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A PROOF OF THE CONJECTURE OF MAZUR-RUBIN-STEIN

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Abstract. We present a concise proof of the conjecture of Mazur-Rubin-Stein on the distribution of modular symbols.

1. Introduction

Let f be a cusp form of a level N and weight 2. For $r \in \mathbb{Q} \cap (0,1)$, one defines period integrals

$$m_{\pm}(r) = \int_{r}^{i\infty} 2\pi i f(z)dz \pm \int_{-r}^{i\infty} 2\pi i f(z)dz.$$

Developing a conjecture on the Diophantine stability, Mazur and Rubin [6] establish heuristics on the distribution of period integrals $m_{\pm}(r)$ for a new form and propose several conjectures:

(1) The random variable $m_{\pm}$ on the rationals with the fixed denominator M, is asymptotically Gaussian.
(2) For a divisor g of N, there exist constants C_f and $D_{f,g}$ called the *variance slope* and the *variance shift*, respectively, such that the difference between variance of m_+ and $C_f \log M$ converges to $D_{f,g}$ when g is the G.C.D. of M and N.
(3) The integer-valued random variable $m_{\pm}/\Omega_f^{\pm}$ for suitable periods $\Omega_f^{\pm}$ is equi-distributed modulo p.

Petridis-Risager [7] prove average versions of (1) and (2) using a theory of Eisenstein series whose coefficients are the moments of period integrals. Even more, they give explicit expressions for the variance slopes C_f and the shifts $D_{f,g}$ in terms of special values of the symmetric square L-function of f. Lee-Sun [5] also presents a proof of the average version of the conjectures including the statement (3) by studying the dynamics of continued fractions. Using a

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theory of shifted convolution, Blomer *et al.* [1] obtain the second moment of $m_\pm$, i.e., the statement (2).

The above conjecture of Mazur-Rubin implies that the period integrals are distributed with a certain regularity. Therefore, Mazur, Rubin, and Stein [6] propose another conjecture:

Conjecture A (Mazur-Rubin-Stein)**.** *For* $0 < x \leq 1$*, one has*

$$\lim_{M\to\infty} \frac{1}{M} \sum_{r \leq Mx} m_\pm\left(\frac{r}{M}\right) = \sum_{n=1}^{\infty} \frac{a_n(f)e_\pm(2nx)}{n^2},$$

where $e_\pm$ *is given by*

$$e_\pm(x) = \int_0^x \exp(2\pi i t) \pm \exp(-2\pi i t) dt.$$

In this paper, we prove:

Theorem 1.1. *Let* N *be square-free. For any* $\epsilon > 0$ *and* $0 \leq x \leq 1$*, we have*

$$\frac{1}{M} \sum_{1 \leq r \leq Mx} m_\pm\left(\frac{r}{M}\right) = \sum_{n=1}^{\infty} \frac{a_n(f)e_\pm(2nx)}{n^2} + O_{f,\epsilon}\left(\frac{N^{1/2+\epsilon}}{M^{1/4-\epsilon}}\right), \tag{1.1}$$

where the implicit constant in the error term depends only on ϵ *and* f; *and independent of* f *if* f *is a newform.*

The condition on N originates from the functional equation of the L-functions. During preparation of the manuscript, N. Diamantis informed us that a proof of Theorem 1.1 even for a general level is obtained by Diamantis-Hoffstein-Kiral-Lee [2] using the functional equation for general levels. Let us remark that even though the level is limited, a virtue of our paper is the brevity of the proof. The role of smooth approximation of bump functions in Diamantis-Hoffstein-Kiral-Lee [2], is played by Lemma 3.1 in present paper.

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2. Approximate functional equations of additive twists

This section is a summary of Kim-Sun [4, Section 2].

Let $f \in S_{2k}(N, \delta)$ for a Nebentypus δ. Let $W_N = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$ be the Fricke involution. Note that $f|W_N \in S_{2k}(N, \overline{\delta})$. For $x \in \mathbb{Q}$, let us set

$$\mathrm{t}(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Let $\mathbf{e}(x) = \exp(2\pi i x)$. For $f(z) = \sum_{n\geq 1} a_n \mathbf{e}(nz)$ and $s \in \mathbb{C}$ with $\Re(s) > k + 1/2$, the partial L-function is given by

$$L(s, f, x) := \sum_{n\geq 1} \frac{a_n(f)\mathbf{e}(nx)}{n^s}.$$

It can be easily verified that

$$L(1, f, x) = -2\pi i \int_x^{i\infty} f(z)dz. \tag{2.1}$$

Let $q > 0$ be an integer that is not necessarily prime to N. Let Q be the least common multiple of N and q^2. Let us set $d = \gcd(q^2, N)$, $N_0 = \frac{N}{d}$ and assume that

$$\gcd(N_0, d) = 1.$$

Hence $Q = \frac{q^2 N}{d}$. Let δ be decomposed as

$$\delta = \delta_1\delta_2$$

corresponding to $(\mathbb{Z}/N\mathbb{Z})^\times \simeq (\mathbb{Z}/N_0\mathbb{Z})^\times \times (\mathbb{Z}/d\mathbb{Z})^\times$. For $x, y \in \mathbb{Z}$ with $xd - \frac{N}{d}y = 1$, let us set

$$\mathrm{W}_d = \begin{pmatrix} dx & y \\ N & d \end{pmatrix}.$$

The matrix W_d is a normalizer of $\Gamma_0(N)$. Let us set

$$\mathrm{W}_{N,d} = \mathrm{W}_N \mathrm{W}_d.$$

Then $\mathrm{W}_{N,d}$ commutes with the Hecke operators $\mathrm{T}(n)$ when $\gcd(n, N) = 1$ and $f\big|\mathrm{W}_{N,d} \in S_{2k}\big(N, \overline{\delta}_1\delta_2\big)$. Furthermore, we have $f\big|\mathrm{W}_{N,d}^2 = \delta(dx - N_0y^2)f = \overline{\delta}_2(-N_0)f$. Note that if f is a newform, then $f|\mathrm{W}_{N,d} = \zeta f$ for a $\zeta \in \mathbb{C}$ with $\zeta^2 = \overline{\delta}_2(-N_0)$.

Let Φ be an infinitely differentiable function on $(0, \infty)$ with compact support and $\int_0^\infty \Phi(y)\frac{dy}{y} = 1$, and set $\kappa(t) = \int_0^\infty \Phi(y)y^t\frac{dy}{y}$. Let us set

$$F_{1,s}(x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \kappa(t)\Gamma(s+t)x^{-t}\frac{dt}{t} \text{ and}$$
$$F_{2,s}(x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \kappa(-t)\Gamma(s+t)x^{-t}\frac{dt}{t}.$$

With those settings, we have the asymptotics of $F_{i,s}$ and the approximate functional equation of which proofs are given in Kim-Sun [4, (2.5), (2.6), (2.7)]:

Proposition 2.1. (1) *For each i, we have*

$$F_{i,s}(x) = O(\Gamma(\Re(s) + j)x^{-j}) \textit{ for all } j \geq 1 \textit{ as } x \to \infty. \tag{2.2}$$

$$F_{i,s}(x) = \Gamma(s) + O\Big(\Gamma\Big(\Re(s) - \frac{1}{2}\Big)x^{\frac{1}{2}}\Big) \textit{ as } x \to 0. \tag{2.3}$$

(2) *For an integer* $a>0$ *with* $\gcd(a,q)=1$*, choose an integer* $u>0$ *such that* $u\equiv -(aN_0)^{-1} \,(\bmod\ \frac{q^2}{d})$*. Then* $L(s,f,\frac{a}{q})$ *satisfies the approximate functional equation*

(2.4)

$$\begin{aligned}&\Gamma(s)L\Big(s,f,\frac{a}{q}\Big)\\ =&\sum_{n=1}^{\infty}\frac{a_n(f)\mathbf{e}(\frac{an}{q})}{n^s}F_{1,s}\left(\frac{n}{y}\right)\\ &+i^{2k}\delta_1(q)\delta_2(N_0)\Big(\frac{Q}{4\pi^2}\Big)^{k-s}\sum_{n=1}^{\infty}\frac{a_n(f\big|\mathrm{W}_{N,d})\mathbf{e}(\frac{un}{q})\delta_2(u)}{n^{2k-s}}F_{2,2k-s}\Big(\frac{4\pi^2ny}{Q}\Big),\end{aligned}$$

where $y>0$ *is a real number.*

3. Proof of Theorem 1.1

Let f be a cusp form of weight 2 and a square-free level N with a Nebentypus δ. Note that we have

$$e_+(x)=\frac{\sin(2\pi x)}{\pi}\ \text{and}\ e_-(x)=\frac{i(1-\cos(2\pi x))}{\pi}.$$

For integers $n>0$ and $M\geq 2$ and $0<x\leq 1$, let us set

$$U(x,M;n):=\frac{1}{M}\sum_{r=1}^{Mx}\left\{\mathbf{e}\left(\frac{rn}{M}\right)\pm\mathbf{e}\left(\frac{-rn}{M}\right)\right\}.$$

We need an estimate on the sum.

Lemma 3.1. *Let* $M\geq 2$ *and* $n\geq 1$ *be integers. Let* $[n]$ *be the least positive residue of* n *modulo* M *and assume that* n *is not divisible by* M*. Then for a real* x *with* $0\leq x\leq 1$ *we have*

$$U(x,M;n)=\frac{e_\pm([n]x)}{[n]}+O\left(\frac{1}{M}\right).$$

Proof. We may assume that $n<M$. Since $U(x,M;n)$ is a geometric series, it is equal to

$$\frac{\mathbf{e}(\frac{n}{M})(\mathbf{e}(\frac{\lfloor Mx\rfloor n}{M})-1)\mp(\mathbf{e}(-\frac{\lfloor Mx\rfloor n}{M})-1)}{M(\mathbf{e}(\frac{n}{M})-1)}.$$

Here note that $[n]=n$ as $n<M$. Since $\lfloor Mx\rfloor/M=x-\theta/M$ with $0\leq\theta<1$, the last expression is equal to

$$\frac{(\mathbf{e}(xn)-1)\mp(\mathbf{e}(-xn)-1)+O(\frac{n}{M})}{2\pi in+O(\frac{n^2}{M})}+O\left(\frac{1}{M}\right).$$

Hence we finish the proof. ∎

We need an estimate on an exponential sum that can be obtained from one on the Kloosterman sum (see Heath-Brown [3]):

Proposition 3.2. *Let I be a sub-interval of $[0,1)$. Let a, b, and $M \geq 2$ be integers. Then one has*

$$\sum_{\frac{r}{M}\in I}{}' \mathbf{e}\left(\frac{ar+br'}{M}\right) \ll \gcd(a,b,M)^{1/2} M^{1/2+\epsilon},$$

where Σ' is the sum over the integers $1 \leq r \leq M$ with $\gcd(r,M)=1$.

For an integer M and Dirichlet character δ_2 of modulus $N/\gcd(M^2,N)$, let us set

$$V(x,M;n) = \sum_{\substack{r\leq Mx \\ \gcd(r,M)=1}} \delta_2(r')\left\{\mathbf{e}\left(\frac{r'n}{M}\right) \pm \mathbf{e}\left(\frac{-r'n}{M}\right)\right\},$$

where r' is the inverse of r modulo M. We also need:

Lemma 3.3. *For two integers $M \geq 2$, $n \geq 1$ and a real $0 \leq x \leq 1$, we have*

$$V(x,M;n) \ll N^{1/2}\gcd(n,M)^{1/2}M^{1/2+\epsilon}.$$

Proof. The case of $M \mid n$ is obvious. Let us assume $M \nmid n$ and set $N_0 = N/\gcd(M^2,N)$, $M = N_0M_0$. Since δ_2 is a periodic function of a period N_0, it can be written as

$$\delta_2(s) = \sum_{j=1}^{N_0} c_j \mathbf{e}\left(\frac{sjM_0}{M}\right)$$

form some c_j with $\sum_{j=1}^{N_0}|c_j|^2 \leq 1$. Then, we obtain

$$V(x,M;n) = \sum_{j=1}^{N_0} c_j \sum_{\substack{r\leq Mx \\ \gcd(r,M)=1}} \left\{\mathbf{e}\left(\frac{r'(n+jM_0)}{M}\right) \pm \mathbf{e}\left(\frac{-r'(n-jM_0)}{M}\right)\right\}.$$

From Cauchy-Schwartz inequality and Proposition 3.2, we finish the proof. □

Weil and Deligne have shown that for a cusp form f and $\epsilon > 0$, there exists a constant $b_f > 0$ dependent only on f such that

$$a_n(f) \ll_\epsilon b_f n^{1/2+\epsilon}. \tag{3.1}$$

Note that if f is a newform, then b_f is independent of f and can be chosen as $b_f = 1$.

We are ready to give:

Proof of Theorem 1.1. First note that

$$\sum_{r\leq Mx} m_\pm\left(\frac{r}{M}\right) = \sum_{g\mid M}\sum_{\substack{s\leq gx \\ \gcd(g,s)=1}} m_\pm\left(\frac{s}{g}\right). \tag{3.2}$$

From (2.1), (2.4), and (3.2), we have

$$\begin{aligned}(3.3)\qquad &\frac{1}{M}\sum_{r=1}^{Mx} m_\pm\left(\frac{r}{M}\right)\\ &= \sum_{n=1}^{\infty}\frac{a_n(f)U(x,M;n)}{n}F_{1,1}\left(\frac{n}{y}\right)\\ &\quad+\frac{i^{2k}}{M}\sum_{g|M}\delta_1(g)\delta_2(N/e)\sum_{n=1}^{\infty}\frac{a_n(f\big|\mathrm{W}_{N,e})V(x,g;n)}{n}F_{2,1}\left(\frac{4\pi^2 ny}{Ng^2/e}\right),\end{aligned}$$

where $e=\gcd(g^2,N)$ and δ_1, δ_2 in the sum $\sum_{g|M}$ are the Dirichlet characters of moduli N/e and e, respectively. Here note that e and N/e are relatively prime since N is square-free. Let us rewrite (3.3) as

$$S_1+\frac{1}{M}\sum_{g|M}\delta_1(g)\delta_2(N/e)S_2(g).$$

First of all, consider the first sum S_1. We split it into two parts,

$$S_1=S_{1,\le y}+S_{1,>y},$$

a sum over $n\le y$ and one over $n>y$, respectively. Observe that by Lemma 3.1 and the estimate $F_{1,1}(x)=O(1)$, the sum over $n\le y$ is equal to

$$\sum_{\substack{M\nmid n\\ n\le y}}\frac{a_n(f)e_\pm([n]x)}{[n]n}F_{1,1}\left(\frac{n}{y}\right)+O\bigg(\frac{1}{M}\sum_{\substack{n\le y\\ M\nmid n}}\frac{|a_n(f)|}{n}\bigg)+O\bigg(\sum_{\substack{n\le y\\ M|n}}\frac{|a_n(f)|}{n}\bigg).$$

By (2.3) and the bound (3.1), this is equal to

$$(3.4)\qquad \sum_{\substack{n\le y\\ M\nmid n}}\frac{a_n(f)e_\pm([n]x)}{[n]n}+O\bigg(b_f\sum_{\substack{n\le y\\ M\nmid n}}\frac{n^{1/2+\epsilon}}{n[n]}\left(\frac{n}{y}\right)^{1/2}\bigg)+O\bigg(\frac{b_f y^{1/2+\epsilon}}{M}\bigg).$$

Observe that for a real number $b>0$ and $y>M$, we obtain

$$\sum_{\substack{n\le y\\ M\nmid n}}\frac{n^b}{[n]}=\sum_{r=1}^{M-1}\frac{1}{r}\sum_{\substack{n\le y\\ n\equiv r(M)}}n^b\ll\sum_{r=1}^{M-1}\frac{1}{r}\sum_{q<y/M}(Mq+r)^b\ll(\log M)M^b\left(\frac{y}{M}\right)^{1+b}.$$

Therefore, with $b=\epsilon$, the error terms of (3.4) are equal to $O\left(\frac{b_f(\log M)y^{1/2+\epsilon}}{M}\right)$. Separating the first summand of (3.4) into three parts, namely (1) $n<M$, (2) $M<n<y$ with $M\nmid n$, and (3) $M<n<y$ with $M\mid n$, the first summand of (3.4) equals

$$\sum_{n<M}\frac{a_n(f)e_\pm(nx)}{n^2}+O\bigg(\sum_{\substack{M<n\le y\\ M\nmid n}}\frac{b_f}{n^{1/2-\epsilon}[n]}\bigg)+O\bigg(\frac{b_f y^{1/2+\epsilon}}{M}\bigg)$$

$$= \sum_{n<M} \frac{a_n(f)e_\pm(nx)}{n^2} + O\left(\frac{b_f(\log M)y^{1/2+\epsilon}}{M}\right).$$

Hence, finally, (3.4) is equal to

$$S_{1,\le y} = \sum_{n<M} \frac{a_n(f)e_\pm(nx)}{n^2} + O\left(\frac{b_f(\log M)y^{1/2+\epsilon}}{M}\right).$$

Note that the sum over $n > y$ in the first sum of (3.3) also can be calculated in a similar way but using (2.2) instead of (2.3). It is equal to

$$S_{1,>y} = \sum_{n>y} \frac{a_n(f)U(M,x;n)}{n} F_{1,1}\left(\frac{n}{y}\right) = O\left(\frac{b_f y^{1/2+\epsilon}\log M}{M}\right).$$

Now let us consider the sum $S_2(g)$. We also divide it into two parts:

$$S_2(g) = S_{2,\le y}(g) + S_{2,>y}(g),$$

a sum over $n \le Ag^2/y$ and one over $n > Ag^2/y$, respectively for $A = N/(4\pi^2 e)$. By Lemma 3.3 and (2.3), we obtain $V(x,g;n) \ll N^{1/2}\gcd(g,n)^{1/2}g^{1/2+\epsilon}$ and the sum over $n \le Ag^2/y$ is

$$S_{2,\le y}(g) \ll b_f N^{1/2} g^{1/2+\epsilon} \sum_{n\le Ag^2/y} \frac{\gcd(n,g)^{1/2}}{n^{1/2-\epsilon}}.$$

The last sum is equal to

$$\sum_{d|g} d^\epsilon \sum_{m\le Ag^2/dy, \gcd(m,g/d)=1} \frac{1}{m^{1/2-\epsilon}} \ll \left(\frac{Ag^2}{y}\right)^{1/2+\epsilon} \sum_{d|g} \frac{1}{d^{1/2}}.$$

Since $\sum_{d|g} d^{-1/2} \ll g^\epsilon$, the sum $S_{2,\le y}(g)$ is equal to

$$S_{2,\le y}(g) = O\left(\frac{N^{1/2} b_f A^{1/2+\epsilon} g^{3/2+3\epsilon}}{y^{1/2+\epsilon}}\right).$$

In a similar way as the last calculations together with (2.2), the sum over $n > Ag^2/y$ is equal to

$$S_{2,>y}(g) = O\left(\frac{N^{1/2} b_f A^{1/2+\epsilon} g^{3/2+3\epsilon}}{y^{1/2+\epsilon}}\right).$$

Therefore, from the inequality $\sum_{g|M} g^a \ll M^{a+\epsilon}$ for $a > 0$, we obtain

$$\frac{1}{M}\sum_{g|M} |S_2(g)| \ll \frac{N^{1/2} b_f}{M} \sum_{g|M} \frac{A^{1/2+\epsilon} g^{3/2+3\epsilon}}{y^{1/2+\epsilon}} \ll \frac{b_f N^{1+\epsilon} M^{1/2+4\epsilon}}{y^{1/2+\epsilon}}.$$

In total, setting

$$y = N^{\frac{1+\epsilon}{1+2\epsilon}} M^{\frac{3/2+3\epsilon}{1+2\epsilon}},$$

we complete the proof with a new $\epsilon > 0$. □

Let $\mathbb{E}_M$ be the average on the rationals in $(0,1)$ with the denominator M.

Corollary 3.4. *Let N be square-free. For $\epsilon > 0$, we have*

$$\mathbb{E}_M[m_\pm] \ll_{f,\epsilon} N^{1/2+\epsilon} M^{-1/4+\epsilon},$$

where the implicit constant in the error term is independent of f if f is a newform.

Proof. Let us set

$$G^\pm(f;x) = \sum_{n=1}^{\infty} \frac{a_n(f) e_\pm(nx)}{n^2}.$$

From Möbius inversion formula, we obtain

$$\phi(M)\mathbb{E}_M[m_\pm] = \sum_{\substack{r=1 \\ (r,M)=1}}^{M} m_\pm\left(\frac{r}{M}\right) = \sum_{d|M} \mu\left(\frac{M}{d}\right) \sum_{s=1}^{d} m_\pm\left(\frac{s}{d}\right).$$

By Theorem 1.1, this equals

$$\sum_{d|M} \mu\left(\frac{M}{d}\right) \left[dG^\pm(f;1) + O\left(\frac{b_f d N^{1/2+\epsilon}}{d^{1/4-\epsilon}}\right)\right].$$

Since $\phi(M) \gg M/\log M$, $q^{3/4+\epsilon} + 1 \ll_\epsilon q^{3/4+2\epsilon}$, and

$$\sum_{d|M} \left|\mu\left(\frac{M}{d}\right)\right| d^{3/4+\epsilon} = \prod_{\substack{q|M \\ q:prime}} \left(1 + q^{3/4+\epsilon}\right) \ll_\epsilon M^{3/4+2\epsilon},$$

we obtain the proof of the Corollary with a new $\epsilon > 0$. □

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