beznosova, *Linear bound for the dyadic paraproduct on weighted Lebesgue space* $L_2(w)$, J. Funct. Anal. **255** (2008), no. 4, 994–1007. https://doi.org/10.1016/j.jfa.2008.04.025

[2] S. M. Buckley, *Estimates for operator norms on weighted spaces and reverse Jensen inequalities*, Trans. Amer. Math. Soc. **340** (1993), no. 1, 253–272. https://doi.org/10.2307/2154555

[3] D. Chung, *Sharp estimates for the commutators of the Hilbert, Riesz transforms and the Beurling-Ahlfors operator on weighted Lebesgue spaces*, Indiana Univ. Math. J. **60** (2011), no. 5, 1543–1588. https://doi.org/10.1512/iumj.2011.60.4453

[4] D. Chung, M. C. Pereyra, and C. Perez, *Sharp bounds for general commutators on weighted Lebesgue spaces*, Trans. Amer. Math. Soc. **364** (2012), no. 3, 1163–1177. https://doi.org/10.1090/S0002-9947-2011-05534-0

[5] R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. **51** (1974), 241–250. https://doi.org/10.4064/sm-51-3-241-250

[6] R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), no. 4, 569–645. https://doi.org/10.1090/S0002-9904-1977-14325-5

[7] O. Dragičević, L. Grafakos, M. C. Pereyra, and S. Petermichl, *Extrapolation and sharp norm estimates for classical operators on weighted Lebesgue spaces*, Publ. Mat. **49** (2005), no. 1, 73–91. https://doi.org/10.5565/PUBLMAT_49105_03

[8] R. Hunt, B. Muckenhoupt, and R. Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc. **176** (1973), 227–251. https://doi.org/10.2307/1996205

[9] T. P. Hytönen, *The sharp weighted bound for general Calderón-Zygmund operators*, Ann. of Math. (2) **175** (2012), no. 3, 1473–1506. https://doi.org/10.4007/annals.2012.175.3.9

[10] J. Li, J. Pipher, and L. A. Ward, *Dyadic structure theorems for multiparameter function spaces*, Rev. Mat. Iberoam. **31** (2015), no. 3, 767–797. https://doi.org/10.4171/RMI/853

[11] J. C. P. Moraes, *Weighted estimates for dyadic operators with complexity*, ProQuest LLC, Ann Arbor, MI, 2011.

[12] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226. https://doi.org/10.2307/1995882

[13] F. Nazarov, S. Treil, and A. Volberg, *The Bellman functions and two-weight inequalities for Haar multipliers*, J. Amer. Math. Soc. **12** (1999), no. 4, 909–928. https://doi.org/10.1090/S0894-0347-99-00310-0

[14] F. Nazarov, A. Volberg, *Bellman function, polynomial estimates of weighted dyadic shifts, and A2 conjecture*, Preprint, 2011.

[15] M. C. Pereyra, *Dyadic harmornic analysis and weighted inequalities*: *the sparse revolution*, New Trends in Applied Harmonic Analysis (Vol 2), A. Aldroubi et al. (eds.) (Springer 2019), 159–239.

[16] S. Petermichl, *The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical A_p characteristic*, Amer. J. Math. **129** (2007), no. 5, 1355–1375. https://doi.org/10.1353/ajm.2007.0036

[17] D. Sarason, *Functions of vanishing mean oscillation*, Trans. Amer. Math. Soc. **207** (1975), 391–405. https://doi.org/10.2307/1997184

Daewon Chung
Major in Mathematics
Faculty of Basic Sciences
Keimyung University
Daegu 42601, Korea
Email address: dwchung@kmu.ac.kr