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ON THE FIXING NUMBER OF FUNCTIGRAPHS

Muhammad Fazil, Imran Javaid, and Muhammad Murtaza

Abstract. The fixing number of a graph G is the smallest order of a subset S of its vertex set $V(G)$ such that the stabilizer of S in G, $\Gamma_S(G)$ is trivial. Let G_1 and G_2 be the disjoint copies of a graph G, and let $g : V(G_1) \to V(G_2)$ be a function. A functigraph F_G consists of the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2) \cup \{uv : v = g(u)\}$. In this paper, we study the behavior of fixing number in passing from G to F_G and find its sharp lower and upper bounds. We also study the fixing number of functigraphs of some well known families of graphs like complete graphs, trees and join graphs.

1. Introduction

The idea of symmetry breaking in graphs was given by Albertson and Collins in [1], which has the applications in the problem of programming a robot to manipulate objects [18]. A number of different methods (like orienting some of the edges, coloring some of the vertices with one or more colors and same for the edges, labeling vertices or edges, adding or deleting vertices or edges) of destroying the symmetries of a graph were given by Harary in [14]. The concept to destroy all automorphisms of a graph by using its vertices was given by Erwin and Harary in [9] where the authors defined the fixing number of a graph G in 2006. Boutin [2] independently, did her research on the fixing number and name this parameter, determining number.

Unless otherwise specified, all the graphs G considered in this paper are simple, non-trivial and connected. The *degree* of a vertex v in G, denoted by $\deg_G(v)$, is the number of edges to which v belongs. The *open neighborhood* of a vertex u of G is $N(u) = \{v \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of u is $N[u] = N(u) \cup \{u\}$. Two vertices u, v are *adjacent twins* if $N[u] = N[v]$ and *non-adjacent twins* if $N(u) = N(v)$. If u, v are adjacent or non-adjacent twins, then u, v are called *twins*. A set of vertices is called a *twin-set* if each of its two vertices are twins. An *automorphism* α of G, $\alpha : V(G) \to V(G)$, is a bijective mapping such that $\alpha(u)\alpha(v) \in E(G)$ if and only if $uv \in E(G)$.

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Thus, each automorphism α of G is a *permutation* of the vertex set $V(G)$ which preserves adjacencies and non-adjacencies. The *automorphism group* of a graph G, denoted by $\Gamma(G)$, is the set of all automorphisms of G. The *stabilizer* of a vertex v, denoted by $\Gamma_v(G)$, is the set $\{\alpha \in \Gamma(G) : v = \alpha(v)\}$. The stabilizer of a set of vertices $S \subseteq V(G)$ is $\Gamma_S(G) = \{\alpha \in \Gamma(G) : v = \alpha(v) \ \forall \ v \in S\}$. Note that $\Gamma_S(G) = \bigcap_{v \in S} \Gamma_v(G)$. The *orbit* of a vertex v, denoted by $\theta(v)$, is the set $\{u \in V(G) : u = \alpha(v) \text{ for some } \alpha \in \Gamma(G)\}$.

A vertex v is *fixed* by a group element $\alpha \in \Gamma(G)$ if $\alpha \in \Gamma_v(G)$. A set of vertices $S \subseteq V(G)$ is a *fixing set* of G if $\Gamma_S(G)$ is trivial. In this case, we say that S fixes G. The *fixing number* of a graph G, denoted by $fix(G)$, is the cardinality of a smallest fixing set of G [9]. The graphs with $fix(G) = 0$ are called *rigid graphs* [1], which have a trivial automorphism group. Every graph G has a fixing set. Trivially, the set of vertices of G itself is a fixing set. It is also clear that any set containing all but one vertex is a fixing set. Thus, for a graph G on n vertices $0 \leq fix(G) \leq n - 1$.

The fixing number of a graph gives us the measure of non-rigidity of the graph. Gibbons and Laison [12] investigated the fixing numbers of Petersen graph and Cayley graph. Bradley et al. [3] found the fixing numbers of graphs having abelian automorphism groups. In the recent past, Koorepazan-Moftakhar et al. [19] investigated the automorphism group and fixing number of six families of $(3, 6)$-fullerene graphs. Javaid et al. [16] found the fixing number of composition and corona product of two graphs.

The idea of a permutation graph was introduced by Chartrand and Harary [4] for the first time. The authors defined a permutation graph as follows: a permutation graph consists of two identical disjoint copies of a graph G, say G_1 and G_2, along with $|V(G)|$ additional edges joining $V(G_1)$ and $V(G_2)$ according to a given permutation on $\{1, 2, \ldots, |V(G)|\}$. Dorfler [6] defined a mapping graph as follows: a *mapping graph* of a graph G on n vertices consists of two disjoint identical copies of graph G with n additional edges between the vertices of two copies, where the additional edges are defined by a function. The mapping graph was rediscovered and studied by Chen et al. [5], where it was called the functigraph. A functigraph is an extension of a permutation graph. Formally, the functigraph is defined as follows: let G_1 and G_2 be disjoint copies of a connected graph G and let $g : V(G_1) \to V(G_2)$ be a function. A *functigraph* F_G of a graph G consists of the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2) \cup \{uv : g(u) = v\}$. Linda et al. [7, 8] and Kang et al. [20] studied the functigraphs for some graph invariants like metric dimension, domination and zero forcing number. In [11], we have studied the distinguishing number of functigraphs. The aim of this paper is to study the fixing number of functigraphs.

Network science and graph theory are two interconnected research fields that have synonymous structures, problems and their solutions. The notions network and graph are identical and these can be used interchangeably subject to the nature of application. The roads network, railway network, social networks, scholarly networks, etc. are among the examples of networks. In the recent past, the network science has imparted to a functional understanding and the analysis of the complex real world networks. The basic premise in these fields is to relate metabolic networks, proteomic and genomic with disease networks [13] and information cascades in complex networks [15]. Real systems of quite a different nature can have the same network representation. Even though these real systems have different nature, appearance or scope, they can be represented as the same network. Since a functigraph consists of two copies of the same graph (network) with the additional edges described by a function, a mathematical model involving two systems with the same network representation and additional links (edges) between nodes (vertices) of two systems can be represented by a functigraph. The present study is useful in breaking the symmetries (by fixing the minimum number of nodes) of such pair of the same networks (systems) that can be represented by a functigraph.

Throughout the paper, we will denote the functigraph of G by F_G, $V(G_1) = A$, $V(G_2) = B$, $g(V(G_1)) = I$, $|g(V(G_1))| = |I| = s$ and the minimum fixing set of F_G by S^*.

A brief plan of the article is the following. Section 2 provides the study of fixing number of functigraphs. We give sharp lower and upper bounds for fixing number of functigraph. This section also establishes the connections between the fixing number of graphs and their corresponding functigraphs in the form of realizable results. Section 3 provides the fixing number of functigraphs of some well known families of graphs likes complete graphs, trees and join graphs. Some useful results related to these families are also part of this section.

2. Some basic results and bounds

We recall some elementary results about the fixing number which are useful for onward discussion.

Proposition 2.1 ([17]). *Suppose that u, v are twins in a connected graph G and S is a fixing set of G. Then either u or v is in S. Moreover, if $u \in S$ and $v \notin S$, then $(S - \{u\}) \cup \{v\}$ is a fixing set of G.*

Proposition 2.2 ([17]). *Let U be a twin-set of order $m \geq 2$ in a connected graph G. Then every fixing set S of G contains at least $m - 1$ vertices of U.*

Proposition 2.3 ([2, 9]). *If $S \subseteq V(G)$ is a resolving set of G, then S is a fixing set of G.*

Theorem 2.4 ([8]). *Let G be a connected graph of order $n \geq 3$, and let $g : A \to B$ be a function. Then $2 \leq \beta(F_G) \leq 2n - 3$ ($\beta(F_G)$ is the metric dimension of F_G). Both bounds are sharp.*

The sharp lower and upper bounds on the fixing number of functigraphs are given in the following result.

Proposition 2.5. *Let G be a connected graph of order $n \geq 3$, and let $g : A \to B$ be a function. Then $0 \leq fix(F_G) \leq 2n - 3$. Both bounds are sharp.*

Proof. Obviously, $0 \leq fix(F_G)$ by definition. The upper bound follows from Proposition 2.3 and Theorem 2.4. Hence, $0 \leq fix(F_G) \leq 2n - 3$. For the sharpness of the lower bound, take $G = P_3$ and $g : A \to B$ be a function such that $v_1 = g(u_i), i = 1, 2$ and $v_3 = g(u_3)$. For the sharpness of the upper bound, take $G = K_n$, the complete graph of order $n \geq 3$, and let $g : A \to B$ be defined by $v_1 = g(u_i)$ for each i, where $1 \leq i \leq n$. Hence, $fix(F_G) = 2n - 3$ and the proof is complete. □

A connected graph G is called *symmetric* if $fix(G) \neq 0$. By using Proposition 2.5, we have the following result.

Proposition 2.6. *Let G be a symmetric connected graph. Then $1 \leq fix(G) + fix(F_G) \leq 3n - 4$. Both bounds are sharp.*

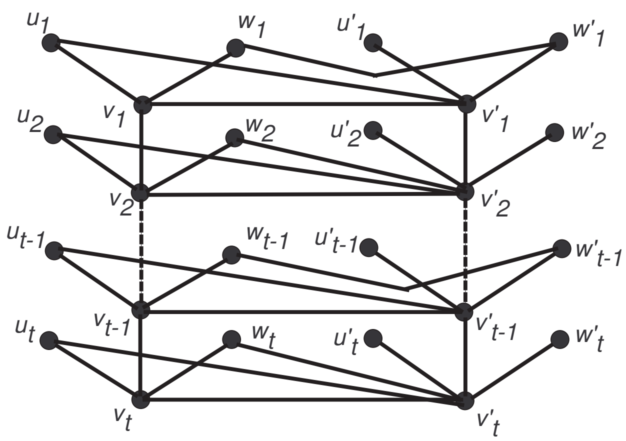

FIGURE 1. The graph with $fix(G) = t = fix(F_G)$.

Lemma 2.7. *For any integer $t \geq 2$, there exist a connected graph G and a function g such that $fix(G) = t = fix(F_G)$.*

Proof. Construct a graph G as follows: let $P_t : v_1v_2 \cdots v_t$ be a path. Join two pendant vertices u_i, w_i with each v_i, where $1 \leq i \leq t$. This completes construction of G. Note that, one of the vertexs from each pair of pendant vertices belongs to a fixing set of G with the minimum cardinality, and hence $fix(G) = t$. Now, we label the corresponding vertices of B as v'_i, u'_i, w'_i for all i, where $1 \leq i \leq t$ and construct functigraph F_G as follows: if t is even, then define $g : A \to B$ as $v'_i = g(v_i)$ for all i, where $1 \leq i \leq t$; $v'_i = g(u_i), w'_i =$

$g(w_i)$ for all $i = 2k+1$, where $0 \leq k \leq \frac{t}{2} - 1$; and $v'_i = g(u_i) = g(w_i)$ for all $i = 2k$, where $1 \leq k \leq \frac{t}{2}$ as shown in Figure 1. Now, consider the set $S^* = \{u_i, u'_i;\ i = 2k,\ 1 \leq k \leq \frac{t}{2}\}$. Note that, $\Gamma_{S^*}(F_G)$ is trivial, and hence S^* is a fixing set of F_G. Thus, $fix(F_G) \leq t$. Moreover, $N(u_i) = N(w_i)$ and $N(u'_i) = N(w'_i)$ for all $i = 2k$, where $1 \leq k \leq \frac{t}{2}$. Thus, we have twin-sets $\{u_i, w_i\}$, $\{u'_i, w'_i\}$ for all $i = 2k$, where $1 \leq k \leq \frac{t}{2}$. By Proposition 2.2, at least one element from these t twin-sets must belongs to every fixing set of F_G. This implies that $fix(F_G) \geq t$. Hence, $fix(F_G) = t$. If t is odd, then we define $g : A \to B$ by $v'_i = g(v_i)$ for all i, where $1 \leq i \leq t$; $v'_i = g(u_i), w'_i = g(w_i)$ for all $i = 2k+1$, $0 \leq k \leq \lfloor \frac{t}{2} \rfloor - 1$; $v'_i = g(u_i) = g(w_i)$ for all $i = 2k$, $1 \leq k \leq \lfloor \frac{t}{2} \rfloor$; and $v'_{t-2} = g(u_t)$, $w'_{t-2} = g(w_t)$. Use same steps as for case when t is even and choosing $S^* = \{u_i, u'_i;\ i = 2k,\ \text{where } 1 \leq k \leq \lfloor \frac{t}{2} \rfloor\}$, we note that $fix(F_G) = t$. Hence, $fix(G) = t = fix(F_G)$. □

Let us now discuss a functigraph of graph G as described in proof of Lemma 2.7 and function $g : A \to B$ defined as: if t is even, then $v'_i = g(v_i)$ for all $1 \leq i \leq t$; $v'_{t-1} = g(u_{t-1}) = g(w_{t-1})$; $v'_{t-3} = g(u_t)$, $w'_{t-3} = g(w_t)$; $v'_i = g(u_i), w'_i = g(w_i)$ for all $i = 2k+1$, $0 \leq k \leq \frac{t}{2} - 2$; $v'_i = g(u_i) = g(w_i)$ for all $i = 2k$, $1 \leq k \leq \frac{t}{2} - 1$. Now, if t is odd and $g : A \to B$ is defined by $v'_i = g(v_i)$ for all $1 \leq i \leq t$; $v'_t = g(u_t) = g(w_t)$; $v'_i = g(u_i), w'_i = g(w_i)$ for all $i = 2k+1$, $0 \leq k \leq \lfloor \frac{t}{2} \rfloor - 1$; $v'_i = g(u_i) = g(w_i)$ for all $i = 2k$, $1 \leq k \leq \lfloor \frac{t}{2} \rfloor$. From this construction and using same arguments as in proof of Lemma 2.7, we conclude that $fix(G) = t$ and $fix(F_G) = t+1$. Hence, we have the following result.

Lemma 2.8. *For any two integers $t_1 \geq 2$ and $t_2 = t_1 + 1$, there exist a connected graph G and a function g such that $fix(G) = t_1, fix(F_G) = t_2$.*

Remark 2.9. Let $t_1, t_2 \geq 2$ be any two integers. Then by definition of functigraph, it is not necessary that there always exists a connected graph G such that $fix(G) = t_1, fix(F_G) = t_2$.

Consider an integer $t \geq 2$. For $t = 2$, we take $G = P_3$ and its functigraph F_G, where function $g : A \to B$ is defined as: $v'_i = g(v_i)$ for all i, where $1 \leq i \leq 3$. For $t > 2$, we take graph G same as in proof of Lemma 2.7 by taking path of order $t-1$ and its functigraph F_G, where $g : A \to B$ is defined as: $v'_i = g(v_i)$ for all $1 \leq i \leq t-1$; $v'_{t-2} = g(u_{t-1}), w'_{t-2} = g(w_{t-1})$; $v'_i = g(u_i), w'_i = g(w_i), 1 \leq i \leq t-2$. From this construction, we have the following result which shows that $fix(G) + fix(F_G)$ can be arbitrary large.

Lemma 2.10. *For any integer $t \geq 2$, there exist a connected graph G and a function g such that $fix(G) + fix(F_G) = t$.*

Consider an integer $t \geq 2$. We take graph G by taking path of order $t+1$ and its functigraph F_G as constructed in Lemma 2.10, we have the following result which shows that $fix(G) - fix(F_G)$ can be arbitrary large.

Lemma 2.11. *For any integer $t \geq 2$, there exist a connected graph G and a function g such that $fix(G) - fix(F_G) = t$.*

For any integer $t \geq 2$, we take graph G same as in the proof of Lemma 2.7 by taking path of order t and its functigraph F_G, where $g : A \to B$ is defined as: $v_i' = g(v_i) = g(u_i) = g(w_i)$ for all $1 \leq i \leq t$. From this type of construction, we have the following result which shows that $fix(F_G) - fix(G)$ can be arbitrary large.

Lemma 2.12. *For any integer $t \geq 2$, there exist a connected graph G and a function g such that $fix(F_G) - fix(G) = t$.*

3. The fixing number of functigraphs of some families of graphs

In this section, we give bounds of the fixing number of functigraphs on complete graphs, trees and join graphs. We also characterize complete graphs for every value of s, where $2 \leq s \leq n-2$ such that $fix(G) = fix(F_G)$.

Following result gives the sharp upper and lower bound for fixing number of functigraphs of complete graphs.

Theorem 3.1. *Let $G = K_n$ be the complete graph of order $n \geq 3$, and let $1 < s < n$. Then*

$$2(n-s) - 1 \leq fix(F_G) \leq 2n - s - 3.$$

Proof. We assume $I = \{v_1, v_2, \ldots, v_s\}$ and $n_i = |\{u \in A : v_i = g(u)\}|$ for all i, where $1 \leq i \leq s$. Also, let $j = |\{n_i : n_i = 1, 1 \leq i \leq s\}|$. There are three possible cases for j in a functigraph F_G:

(1) If $j = 0$, then $2 \leq n_i \leq n-2$ for all i, where $1 \leq i \leq s$. Thus, by definitions of K_n and n_i, there are s twin-sets of vertices in A and a twin-set has n_i number of vertices for each i, where $1 \leq i \leq s$. Hence, S^* contains $\sum_{i=1}^{s}(n_i - 1)$ vertices from A. Moreover, B contains $|B \setminus I|$ twin vertices, and hence S^* contains $n-s-1$ vertices from B. Hence, $|S^*| = \sum_{i=1}^{s}(n_i - 1) + (n-s-1) = 2(n-s)-1$.

(2) If $j = 1$, then without loss of generality, we assume that $n_s = 1$. Thus, there are $s-1$ twin-sets of vertices in A and a twin-set has n_i number of vertices for each i, where $1 \leq i \leq s-1$. Thus, S^* contains $\sum_{i=1}^{s-1}(n_i - 1)$ vertices from A and $n-s-1$ vertices from B as in the previous case. Hence, $|S^*| = \sum_{i=1}^{s-1}(n_i - 1) + (n-s-1) = 2(n-s) - 1$.

(3) If $2 \leq j \leq s-1$. Let $N = \{n_1, n_2, \ldots, n_s\}$. We partition the set N into N_1 and N_2, where N_1 contains all those elements of N in which $n_i > 1$ and N_2 contains all those elements of N in which $n_i = 1$. Let $|N_1| = l$, then $j + l = s$ where $j = |N_2|$. We re-index elements of N_1 and N_2 as follows: $N_1 = \{n_1^{(1)}, n_2^{(1)}, \ldots, n_l^{(1)}\}$, $N_2 = \{n_1^{(2)}, n_2^{(2)}, \ldots, n_j^{(2)}\}$ where superscripts shows associations of an element in N_1 or N_2. Now, A contains l twin-sets of vertices and each twin-set has $n_k^{(1)}$ number of vertices for each k, where $1 \leq k \leq l$. Also, remaining j vertices of A are those having exactly j images under g, and hence $j-1$ such vertices must belong to S^*. For otherwise, let $u, u' \in A$ be two such vertices, then there exists an automorphism $(uu')(g(u)g(u'))$ in $\Gamma(F_G)$. Hence, S^* contains $[\sum_{k=1}^{l}(n_k^{(1)} - 1)] + (j-1)$ vertices from A. Again S^* contains

$n-s-1$ vertices from B. Hence, $|S^*| = [\sum_{k=1}^{l}(n_k^{(1)}-1)]+(j-1)+(n-s-1) = 2n-2s+j-2$. $\square$

Corollary 3.2. *Let $G = K_{n\geq 3}$ be the complete graph and $2 < s < n$ in a functigraph of G. If $n-s$ vertices of A have same image under g, then $fix(F_G) = 2n-(s+4)$.*

Corollary 3.3. *Let $G = K_{n\geq 3}$ be the complete graph and $2 < s < n$ in a functigraph of G. If $n-s+1$ vertices of A have the same image under g, then $fix(F_G) = 2n-(s+3)$.*

Corollary 3.4. *Let $G = K_{n\geq 3}$ be the complete graph and $2 < s < n$ in a functigraph of G. If $|g^{-1}(v)| = \frac{n}{s}$ for all $v \in I$, then $fix(F_G) = 2(n-s)-1$.*

Proof. Since there are exactly s twin-sets of vertices of A each of cardinality $\frac{n}{s}$ and one twin-set of B of cardinality $n-s$. Hence, $fix(F_G) = s(\frac{n}{s}-1)+n-s-1 = 2(n-s)-1$. $\square$

Proposition 3.5. *For every pair of integers n and s, where $2 \leq s \leq n-2$, there are exactly $s-1$ complete graphs G such that $fix(G) = fix(F_G)$ for some function g.*

Proof. We claim that for every s where $2 \leq s \leq n-2$, the required $s-1$ complete graphs are $\{K_{s+i+2} : 0 \leq i \leq s-2\}$ by Theorem 3.1. For otherwise if $G \in \{K_{s+i+2} : s-1 \leq i \leq n-(s+2)\}$, then $fix(G = K_{s+i+2}) = s+i+1$ and by Theorem 3.1, $2i+3 \leq fix(F_{G=K_{s+i}}) \leq s+2i+1$. Since $i > s-2$, so $fix(G) \neq fix(F_G)$.

Next, we define those functions $g_i : A \to B$ in functigraph of G, where $G \in \{K_{s+i+2}, 0 \leq i \leq s-2\}$ such that $fix(G) = fix(F_G)$. We discuss the following cases:

(1) For $s = 2$, we have $G = K_4$ and there are two definitions of function g satisfying hypothesis, one is defined as $g^{-1}(v_1) = \{u_j : 1 \leq j \leq 3\}$, $g^{-1}(v_2) = \{u_4\}$. Other definition of g is $g^{-1}(v_j) = \{u_{2j-1}, u_{2j}\}$ for all j, where $1 \leq j \leq 2$.

(2) For $s = 3$, we have $G = K_5, K_6$. In K_5, g is defined as $g^{-1}(v_1) = \{u_j : 1 \leq j \leq 3\}$, $g^{-1}(v_2) = \{u_4, u_5\}$. In K_6, again there are two definitions of g satisfying hypothesis. One definition is $g^{-1}(v_1) = \{u_j : 1 \leq j \leq 3\}$, $g^{-1}(v_2) = \{u_4, u_5\}$, $g(u_6) = v_3$. Other definition of g is $g^{-1}(v_j) = \{u_{2j-1}, u_{2j}\}$ for all j, where $1 \leq j \leq 3$.

(3) For $s = 4$, we have $G = K_6, K_7, K_8$. In K_6, g is defined as $g^{-1}(v_1) = \{u_j : 1 \leq j \leq 3\}$, $g^{-1}(v_j) = \{u_j\}$ for all j, where $4 \leq j \leq 6$. In K_7, $g^{-1}(v_1) = \{u_j : 1 \leq j \leq 3\}$, $g^{-1}(v_2) = \{u_j : 4 \leq j \leq 5\}$, $g(u_j) = v_{j-3}$ for all j, where $6 \leq j \leq 7$. In K_8, again there are two definitions of g. One definition is $g^{-1}(v_1) = \{u_j : 1 \leq j \leq 3\}$, $g^{-1}(v_2) = \{u_4, u_5\}$, $g^{-1}(v_3) = \{u_6, u_7\}$, $g(u_8) = v_4$. Other definition of g is $g^{-1}(v_j) = \{u_{2j-1}, u_{2j}\}$ for all j, where $1 \leq j \leq 4$.

Continuing the same way we have:

(4) For $s = n - 2$, we have $G = K_n, K_{n+1}, \dots, K_{2n-5}, K_{2n-4}$. In K_n, g is defined as $g^{-1}(v_1) = \{u_j : 1 \leq j \leq 3\}$, $g^{-1}(v_j) = \{u_j\}$ for all j, where $4 \leq j \leq n$. In K_{n+1}, $g^{-1}(v_1) = \{u_j : 1 \leq j \leq 3\}$, $g^{-1}(v_2) = \{u_j : 4 \leq j \leq 5\}$, $g(u_j) = v_{j-3}$ for all j, where $6 \leq j \leq n+1$. In K_{n+2}, $g^{-1}(v_1) = \{u_j : 1 \leq j \leq 3\}$, $g^{-1}(v_2) = \{u_j : 4 \leq j \leq 5\}$, $g^{-1}(v_3) = \{u_j : 6 \leq j \leq 7\}$, $g(u_j) = v_{j-3}$ for all j, where $8 \leq j \leq n+2$. Continuing in the similar way till $G = K_{2n-5}$, we can find g by the similar definitions. In K_{2n-4}, again there are two definitions of g. One definition is $g^{-1}(v_1) = \{u_j : 1 \leq j \leq 3\}$, $g^{-1}(v_j) = \{u_{2j}, u_{2j+1}\}$, where $2 \leq j \leq n-3$, $g(u_{2n-4}) = v_{n-2}$. Other definition of g is $g^{-1}(v_j) = \{u_{2j-1}, u_{2j}\}$ for all j, where $1 \leq j \leq n-2$. □

Remark 3.6. For each $2 \leq s \leq n-2$, there are exactly s mappings $g : V(K_n) \rightarrow V(K_n)$ such that $fix(G) = fix(F_G)$.

Remark 3.7. K_4 is the only complete graph such that $fix(G) = fix(F_G)$ for all functions g.

Let e^* be an edge of a connected graph G. Let $G - e^*$ is the graph obtained by deleting edge e^* from graph G. A vertex v of a graph G is called *saturated* if it is adjacent to all other vertices of G.

Theorem 3.8. *Let G be a complete graph of order $n \geq 3$ and $G_i = G - ie^*$ for all i where $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and e^* joins two saturated vertices of the graph G. If g is constant function, then*

$$fix(F_{G_i}) = \begin{cases} 2n - 2i - 3, & \text{if } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1, \\ n - 1, & \text{if } i = \frac{n}{2} \text{ and } n \text{ is even}, \\ 2\lfloor \frac{n}{2} \rfloor - 1, 2\lfloor \frac{n}{2} \rfloor & \text{if } i = \lfloor \frac{n}{2} \rfloor \text{ and } n \text{ is odd.} \end{cases}$$

Proof. We consider the following three cases for i:

(1) For $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$. On deleting edge e^* between two saturated vertices, the two vertices will no longer remain saturated, however these will remain twin. Hence, for each i, where $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$, G_i contains $n - 2i$ saturated vertices and i twin-sets of vertices each of cardinality two. Hence, $fix(G_i) = n - i - 1$. Now if g is constant, then $|S^*| = (n - i - 1) + (n - i - 2) = 2n - 2i - 3$.

(2) If $i = \frac{n}{2}$ and n is even, then G_i contains no saturated vertex and $\frac{n}{2}$ twin-sets of vertices each of cardinality two. Hence, $fix(G_i) = \frac{n}{2}$. Now, if g is constant, then $|S^*| = \frac{n}{2} + (\frac{n}{2} - 1) = n - 1$.

(3) If $i = \lfloor \frac{n}{2} \rfloor$ and n is odd, then G_i contains one saturated vertex and $\lfloor \frac{n}{2} \rfloor$ twin-sets of vertices each of cardinality two. Hence, $fix(G_i) = \lfloor \frac{n}{2} \rfloor$. Suppose that, $u' = g(u_i)$ for all $u_i \in V(G_i)$ where $1 \leq i \leq n$. Now if u' is a twin vertex, then $|S^*| = \lfloor \frac{n}{2} \rfloor + (\lfloor \frac{n}{2} \rfloor - 1) = 2\lfloor \frac{n}{2} \rfloor - 1$. However, if u' ia saturated vertex, then $|S^*| = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2} \rfloor = 2\lfloor \frac{n}{2} \rfloor$. □

From Theorem 3.8, we can establish the sharp bounds for the fixing number of a functigraph of $G_i = G - ie^*$ in the following corollary.

Corollary 3.9. *Let G be a complete graph of order $n \geq 3$ and $G_i = G - ie^*$ for all i where $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and e^* joins two saturated vertices of the graph G. If g is a constant function, then $fix(G) \leq fix(F_{G_i}) \leq fix(F_G)$. Both bounds are sharp.*

Let T be a tree graph and $v \in V(T)$. If $\deg_T(v) = 1$, then v is called a *pendant vertex*. A vertex $v \in T$ that adjacent to a pendant vertex is called a *support vertex*. We denote the total number of pendant vertices in a tree T by $p(T)$. We denote the total number of support vertices in a tree T by $s(T)$.

Proposition 3.10. *If T is a symmetric tree of order $n \geq 2$, then $fix(F_T) \leq 2fix(T)$. This bound is sharp.*

Corollary 3.11. *If P_n is a path of order $n \geq 2$, then $fix(F_{P_n}) \leq 2$. This bound is sharp.*

Theorem 3.12. *Let T be a symmetric tree and F_T be its symmetric functigraph. Then $fix(F_T) = 2|T| - t$, $2 \leq t \leq 3$ if and only if $T = P_2$.*

Proof. If $T = P_2$, then F_T is either C_4 or K_3 with a pendant vertex. Hence, $fix(F_T) = 2|T| - t,\ 2 \leq t \leq 3$. Conversely, suppose that, $fix(F_T) = 2|T| - t,\ 2 \leq t \leq 3$. We discuss the following cases for $s(T)$.

(1) If $s(T) \geq 3$, we partition $V(T)$ into the sets X_1, X_2 and X_3 where $X_1 = \{u \in V(T) : u$ is a pendant vertex of $T\}$, $X_2 = \{u \in V(T) : u$ is a support vertex of $T\}$ and $X_3 = V(T) \setminus \{X_1 \cup X_2\}$. Let $X_4 = X_2 \cup X_3$, then $fix(T) \leq |T| - |X_4| - 1$ and from Proposition 3.10, $fix(F_T) \leq 2[|T| - |X_4| - 1]$ which leads to a contradiction as $|X_4| \geq 3$.

(2) If $s(T) = 2$, then we have the following two subcases:

(a) If $s(T) = p(T)$, then $T = P_{n\geq 2}(n \neq 3)$, and hence $fix(F_{P_n}) \leq 2$. Thus, by hypothesis, $T = P_2$.

(b) If $s(T) \neq p(T)$, then either $p(T) = 1$ or $p(T) > 2$. However, $p(T) = 1$ and $s(T) = 2$ is impossible in a tree, so $p(T) > 2$. This also leads to a contradiction as in Case (1).

(3) If $s(T) = 1$, then $T = K_{1,n}, n \geq 2$, and hence $fix(F_T) \leq 2[|T| - 2]$ which is again a contradiction. □

The following corollary can be proved by using similar arguments as in the proof of Theorem 3.12.

Corollary 3.13. *Let T be a symmetric tree and F_T be its symmetric functigraph.*

(1) *If $fix(F_T) = 2|T| - t$, $4 \leq t \leq 5$, then $T = K_{1,n}, n \geq 2$.*
(2) *If $fix(F_T) = 2|T| - 6$, then $T = K_{1,n}, n \geq 3$.*
(3) *If $fix(F_T) = 2|T| - 7$, then $T \in \{P_4, K_{1,n}, n \geq 3\}$.*
(4) *If $fix(F_T) = 2|T| - 8$, then $T \in \{P_5,\ K_{1,n},\ n \geq 4, (K_{1,n_1}, K_{1,n_2}) + e, n_1, n_2 \geq 2, K_{1,n}, n \geq 2$ and a vertex adjacent with one pendant vertex of $K_{1,n}\}$.*

Suppose that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with disjoint vertex sets V_1 and V_2 and disjoint edge sets E_1 and E_2. The *union* of G_1 and G_2 is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The *join* of G_1 and G_2 is the graph $G_1 + G_2$ that consists of $G_1 \cup G_2$ and all edges joining all vertices of V_1 with all vertices of V_2.

Theorem 3.14 ([10]). *Let G_1 and G_2 be two connected graphs. Then $fix(G_1 + G_2) \geq fix(G_1) + fix(G_2)$. This bound is sharp.*

Proposition 3.15. *Let G_1 and G_2 be two connected graphs and $g : V(G_1 + G_2) \to V(G_1 + G_2)$ be a constant function. Then*

$$fix(F_{G_1+G_2}) = 2fix(G_1 + G_2) - i, \ \ 0 \leq i \leq 1.$$

Proof. Let A and B be two copies of $G_1 + G_2$. Let S_1 and S_2 be minimum fixing sets of A and B, respectively. Define $g : A \to B$ by $u' = g(u)$ for all $u \in A$. We discuss the following two cases:

(1) If G_1 does not contain any saturated vertex, then $G_1 + G_2$ also does not contain any saturated vertex. In this case $fix(A) = fix(B) = fix(G_1) + fix(G_2)$ by Theorem 3.14. Now, if $u' \in S_2$, then $S^* = S_1 \cup \{S_2 \setminus \{u'\}\}$ because $\theta(u') = \{u'\}$ in $F_{G_1+G_2}$. If $u' \notin S_2$, then $S^* = S_1 \cup S_2$.

(2) If G_1 contains any saturated vertex, then both A and B have saturated vertices. In this case $fix(A) = fix(B) > fix(G_1) + fix(G_2)$ by Theorem 3.14. Now if u' is a saturated vertex of B, then $u' \in S_2$, and hence $S^* = S_1 \cup \{S_2 \setminus \{u'\}\}$. If u' is not a saturated vertex, then either $u' \in S_2$ or $u' \notin S_2$. If $u' \in S_2$, then $S^* = S_1 \cup \{S_2 \setminus \{u'\}\}$ and if $u' \notin S_2$, then $S^* = S_1 \cup S_2$. $\square$

References

[1] M. O. Albertson and K. L. Collins, *Symmetry breaking in graphs*, Electron. J. Combin. **3** (1996), no. 1, Research Paper 18, approx. 17 pp.

[2] D. L. Boutin, *Identifying graph automorphisms using determining sets*, Electron. J. Combin. **13** (2006), no. 1, Research Paper 78, 12 pp.

[3] J. Bradley, P. Dankelmann, D. Erwin, and B. G. Rodrigues, *On the fixing number of graphs with abelian automorphism group*, Util. Math. **86** (2011), 71–78.

[4] G. Chartrand and F. Harary, *Planar permutation graphs*, Ann. Inst. H. Poincaré Sect. B (N.S.) **3** (1967), 433–438.

[5] A. Chen, D. Ferrero, R. Gera, and E. Yi, *Functigraphs: an extension of permutation graphs*, Math. Bohem. **136** (2011), no. 1, 27–37.

[6] W. Dörfler, *On mapping graphs and permutation graphs*, Math. Slovaca **28** (1978), no. 3, 277–288.

[7] L. Eroh, R. Gera, C. X. Kang, C. E. Larson, and E. Yi, *Domination in functigraphs*, Discuss. Math. Graph Theory **32** (2012), no. 2, 299–319. https://doi.org/10.7151/dmgt.1600

[8] L. Eroh, C. X. Kang, and E. Yi, *On metric dimension of functigraphs*, Discrete Math. Algorithms Appl. **5** (2013), no. 4, 1250060, 13 pp. https://doi.org/10.1142/S1793830912500607

[9] D. Erwin and F. Harary, *Destroying automorphisms by fixing nodes*, Discrete Math. **306** (2006), no. 24, 3244–3252. https://doi.org/10.1016/j.disc.2006.06.004

[10] M. Fazil and I. Javaid, *On the fixatic number of graphs*, Accepted for publication in Ars. Combin.

[11] M. Fazil, M. Murtaza, Z. Ullah, U. Ali, and I. Javaid, *On the distinguishing number of functigraphs*, Symmetry **10** (2018).
[12] C. R. Gibbons and J. D. Laison, *Fixing numbers of graphs and groups*, Electron. J. Combin. **16** (2009), no. 1, Research Paper 39, 13 pp.
[13] M. Gosak, R. Markovi, J. Dolenek, M. S. Rupnik, M. Marhl, A. Stoer, and M. Perc, *Network science of biological systems at different scales*, A review, Phys. Life Rev. **24** (2018), 118–135.
[14] F. Harary, *Methods of destroying the symmetries of a graph*, Bull. Malays. Math. Sci. Soc. (2) **24** (2001), no. 2, 183–191 (2002).
[15] M. Jalili and M. Perc, *Information cascades in complex networks*, J. Complex Netw. **5** (2017), no. 5, 665–693. https://doi.org/10.1093/comnet/cnx019
[16] I. Javaid, M. S. Aasi, I. Irshad, and M. Salman, *On fixing sets of composition and corona product of graphs*, Util. Math. **109** (2018), 17–28.
[17] I. Javaid, M. Fazil, U. Ali, and M. Salman, *On some parameters related to fixing sets in graphs*, J. Prime Res. Math. **14** (2018), 1–12.
[18] K. Lynch, *Determining the orientation of a painted sphere from a single image*, A graph coloring problem. Preprint.
[19] F. K. Moftakhar, A. R. Ashrafi, Z. Mehranian, and M. Ghorbani, *Automorphism group and fixing number of* $(3,6)$*- and* $(4,6)$*-fullerene graphs*, Electron. Notes in Disc. Math. **45** (2014), 113–120.
[20] D. D. Row, *Zero forcing number: Results for computation and comparison with other graph parameters*, ProQuest LLC, Ann Arbor, MI, 2011.

Muhammad Fazil
Centre for Advanced Studies in Pure and Applied Mathematics
Bahauddin Zakariya University Multan
Multan 60800, Pakistan
Email address: mfazil@bzu.edu.pk

Imran Javaid
Centre for Advanced Studies in Pure and Applied Mathematics
Bahauddin Zakariya University Multan
Multan 60800, Pakistan
Email address: imran.javaid@bzu.edu.pk

Muhammad Murtaza
Centre for Advanced Studies in Pure and Applied Mathematics
Bahauddin Zakariya University Multan
Multan 60800, Pakistan
Email address: mahru830@gmail.com