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GRADED UNIFORMLY *pr*-IDEALS

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Abstract. Let R be a G-graded commutative ring with a nonzero unity and P be a proper graded ideal of R. Then P is said to be a graded uniformly pr-ideal of R if there exists $n \in \mathbb{N}$ such that whenever $a, b \in h(R)$ with $ab \in P$ and $Ann(a) = \{0\}$, then $b^n \in P$. The smallest such n is called the order of P and is denoted by $ord_R(P)$. In this article, we study the characterizations on this new class of graded ideals, and investigate the behaviour of graded uniformly pr-ideals in graded factor rings and in direct product of graded rings.

1. Introduction

Let G be a group with identity e and R be a commutative ring with unity 1. Then R is said to be G-graded ring if there exist additive subgroups R_g of R such that $R = \bigoplus_{g\in G} R_g$ and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called homogeneous of degree g and R_e (the identity component of R) is a subring of R and $1 \in R_e$. For $x \in R$, x can be written uniquely as $\sum_{g\in G} x_g$ where x_g is the component of x in R_g. The support of (R, G) is defined by $supp(R, G) = \{g \in G : R_g \neq 0\}$ and $h(R) = \bigcup_{g\in G} R_g$. If R is a G-graded ring and I is an ideal of R, then R/I is a G-graded ring by $(R/I)_g = R_g + I$ for all $g \in G$. If R and S are two G-graded rings, then $R \times S$ is a G-graded ring by $(R \times S)_g = R_g \times S_g$ for all $g \in G$. For more details, one can look in [8].

Let R be a G-graded ring and I an ideal of R. Then I is said to be G-graded ideal if $I = \bigoplus_{g\in G}(I \cap R_g)$, i.e., if $x \in I$ and $x = \sum_{g\in G} x_g$, then $x_g \in I$ for all $g \in G$. An ideal of a G-graded ring need not be G-graded, and one can look in [2].

Let R be a G-graded ring and P be a graded ideal of R. Then the graded radical of P is denoted by $Grad(P)$ and it is defined to be the set of all $x \in R$ such that for each $g \in G$, there exists $n_g \in \mathbb{N}$ satisfies $x_g^{n_g} \in P$. One can see that if x is a homogeneous element, then $x \in Grad(P)$ if and only if $x^n \in P$ for some $n \in \mathbb{N}$.

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The aim of this article is to introduce graded uniformly pr-ideals of graded commutative rings, and to give relations with some classical graded ideals such as graded uniformly primary ideals, graded strongly primary ideals and graded r-ideals.

Graded r-ideals and graded pr-ideals of graded commutative rings have been introduced in [3], a proper graded ideal P of R is said to be graded r-ideal (graded pr-ideal) if whenever $a, b \in h(R)$ such that $ab \in P$ with $Ann(a) = \{0\}$, then $b \in P$ ($b^n \in P$ for some $n \in \mathbb{N}$).

In graded commutative rings, graded prime ideals and its generalizations have an important role. There have been lots of studies on this issue, one can look in ([1], [4], [6]). Graded primary ideals have been introduced in [10], a proper graded ideal P of R is said to be graded primary if whenever $a, b \in h(R)$ such that $ab \in P$, then either $a \in P$ or $b^n \in P$ for some $n \in \mathbb{N}$. In [5], the author studies a special class of graded primary ideals fixing the power of an element $b \in R$ in the above definition, a proper graded ideal P of R is said to be graded uniformly primary if there exists $n \in \mathbb{N}$ such that whenever $a, b \in h(R)$ with $ab \in P$, then either $a \in P$ or $b^n \in P$. The smallest such n is called the order of P and is denoted by $ord(P)$. Also, P is called graded strongly primary if P is graded primary and $(Grad(P))^n \subseteq P$ for some $n \in \mathbb{N}$. The smallest such n is called the exponent of P and is denoted by $\exp(P)$.

Note that the class of graded primary ideals contains the class of graded uniformly primary ideals, and also the class of graded uniformly primary ideals contains the class of graded strongly primary ideals. With these motivations, in this article, graded uniformly pr-ideals and graded strongly pr-ideals are investigated, a proper graded ideal P of R is said to be graded uniformly pr-ideal if there exists $n \in \mathbb{N}$ such that whenever $a, b \in h(R)$ with $ab \in P$ and $Ann(a) = \{0\}$, then $b^n \in P$. Also, P is said to be graded strongly pr-ideal if P is graded pr-ideal and $(Grad(P))^n \subseteq P$ for some $n \in \mathbb{N}$.

A non-empty set X of a ring R is said to be a sub-semigroup of R if $RX \subseteq X$. An element of a ring R is said to be regular if it is not a zero divisor. If $f(x) = a_0 + a_1x + \cdots + a_mx^m \in R[x]$, then the content of f is defined as $c(f) = (a_0, a_1, \ldots, a_m)$. By [7, Theorem 28.1], $c(f)^{m+1}c(g) = c(f)^m c(fg)$. A ring R is said to be Armendariz ring if whenever $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_kx^k \in R[x]$ such that $f(x)g(x) = 0$, then $a_ib_j = 0$ for all i, j, one can look in [11]. The set of graded maximal ideals, graded prime ideals and graded minimal prime ideals of R are denoted by $Gmax(R)$, $Gspec(R)$ and $Gmin(R)$ respectively. Also, the set of all zero divisors of R is denoted by $Zd(R)$. A graded ring R is said to be graded π-regular if for all family of graded ideals $\{P_k\}_{k\in\Delta}$, $Grad\left(\bigcap_{k\in\Delta} P_k\right) = \bigcap_{k\in\Delta} Grad(P_k)$, one can look in [8].

In this article, we will prove that the class of graded pr-ideals contains the class of graded uniformly pr-ideals and also the class of graded uniformly pr-ideals contains the class of graded strongly pr-ideals. Also, we will prove that graded pr-ideals, graded uniformly pr-ideals and graded strongly pr-ideals are equivalent in any $\mathbb{Z}$-graded Noetherian ring. Moreover, we will prove that any power of a graded minimal prime ideal of a $\mathbb{Z}$-graded ring is a graded strongly pr-ideal. When a graded primary ideal becomes a graded pr-ideal and a graded uniformly primary ideal becomes a graded uniformly pr-ideal have been demonstrated. Further, the behavior of graded uniformly pr-ideals in graded factor rings and in direct product of graded rings are investigated. Finally, graded uniformly pr-ideals in graded polynomial rings are examined.

2. Graded uniformly *pr*-ideals

In this section, we introduce and study the concept of graded uniformly pr-ideals.

Definition. Let R be a graded ring and P be a proper graded ideal of R. Then P is said to be a graded uniformly pr-ideal of R if there exists $n \in \mathbb{N}$ such that whenever $a, b \in h(R)$ with $ab \in P$ and $Ann(a) = \{0\}$, then $b^n \in P$. The smallest such n is called the order of P and is denoted by $ord_R(P)$.

Proposition 2.1. *If R is a finite G-graded ring, then every homogeneous element of R is either a unit or a zero divisor.*

Proof. Let R be a G-graded finite ring. Assume that $a \in h(R)$. Then $a \in R_g$ for some $g \in G$. Define $\phi : R_{g^{-1}} \to R_e$ by $\phi(b) = ab$. If ϕ is injective, then since R is finite, ϕ is surjective and as $1 \in R_e$, $1 = ab$ for some $b \in R_{g^{-1}}$ and then a is a unit. Suppose that ϕ is not injective. Then there exist $b, c \in R_{g^{-1}}$ with $b \neq c$ such that $ab = ac$. But then $a(b - c) = 0$ and $b - c \neq 0$, so a is a zero divisor. □

Example 2.2. Every proper graded ideal of a finite graded ring is a graded uniformly pr-ideal with order 1. To see this, let R be a finite graded ring and P be a proper graded ideal of R. Assume that $a, b \in h(R)$ such that $ab \in P$ and $Ann(a) = \{0\}$. Then by Proposition 2.1 and since $Ann(a) = \{0\}$, a is a unit and then $b = a^{-1}(ab) \in P$. Hence, P is a graded uniformly pr-ideal of R with $ord_R(P) = 1$.

Example 2.3. Let $R = \mathbb{Z}_n[i]$ and $G = \mathbb{Z}_4$. Then R is G-graded by $R_0 = \mathbb{Z}_n$, $R_2 = i\mathbb{Z}_n$ and $R_1 = R_3 = \{0\}$. Then by Example 2.2, every proper graded ideal of R is a graded uniformly pr-ideal of R with order 1.

Definition. Let R be a graded ring and P be a proper graded ideal of R. Then P is said to be a graded strongly pr-ideal of R if P is a graded pr-ideal of R and $(Grad(P))^n \subseteq P$ for some $n \in \mathbb{N}$. The smallest such n is called the exponent of P and is denoted by $\exp_R(P)$.

Proposition 2.4. *Let R be a graded ring and P be a graded ideal of R. If P is a graded strongly pr-ideal of R, then P is a graded uniformly pr-ideal of R with $ord_R(P) \leq \exp_R(P)$.*

Proof. Since P is a graded strongly pr-ideal of R, P is a graded pr-ideal of R and there exists $n \in \mathbb{N}$ such that $(Grad(P))^n \subseteq P$ (we can assume that $\exp_R(P) = n$) . Let $a, b \in h(R)$ such that $ab \in P$ and $Ann(a) = \{0\}$. Since P is a graded pr-ideal of R, $b^m \in P$ for some $m \in \mathbb{N}$ which implies that $b \in Grad(P)$. Since $(Grad(P))^n \subseteq P$, $b^n \in P$ and hence P is a graded uniformly pr-ideal of R. Moreover, $ord_R(P) \leq n$ and so $ord_R(P) \leq \exp_R(P)$. □

If P is a graded ideal of a G-graded ring R, then $Grad(P)$ need not to be a graded ideal of R; see [9].

Lemma 2.5 ([3, Lemma 2.13])**.** *If P is a graded ideal of a $\mathbb{Z}$-graded ring R, then $Grad(P)$ is a graded ideal of R.*

Proposition 2.6 ([3, Theorem 2.14])**.** *Let R be a $\mathbb{Z}$-graded ring and P be a graded ideal of R. Then P is a graded pr-ideal of R if and only if $Grad(P)$ is a graded r-ideal of R.*

Corollary 2.7. *Let R be a $\mathbb{Z}$-graded ring and P be a graded ideal of R such that $Grad(P)$ is finitely generated. Then P is a graded pr-ideal if and only if P is a graded uniformly pr-ideal of R if and only if P is a graded strongly pr-ideal of R if and only if $Grad(P)$ is a graded r-ideal of R.*

Corollary 2.8. *Let R be a $\mathbb{Z}$-graded Noetherian ring and P be a graded ideal of R. Then the following are equivalent.*

1. *P is a graded pr-ideal of R.*
2. *$Grad(P)$ is a graded r-ideal of R.*
3. *P is a graded uniformly pr-ideal of R.*
4. *P is a graded strongly pr-ideal of R.*

Lemma 2.9. *If R is a graded ring contains a sub-semigroup X does not contain the zero, then R has a graded prime ideal P which is maximal with respect to the property $P \bigcap X = \emptyset$.*

Proof. The existence of the graded ideal P follows from Zorn's lemma in the graded case. Let $a \in h(R)$ and consider the ideal generated by P and a that is $(P, a) = P \bigcup Ra \bigcup \{a\}$. To prove that P is a graded prime ideal of R, let $x, y \in h(R) - P$. Then there exist $s, t \in R$ such that $s \in X \bigcap (P, x)$ and $t \in X \bigcap (P, y)$. Since $P \bigcap X = \emptyset$, we have the following cases: (1) $s = \alpha x$ and $t = \beta y$ for some $\alpha, \beta \in R$, (2) $s = \alpha x$, $t = y$, (3) $s = x$, $t = \beta y$, (4) $s = x$, $t = y$. If (1) holds, then $st = \alpha\beta xy$, and since $st \in X$, $xy \notin P$. Similarly for the other cases. □

Proposition 2.10. *Let R be a graded ring and $P \in Gmin(R)$. Then $R - P$ is a sub-semigroup of R which is maximal with respect to the property $(R - P) \bigcap \{0\} = \emptyset$.*

Proof. Clearly, $R - P$ is a sub-semigroup of R such that $(R - P) \bigcap \{0\} = \emptyset$. As an easy consequence of Zorn's lemma, $R - P$ is contained in a sub-semigroup

X of R which is maximal with respect to the property $X\bigcap\{0\}=\emptyset$ and then $\{0\}\subseteq R-X\subseteq P$. By Lemma 2.9, there exists a graded prime ideal Q of R such that $\{0\}\subseteq Q\subseteq R-X$. Since $(R-Q)\bigcap\{0\}=\emptyset$ and since $R-Q$ is a sub-semigroup of R, the maximal property of X ensures that $X=R-Q$ and then $R-X=Q$. Hence, $R-X$ is a graded prime ideal of R. Since $\{0\}\subseteq R-X\subseteq P$ and $P\in Gmin(R)$, $R-X=P$. Hence, $R-P$ is maximal with respect to the property $(R-P)\bigcap\{0\}=\emptyset$. □

Corollary 2.11. *Let R be a graded ring and $P\in Gmin(R)$. Then for every $x\in P$, there exists $r\in R-P$ such that rx is a nilpotent.*

Proof. By Proposition 2.10, $R-P$ is maximal with respect to the property $(R-P)\bigcap\{0\}=\emptyset$. Let $x\in P$ and assume that $X=\{rx^n : r\in R-P, n=0,1,2,\dots\}$. Since $rx^0=r$, X is a sub-semigroup of R which properly includes $R-P$. By the maximal property of $R-P$, we should have $rx^n=0$ for some $r\in R-P$ and some positive integer n, and then $(rx)^n=r^nx^n=r^{n-1}(rx^n)=r^{n-1}.0=0$. Hence, rx is a nilpotent. □

Corollary 2.12. *Let R be a graded ring and $P\in Gmin(R)$. Then $P\subseteq Zd(R)$.*

Proof. Let $x\in P$. Then by Corollary 2.11, there exists $r\in R-P$ such that rx is a nilpotent which implies that $(rx)^n=0$ for some $n\in\mathbb{N}$. Now, $r^n\neq 0$ since $r^n\in R-P$. Let t be the smallest positive integer such that $r^nx^t=0$. If $t=1$, then we are done. Otherwise, $r^nx^{t-1}\neq 0$ and $(r^nx^{t-1})x=0$. Hence, $x\in Zd(R)$. □

Lemma 2.13. *Let R be a graded ring and P be a graded prime ideal of R. If $P\subseteq Zd(R)$, then P is a graded r-ideal of R.*

Proof. Let $a,b\in h(R)$ such that $ab\in P$ and $Ann(a)=\{0\}$. Then since P is graded prime, either $a\in P$ or $b\in P$. If $a\in P$, then $a\in Zd(R)$ which is a contradiction since $Ann(a)=\{0\}$. So, $b\in P$, and hence P is a graded r-ideal of R. □

Proposition 2.14. *Let R be a $\mathbb{Z}$-graded ring and $P\in Gmin(R)$. Then P^n is a graded strongly pr-ideal of R for every $n\in\mathbb{N}$.*

Proof. By Corollary 2.12, $P\subseteq Zd(R)$ and then by Lemma 2.13, P is a graded r-ideal of R. Thus, by Proposition 2.6, P^n is a graded strongly pr-ideal of R for every $n\in\mathbb{N}$. □

Lemma 2.15. *Let R be a G-graded ring and P be a graded ideal of R. Then $(P:X)=\{r\in R : rX\subseteq P\}$ is a graded ideal of R for every nonempty set X of $h(R)$.*

Proof. Let $X\subseteq h(R)$ such that $X\neq\emptyset$ and $r\in(P:X)$. Then $r\in R$ such that $rX\subseteq P$. Now, $r=\sum_{g\in G}r_g$ where $r_g\in R_g$ for all $g\in G$. Let $x\in X$. Then $x\in h(R)$ and then $r_gx\in h(R)$ for all $g\in G$ such that $\sum_{g\in G}r_gx=$

$\left(\sum_{g\in G} r_g\right) x = rx \in P$. Since P is graded, $r_g x \in P$ for all $g \in G$. So, $r_g X \subseteq P$ for all $g \in G$ which implies that $r_g \in (P : X)$ for all $g \in G$. Hence, $(P : X)$ is a graded ideal of R. □

Proposition 2.16. *Let P be a graded uniformly pr-ideal of a graded ring R. Assume that X is a nonempty set of $h(R)$ such that $X \nsubseteq P$. If X contains a regular element, then $(P : X)$ is a graded uniformly pr-ideal of R.*

Proof. By Lemma 2.15 and since $X \nsubseteq P$, $(P : X)$ is a proper graded ideal of R. Let $x \in X \subseteq h(R)$ be a regular element. Assume that $a, b \in h(R)$ such that $ab \in (P : X)$ with $Ann(a) = \{0\}$. Since x is regular and $Ann(a) = \{0\}$, ax is regular which implies that $Ann(ax) = \{0\}$. Now, $axb = abx \in P$ and P is a graded uniformly *pr*-ideal, so $b^n \in P \subseteq (P : X)$ for some $n \in \mathbb{N}$. Hence, $(P : X)$ is a graded uniformly *pr*-ideal of R. □

Lemma 2.17. *Let $\{P_i\}_{i\in\Delta}$ be graded uniformly pr-ideals of R with $ord_R(P_i) = n_i$. If $\sup\{n_i : i \in \Delta\} < \infty$, then $P = \bigcap_{i\in\Delta} P_i$ is a graded uniformly pr-ideal of R with $ord_R(P) \leq \sup\{n_i : i \in \Delta\}$.*

Proof. Clearly, P is a graded ideal of R. Let $n = \sup\{n_i : i \in \Delta\}$. Then $n_i \leq n$ for all $i \in \Delta$. Assume that $a, b \in h(R)$ such that $ab \in P$ with $Ann(a) = \{0\}$. Then $ab \in P_i$ for all $i \in \Delta$, and then $b^{n_i} \in P_i$ for all $i \in \Delta$ which implies that $b^n \in P_i$ for all $i \in \Delta$, so $b^n \in P$. Hence, P is a graded uniformly *pr*-ideal of R. Moreover, $ord_R(P) \leq n = \sup\{n_i : i \in \Delta\}$. □

Proposition 2.18. *Let $P_1, \dots, P_k$ be graded strongly pr-ideals of R with $\exp_R(P_i) = n_i$. Then $P = \bigcap_{i=1}^k P_i$ is a graded strongly pr-ideal of R with $\exp_R(P) \leq \max\{n_1, \dots, n_k\}$.*

Proof. Let $n = \max\{n_1, \dots, n_k\}$. Then $n_i \leq n$ for all $i = 1, \dots, k$. By Lemma 2.17, P is a graded uniformly *pr*-ideal of R. Now,

$$Grad(P) = Grad\left(\bigcap_{i=1}^k P_i\right) = \bigcap_{i=1}^k Grad(P_i),$$

so $(Grad(P_i))^{n_i} \subseteq P_i$ for all $i = 1, \dots, k$, and then $(Grad(P_i))^n \subseteq P_i$ for all $i = 1, \dots, k$. Since $Grad(P) \subseteq Grad(P_i)$ for all $i = 1, \dots, k$, $(Grad(P))^n \subseteq (Grad(P_i))^n \subseteq P_i$ for all $i = 1, \dots, k$, and then $(Grad(P))^n \subseteq \bigcap_{i=1}^k P_i = P$. Hence, P is a graded strongly *pr*-ideal of R. Moreover, $\exp_R(P) \leq n = \max\{n_1, \dots, n_k\}$. □

Corollary 2.19. *Suppose that R is a graded π-regular ring. Let $\{P_i\}_{i\in\Delta}$ be graded strongly pr-ideals of R with $\exp_R(P_i) = n_i$. If $\sup\{n_i : i \in \Delta\} < \infty$, then $P = \bigcap_{i\in\Delta} P_i$ is a graded strongly pr-ideal of R with $\exp_R(P) \leq \sup\{n_i : i \in \Delta\}$.*

Lemma 2.20. *If P is a graded r-ideal of a G-graded ring R, then $P \subseteq Zd(R)$.*

Proof. Suppose that there exists $a \in P$ such that a is not a zero divisor. Then there exists $g \in G$ such that a_g is not a zero divisor, and then $Ann(a_g) = \{0\}$ with $a_g \cdot 1 = a_g \in P$ as P is graded. Since P is a graded r-ideal of R, $1 \in P$ and then $P = R$ which is a contradiction. Hence, $P \subseteq Zd(R)$. $\square$

Proposition 2.21. *If P is a graded pr-ideal of a $\mathbb{Z}$-graded ring R, then $P \subseteq Zd(R)$.*

Proof. By Proposition 2.6, $Grad(P)$ is a graded r-ideal of R, and then by Lemma 2.20, $Grad(P) \subseteq Zd(R)$, and so $P \subseteq Grad(P) \subseteq Zd(R)$. $\square$

Proposition 2.22. *If P is a graded primary ideal of a graded ring R and $P \subseteq Zd(R)$, then P is a graded pr-ideal of R.*

Proof. Let $a, b \in h(R)$ such that $ab \in P$ with $Ann(a) = \{0\}$. Since P is graded primary, either $a \in P$ or $b^n \in P$ for some $n \in \mathbb{N}$. If $a \in P$, then $a \in Zd(R)$ which is a contradiction since $Ann(a) = \{0\}$. So, $b^n \in P$. Hence, P is a graded pr-ideal of R. $\square$

Corollary 2.23. *Let P be a graded uniformly primary ideal of a $\mathbb{Z}$-graded ring R. Then P is a graded uniformly pr-ideal of R if and only if $P \subseteq Zd(R)$.*

Corollary 2.24. *Let P be a graded strongly primary ideal of a $\mathbb{Z}$-graded ring R. Then P is a graded strongly pr-ideal of R if and only if $P \subseteq Zd(R)$.*

Proposition 2.25. *Let P be a graded r-ideal of a graded ring R. Assume that K is a graded ideal of R such that $P \subseteq K$. If K/P is a graded uniformly pr-ideal of R/P, then K is a graded uniformly pr-ideal of R.*

Proof. Let $a, b \in h(R)$ such that $ab \in K$ with $Ann(a) = \{0\}$. If $ab \in P$, then since P is a graded r-ideal of R, $b \in P$ and then $b \in K$ and we are done. Suppose that $ab \notin P$. We show that $Ann(a + P) = \{0_{R/P}\}$, let $x + P \in R/P$ such that $(a + P)(x + P) = 0 + P$. Then $ax + P = 0 + P$ and then $ax \in P$. Since P is a graded r-ideal of R, $x \in P$ and then $x+P = 0+P$. Thus, $Ann(a+P) = \{0_{R/P}\}$. Now, $(a+P)(b+P) = ab+P \in K/P$ and K/P is a graded uniformly pr-ideal of R/P, so there exists $n \in \mathbb{N}$ such that $(b+P)^n \in K/P$, and then $b^n + P \in K/P$ which implies that $b^n \in K$. Hence, K is a graded uniformly pr-ideal of R. $\square$

Lemma 2.26. *Let P be an ideal of a G-graded ring R and K be an ideal of a G-graded ring S. Then $P \times K$ is a graded ideal of $R \times S$ if and only if P is a graded ideal of R and K is a graded ideal of S.*

Proof. Suppose that P is a graded ideal of R and K is a graded ideal of S. Clearly, $P \times K$ is an ideal of $R \times S$. Let $(x, y) \in P \times K$. Then $x \in P$ and $y \in K$, and since P, K are graded, $x_g \in P$ and $y_g \in K$ for all $g \in G$, which implies that $(x, y)_g = (x_g, y_g) \in P \times K$ for all $g \in G$. Hence, $P \times K$ is a graded ideal of $R \times S$. Conversely, let $x \in P$. Then $(x, 0_S) \in P \times K$, and since $P \times K$ is graded, $(x_g, 0_S) = (x, 0_S)_g \in P \times K$ for all $g \in G$, which implies that $x_g \in P$ for all $g \in G$. Hence, P is a graded ideal of R. Similarly, K is a graded ideal of S. $\square$

Proposition 2.27. *Let P be an ideal of a G-graded ring R and K be an ideal of a G-graded ring S. Then the following are equivalent.*

(1) *$P \times K$ is a graded uniformly pr-ideal of $R \times S$.*
(2) *$P = R$ and K is a graded uniformly pr-ideal of S or $K = S$ and P is a graded uniformly pr-ideal of R or P and K are graded uniformly pr-ideals of R and S respectively.*

Proof. (1) $\Rightarrow$ (2): Suppose that $K = S$. By Lemma 2.26, P is a graded ideal of R. Let $a, b \in h(R)$ such that $ab \in P$ with $Ann(a) = \{0\}$. Then $(a, 1_S), (b, 0_S) \in h(R \times S)$ such that $(a, 1_S)(b, 0_S) = (ab, 0_S) \in P \times K$ with $Ann((a, 1_S)) = \{0_{R\times S}\}$. Since $P \times K$ is a graded uniformly pr-ideal of $R \times S$, there exists $n \in \mathbb{N}$ such that $((b, 0_S))^n \in P \times K$, and then $(b^n, 0_S) \in P \times K$, which implies that $b^n \in P$. Hence, P is a graded uniformly pr-ideal of R. Similarly, K is a graded uniformly pr-ideal of S when $P = R$. Similarly, P and K are graded uniformly pr-ideals of R and S respectively when P and K are proper ideals of R and S respectively. Moreover, $ord_{R\times S}(P \times K) \leq \max\{ord_R(P), ord_S(K)\}$.

(2) $\Rightarrow$ (1): Let P and K be graded uniformly pr-ideals of R and S respectively with $ord_R(P) = n_1$ and $ord_S(K) = n_2$. By Lemma 2.26, $P \times K$ is a graded ideal of $R \times S$. Assume that $n = \max\{n_1, n_2\}$. Suppose that $(a, x), (b, y) \in h(R \times S)$ such that $(a, x)(b, y) \in P \times K$ with $Ann((a, x)) = \{0_{R\times S}\}$. Then $a, b \in h(R)$ and $x, y \in h(S)$ such that $ab \in P$ with $Ann(a) = \{0_R\}$ and $xy \in K$ with $Ann(x) = \{0_S\}$. So, $b^{n_1} \in P$ and $y^{n_2} \in K$, and then $b^n \in P$ and $y^n \in K$, which implies that $((b, y))^n \in P \times K$. Hence, $P \times K$ is a graded uniformly pr-ideal of $R \times S$. Similarly for the other cases. □

Proposition 2.28. *Let $R_1, \ldots, R_n$ be G-graded rings. Assume that $P_1, \ldots, P_n$ be ideals of $R_1, \ldots, R_n$ respectively. Then the following are equivalent.*

(1) *$P_1 \times \ldots \times P_n$ is a graded uniformly pr-ideal of $R_1 \times \cdots \times R_n$.*
(2) *There exist $t_1, \ldots, t_k \in \{1, \ldots, n\}$ such that $P_t = R_t$ for each $t \in \{t_1, \ldots, t_k\}$ and P_t is a graded uniformly pr-ideal of R_t for each $t \in \{1, \ldots, n\} - \{t_1, \ldots, t_k\}$.*

Proposition 2.29. *Let P be an ideal of a G-graded ring R and K be an ideal of a G-graded ring S. Then the following are equivalent.*

(1) *$P \times K$ is a graded strongly pr-ideal of $R \times S$.*
(2) *$P = R$ and K is a graded strongly pr-ideal of S or $K = S$ and P is a graded strongly pr-ideal of R or P and K are graded strongly pr-ideals of R and S respectively.*

Proof. Similar to the proof of Proposition 2.27 using the fact that $Grad(P \times K) = Grad(P) \times Grad(K)$. □

Proposition 2.30. *Let $R_1, \ldots, R_n$ be G-graded rings. Assume that $P_1, \ldots, P_n$ be ideals of $R_1, \ldots, R_n$ respectively. Then the following are equivalent.*

(1) *$P_1 \times \ldots \times P_n$ is a graded strongly pr-ideal of $R_1 \times \cdots \times R_n$.*

(2) *There exist* $t_1, \ldots, t_k \in \{1, \ldots, n\}$ *such that* $P_t = R_t$ *for each* $t \in \{t_1, \ldots, t_k\}$ *and* P_t *is a graded strongly pr-ideal of* R_t *for each* $t \in \{1, \ldots, n\} - \{t_1, \ldots, t_k\}$.

The usual graduation of the ring $S = R[x]$ by $\mathbb{Z}$ is given by $S_j = Rx^j$ if $j \geq 0$ and $S_j = \{0\}$ otherwise, one can look in [8].

Lemma 2.31. *If* $R[x]$ *is graded by the usual graduation by* $\mathbb{Z}$ *and* I *is an ideal of* R, *then* $I[x]$ *is a graded ideal of* $R[x]$.

Proof. Clearly, $I[x]$ is an ideal of $R[x]$. Let $f(x) = a_0 + a_1x + \cdots + a_mx^m \in I[x]$. Then $f_0 = a_0 \in I \subseteq I[x]$, $f_1 = a_1x \in I[x], \ldots, f_m = a_mx^m \in I[x]$ and $f_j = 0 \in I[x]$ otherwise. So, $f_j \in I[x]$ for all $j \in \mathbb{Z}$. Hence, $I[x]$ is a graded ideal of $R[x]$. $\square$

Proposition 2.32. *Let* R *be a graded Armendariz ring by the trivial graduation* ($R_e = R$ *and* $R_g = \{0\}$ *otherwise) and* $S = R[x]$ *be graded by the usual graduation by* $\mathbb{Z}$. *Then* P *is a graded uniformly pr-ideal of* R *if and only if* $P[x]$ *is a graded uniformly pr-ideal of* S.

Proof. Suppose that $P[x]$ is a graded uniformly pr-ideal of S. Let $a, b \in h(R)$ such that $ab \in P$ with $Ann(a) = \{0\}$. Then $f(x) = a$, $g(x) = b \in S_0 \subseteq h(S)$ such that $f(x)g(x) \in P[x]$ with $Ann(f(x)) = \{0_S\}$. So, there exists $n \in \mathbb{N}$ such that $(g(x))^n \in P[x]$, and then $b^n \in P$. Hence, P is a graded uniformly pr-ideal of R. Conversely, by Lemma 2.31, $P[x]$ is a graded ideal of S. Assume that $ord_R(P) = n$. Let $f(x)$, $g(x) \in h(S)$ such that $f(x)g(x) \in P[x]$ with $Ann(f(x)) = \{0_S\}$. Since R is an Armendariz ring, $c(f) \subsetneqq Zd(R)$ and so there exists $t \in c(f)$ such that $Ann(t) = \{0\}$. Also, $c(fg) \subseteq P$ and then by [[7], Theorem 28.1], $c(f)^{m+1}c(g) = c(f)^m c(fg) \subseteq P$ where $m = \deg(f)$. Since $t^{m+1} \in c(f)^{m+1}$ and $Ann(t^{m+1}) = \{0\}$, $t^{m+1}c(g) \subseteq P$. Suppose that $c(g) = (b_0, \ldots, b_k)$. Then for each $b_i \in c(g)$, $b_i^n \in P$. Clearly, $c(g)^{(k+1)n} \subseteq P$, so $c(g^{(k+1)n}) \subseteq c(g)^{(k+1)n} \subseteq P$, which implies that $g(x)^{(k+1)n} \in P[x]$. Hence, $P[x]$ is a graded uniformly pr-ideal of S. $\square$

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