Bull. Korean Math. Soc. **58** (2021), No. 1, pp. 147–161
https://doi.org/10.4134/BKMS.b200128
pISSN: 1015-8634 / eISSN: 2234-3016

SA-SUPPLEMENT SUBMODULES

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Abstract. In this paper, we introduced and studied sa-supplement submodules. A submodule U of a module V is called an sa-supplement submodule in V if there exists a submodule T of V such that $V = T + U$ and $U \cap T$ is semiartinian. The class of sa-supplement sequences $\mathcal{SAS}$ is a proper class which is generated by socle-free modules injectively. We studied modules that have an sa-supplement in every extension, modules whose all submodules are sa-supplement and modules whose all sa-supplement submodules are direct summand. We provided new characterizations of right semiartinian rings and right SSI rings.

All rings considered in this paper will be associative with an identity element. Unless otherwise stated, R denotes an arbitrary ring and all modules will be right unitary R-modules. For a module M, by $X \leq M$ we mean X is a submodule of M or M is an extension of X. For a module M, the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+. We use the notation $E(M)$, $soc(M)$, $Rad(M)$, $Z(M)$ for the injective hull, socle, radical, singular submodule of a module M respectively. For all other basic or background material, we refer the reader to [7, 24].

A large number of rings have characterizations in terms of homological algebra. For example, R is semisimple if and only if every short exact sequence of right (or left) R-modules splits if and only if every short exact sequence of right (or left) R-modules closed; R is artinian serial with $J^2(R) = 0$ if and only if every closed exact sequence of right (or left) R-modules splits; R is right perfect if and only if every short exact sequence of right R-modules supplement; R is von Neumann regular if and only if every short exact sequence of right (or left) R-modules is pure. The classes of all short exact sequences, pure short exact sequences, closed short exact sequences, split short exact sequences and supplement short exact sequences have some common properties. Based on these properties, axioms of the notion of proper classes were determined.

Received February 9, 2020; Revised May 25, 2020; Accepted September 15, 2020.
2010 *Mathematics Subject Classification.* Primary 16D10, 18G25.
Key words and phrases. Proper class of short exact sequences, sa-supplement submodule, sa-supplementing modules, semiartinian modules.
This work was financially supported by the Scientific and Technological Research Council of Turkey (TUBITAK) (Project number: 119F176).

Proper classes were introduced by Buchsbaum in [4] for an exact category. We use the axioms given by Mac Lane in [16] for abelian categories. A proper class $\mathcal{P}$, in an abelian category $\mathcal{A}$, defines a closed subbifunctor of the $\mathrm{Ext}^1_{\mathcal{A}}$ functor, i.e.,

$$\mathrm{Ext}^1_{\mathcal{P}}(-,-) \subseteq \mathrm{Ext}^1_{\mathcal{A}} : A^{op} \times A \to Ab;$$

where Ab denotes the category of abelian groups, and for $A, C \in \mathcal{A}$, $\mathrm{Ext}^1_{\mathcal{A}}(C, A)$ denotes the class of isomorphism classes of short exact sequences $0 \to A \to B \to C \to 0$. $\mathrm{Ext}^1_{\mathcal{A}}(C, A)$ together with the Baer sum is an abelian group and for any proper class $\mathcal{P}$, $\mathrm{Ext}^1_{\mathcal{P}}(C, A)$ is a subgroup (see [16]). These ideas were the starting point of relative homological algebra. Proper classes are investigated in [17, 18] for abelian groups, and in [11, 21] for module categories.

Pure submodules, supplement submodules and closed submodules have offered rich topics of research due to their important roles played in ring and module theory and relative homological algebra. These submodules respectively induce proper classes. This was noted for abelian groups in [12] and for modules in [11, 22]. Various generalizations of these proper classes have been considered. Recent examples of these proper classes of short exact sequences on module categories include FC-pure and I-pure ([3]), weak supplement ([1]), neat and coneat ([10]) short exact sequences. Moreover, through torsion theories, the investigations of proper classes were further developed in [2, 8, 9, 23].

Let $\mathcal{P}$ be a proper class. An R-module M is said to be an $\mathcal{P}$-*projective* (resp., $\mathcal{P}$-*injective*) if it is projective (resp., injective) with respect to all short exact sequences in $\mathcal{P}$. A module M is called $\mathcal{P}$-*coprojective* (resp., $\mathcal{P}$-*coinjective*) if every short exact sequence of the form $0 \to A \to B \to M \to 0$ (resp., $0 \to M \to B \to C \to 0$) is in $\mathcal{P}$. Let $\mathcal{M}$ and $\mathcal{J}$ be classes of modules over some ring R. The smallest proper class $\overline{k}(\mathcal{M})$ such that all modules in $\mathcal{M}$ are $\overline{k}(\mathcal{M})$-coprojective is said to be *coprojectively generated* by $\mathcal{M}$. The proper class $\underline{k}(\mathcal{J})$ which is *coinjectively generated* by $\mathcal{J}$ is defined dually. A module L is called $\mathcal{P}$-*regular*, if every exact sequence $0 \to A \to L \to M \to 0$ is in $\mathcal{P}$. Note that if $\mathcal{P}$ is the proper class of pure short exact sequences, then $\mathcal{P}$-coprojective, $\mathcal{P}$-coinjective and $\mathcal{P}$-regular modules are called as flat, fp-injective and regular modules, respectively (see [21, 24]). Moreover, if $\mathcal{P}$ is the proper class of supplement short exact sequences, then $\mathcal{P}$-regular modules and $\mathcal{P}$-coinjective modules are called as supplemented modules and supplementing modules, respectively (see [7, 24]). For more detail on homological objects of proper classes, we refer to [15, 17, 21, 24].

The notion of a supplement submodule was introduced by Kach and Mares in [14] in order to characterize semiperfect modules, that is projective modules whose factor modules have projective covers. A submodule N of M is called *small* in M if $M \neq N + K$ for every proper submodule K of M. Let M be a module and N, K be submodules of M. Then K is said to be a *supplement* of N in M or N is said to have a supplement K in M if $N + K = M$ and $N \cap K$ is small in K. A module M is called *supplemented* if every submodule of M has a supplement in M. See [7, 24] for a nice presentation of results and the definitions related to supplements and supplemented modules. Recently, several authors have studied different generalizations of supplemented modules. In [2], τ-supplemented modules were defined for an arbitrary preradical τ for the category of right R-modules. For submodules U and V of a module M, V is said to be a *τ-supplement* of U in M or U is said to have a τ-supplement V in M if $U + V = M$ and $U \cap V \leq \tau(V)$. M is called a *τ-supplemented* module if every submodule of M has a τ-supplement in M. For the particular case $\tau = Rad$, *Rad*-supplement submodules have been studied in [5]. A submodule U of a module M is said to be have a Z^*-supplement V in M if $M = U + V$ and $U \cap V \leq Z^*(V)$, where $Z^*(V) = \{m \in V \mid Rm \text{ is small in } E(Rm)\}$. Z^*-supplement submodule was studied in [23].

The objective of the present paper is to introduce a new type of supplement by replacing "small submodule" with "semiartinian submodule". A module M is called *semiartinian* if every non-zero homomorphic image of M contains a simple submodule, that is, $soc(M/N) \neq 0$ for every submodule $N \lneqq M$. A module M is called *socle-free* if $soc(M) = 0$. The torsion theory $t = (\mathbb{T}_D, \mathbb{F}_D)$ generated by the class of semisimple (or even simple) modules is a hereditary torsion theory, called the Dickson torsion theory. Its torsion and torsionfree classes are respectively $\mathbb{T}_D = \{M \mid M \text{ is semiartinian}\}$ and $\mathbb{F}_D = \{M \mid M \text{ is socle-free}\}$. Note that $\mathbb{T}_D$ is closed under submodules, homomorphic images, direct sums and extensions, while $\mathbb{F}_D$ is closed under submodules, direct products, extensions and injective hulls. For any module M, since any sum of semiartinian submodules of M is semiartinian, M contains a unique maximal semiartinian submodule, called the torsion submodule with respect to this torsion theory and denoted by $sa(M)$. Clearly, $soc(M) = 0$ if and only if $sa(M) = 0$.

In Section 1, we present and study a new concept namely sa-supplement submodule. We will say a submodule X of a module B has a semiartinian supplement (shortly, sa-supplement) if there exists $S \leq B$ such that $B = S + X$ and $S \cap X$ is semiartinian. A sequence $0 \to A \to^f M \to C \to 0$ is called SAS if $f(A)$ is an sa-supplement submodule of M. The class $\mathcal{SAS}$ of SAS sequences is a proper class which is coinjectively generated by the class of semiartinian modules (see [15, Theorem 3.1]). The proper class $\mathcal{SAS}$ is injectively generated by socle-free modules (Proposition 1.1). A ring R is right semiartinian if and only if every maximal right ideal of R has an sa-supplement in R (Corollary 1.4). Moreover, in this section, we deal with modules which are sa-supplement in every containing module, namely sa-supplementing. Injective modules and semiartinian modules are obvious examples of sa-supplementing modules. It is shown that (1) an sa-supplementing module M is closed under sa-supplement quotients if and only if $M/sa(M)$ is injective; (2) a projective module P is sa-supplementing if and only if $P/sa(P)$ is a quotient of an injective module; (3) R is right semiartinian if and only if every right R-module is sa-supplementing; (4) R is right SSI ring, i.e., every semisimple right module is injective if and only if every right module is sa-supplementing. Section 2 deals with sa-supplemented modules. A module M is said to be *sa-supplemented* module if all its submodules have sa-supplements in M. We show that a module is sa-supplemented if and only if it is semiartinian. In Section 3, we introduce the notion of $\oplus$-sa-supplemented modules. An R-module M is called *$\oplus$-sa-supplemented* if its all sa-supplement submodules are direct summands. It is shown that (1) a module W is a $\oplus$-sa-supplemented module if and only if $W = sa(W) \oplus X$, where X is socle-free and $sa(W)$ is semisimple; (2) R is a right SSI-ring if and only if every right R-module is $\oplus$-sa-supplemented if and only if every injective right R-module is $\oplus$-sa-supplemented; (3) every projective right R-module is $\oplus$-sa-supplemented if and only if there is a ring decomposition $R \cong A \times B$, where A is semisimple and B is right socle-free. Finally, we prove that, over commutative C rings, any module which is projective with respect to all sa-supplement exact sequences is flat.

1. Sa-supplement submodules

We begin this section by introducing the notion of sa-supplement submodules.

Definition. Let $N_1 \leq N$. Then N_1 is an *sa-supplement* in N if there exists $S \leq N$ such that $N = S + N_1$ and $S \cap N_1$ is a semiartinian module.

Obviously, every semiartinian submodule of a module is sa-supplement. A sequence $0 \to M_1 \to^f M \to M_2 \to 0$ is called SAS if $f(M_1)$ is an sa-supplement submodule of M. The class $\mathcal{SAS}$ of SAS sequences is a proper class which is coinjectively generated by the class of semiartinian modules (see [15, Theorem 3.1]). In next result, we show that the class $\mathcal{SAS}$ is injectively generated by socle-free modules.

Proposition 1.1. *A sequence $\mathcal{E} : 0 \to X \to H \to Z \to 0$ is SAS if and only if $\mathrm{Hom}(H, F) \to \mathrm{Hom}(X, F) \to 0$ is exact for each socle-free module F.*

Proof. ($\Rightarrow$) Let $f : X \to W$ be a homomorphism with W socle-free. It is enough to show that $f_*(\mathcal{E}) : 0 \to W \to^g T \to Z \to 0$ is splitting. Since $\mathcal{SAS}$ is a proper class, $f_*(\mathcal{E}) \in \mathcal{SAS}$, and hence there exists $S \leq T$ such that $g(W) + S = T$ and $g(W) \cap S$ is semiartinian. But $g(W)$ is socle-free, and hence $g(W) \cap S$ must be zero. Therefore, $f_*(\mathcal{E})$ is splitting, as desired.

($\Leftarrow$) By our assumption, $X/sa(X)$ is injective to the sequence $\mathcal{E}$. Then, for some $sa(X) \leq T \leq H$, $(X/sa(X)) \oplus (T/sa(X)) = H/sa(X)$. This shows that $X + T = H$ and $X \cap T = sa(X)$. Thus our claim is established. □

A non-zero module M is called *local* if the sum of all proper submodules of M is a proper submodule of M. Obviously, simple modules are local. Recall that for a maximal submodule U of a module M, a submodule V of M is a supplement of U in M if and only if $M = U + V$ and V is local (see [24, 41.1(3)]). Analogously we have:

Proposition 1.2. *Let M be a module and U be a maximal submodule of M. A submodule V of M is an sa-supplement of U in M if and only if $M = U + V$ and V is semiartinian.*

Proof. Let V be an sa-supplement of U in M. Then $M = U + V$ and $U \cap V$ is semiartinian. Then since U is a maximal submodule of M, $V/(U \cap V)$ is simple. We have that V is semiartinian since semiartinian modules are closed under extensions. Conversely, since V is semiartinian and $M = U + V$, we can write $U \cap V$ is semiartinian. Hence, V is an sa-supplement of U in M. □

A module M is called *coatomic* if every proper submodule of M is contained in a maximal submodule of M. Semisimple modules and finitely generated modules are well-known examples of coatomic modules (see, [13, 9.7. Exercises]).

Proposition 1.3. *Let M be a coatomic module. Then M is semiartinian if and only if every maximal submodule of M has an sa-supplement in M.*

Proof. The necessity is clear. For sufficiency, assume that the largest semiartinian submodule $sa(M)$ of M is proper, that is $sa(M) \neq M$. Since M is coatomic, there exists a maximal submodule K of M with $sa(M) \leq K$. By our assumption, K has an sa-supplement, say V, in M. It follows from Proposition 1.2 that V is semiartinian, and this means that $V \leq sa(M) \leq K$. But $M = V + K$, and so $M = K$, a contradiction. □

Corollary 1.4. *A ring R is right semiartinian if and only if every maximal right ideal of R has an sa-supplement in R.*

A ring R is called a right C ring if $soc(R/I) \neq 0$ for every essential right ideal I of R, or equivalently, if every singular module is semiartinian. Left perfect rings, right semiartinian rings, two-sided hereditary Noetherian rings are examples of right C rings. (See [7, 10.10]).

Proposition 1.5. *If R is a right C ring, then M is semisimple if and only if $soc(M) = sa(M)$ and every essential submodule of M has an sa-supplement in M.*

Proof. The necessity is clear. For sufficiency, it is enough to show that M has no proper essential submodules. Assume on the contrary that M has an essential proper submodule, say A. By our assumption, $M = A + V$ and $A \cap V$ is semiartinian. Then since R is a C ring, $V/(A \cap V) \cong M/A$ is semiartinian. By the fact that semiartinian modules are closed under extensions, V is semiartinian. But then $V \leq soc(M) = sa(M) \leq A$, and hence $M = A + V = A$, a contradiction. Therefore, M has no proper essential submodules, that is, M is semisimple. □

Recall that a module M is called *supplementing* if it has a supplement in any module in which it is contained as a submodule ([7]). In the remaining part of this section, we investigated sa-supplementing modules. We call a module M *sa-supplementing* if it has an sa-supplement in every extension. Note that a module T is sa-supplementing if and only if T has an sa-supplement in $E(N)$ if and only if T has an sa-supplement in any sa-supplementing module. Sa-supplementing modules are closed under extensions and sa-supplement submodules (see [17, Propositions 1.7-1.8]).

Remark 1.6. (1) Injective modules and semiartinian modules are obvious examples of sa-supplementing modules.
(2) A socle-free module is sa-supplementing if and only if it is injective by Proposition 1.1.

Proposition 1.7. *The following are equivalent.*
(1) *All sa-supplementing right R-modules are semiartinian.*
(2) *All injective right R-modules are semiartinian.*
(3) *R is a right semiartinian ring.*

Proof. (1) ⇒ (2) and (3) ⇒ (1) are obvious. (2) ⇒ (3) Since the class of semiartinian modules is closed under submodules, R is semiartinian as a submodule of $E(R)$. □

Note that R is right hereditary if and only if quotients of injective modules are injective. Next we consider when quotients of sa-supplementing modules are sa-supplementing.

Lemma 1.8. *Sa-supplementing modules are closed under quotients if and only if quotients of injective modules are sa-supplementing.*

Proof. (⇒) is clear.
(⇐) Let U be an sa-supplementing module and $K \leq U$. Consider the following diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & = & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
\mathbb{E}: 0 & \longrightarrow & U & \longrightarrow & E(U) & \xrightarrow{gf} & E(U)/U & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle f} & & \| & & \\
\mathbb{E}_1: 0 & \longrightarrow & U/K & \longrightarrow & E(U)/K & \xrightarrow{g} & E(U)/U & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Since U is an sa-supplementing module, it is an sa-supplement in $E(U)$. Then, by properties of proper classes, U/K is an sa-supplement in $E(U)/K$. By our hypothesis, $E(U)/K$ is sa-supplementing. Thereby, by [17, Proposition 1.8], U/K is sa-supplementing, as desired. □

Lemma 1.9. *The following are equivalent for an sa-supplementing module M.*

(1) *$M/sa(M)$ is sa-supplementing.*
(2) *$M/sa(M)$ is injective.*
(3) *M/N is sa-supplementing for each semiartinian submodule N of M.*
(4) *M/N is sa-supplementing for each sa-supplement submodule N of M.*

Proof. (1) ⇔ (2) By Proposition 1.1, a socle-free sa-supplementing module is injective. Therefore, $M/sa(M)$ is sa-supplementing if and only if $M/sa(M)$ is injective.

(2) ⇒ (3) Let N be a semiartinian submodule of M. Consider the exact sequence $0 \to sa(M)/N \to M/N \to M/sa(M) \to 0$. Since semiartinian modules are closed under quotients, $sa(M)/N$ is sa-supplementing. By our assumption, since sa-supplementing modules are closed under extensions, M/N is sa-supplementing.

(3) ⇒ (4) Let N be an sa-supplement submodule of M. Then there is a submodule K in M such that $N+K = M$ and $N \cap K$ is semiartinan. Since $N \cap K$ is semiartinian, $N \cap K \subseteq sa(M)$. Consider the exact sequence $0 \to sa(M)/N \cap K \to M/N \cap K \to M/sa(M) \to 0$. Since semiartinian modules are closed under quotients, $sa(M)/N$ is sa-supplementing. By our assumption $M/sa(M)$ is sa-supplementing. Then since sa-supplementing modules are closed under extensions, M/N is sa-supplementing.

(4) ⇒ (1) In particular, consider the exact sequence $0 \to sa(M) \to M \to M/sa(M) \to 0$. Since $sa(M)$ is a trivial sa-supplement submodule of M, $M/sa(M)$ is sa-supplementing by our assumption. □

Corollary 1.10. *The following are equivalent.*

(1) *$E/sa(E)$ is injective for each injective module E.*
(2) *$M/sa(M)$ is injective for each sa-supplementing module M.*
(3) *The class of sa-supplementing modules is closed under sa-supplement quotients.*

Proof. (2) ⇔ (3) follows by Lemma 1.9.

(2) ⇒ (1) is obvious.

(1) ⇒ (2) Let M be an sa-supplementing module. There is a submodule K in $E(M)$ such that $M + K = E(M)$ and $M \cap K$ is semiartinan. Then $(M/sa(M)) + (K + sa(M)/sa(M)) = E(M)/sa(M)$. But $(M/sa(M)) \cap (K + sa(M)/sa(M)) = (M \cap K) + sa(M)/sa(M) = 0$, and hence $M/sa(M)$ is a direct summand of $E(M)/sa(M)$. By Lemma 1.9, $E(M)/sa(M)$ is sa-supplementing. Therefore, $M/sa(M)$ is also sa-supplementing. Now, the claim follows by Proposition 1.1. □

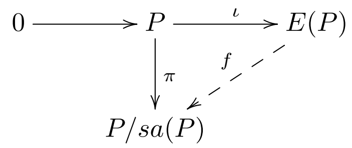
Proposition 1.11. *If R is a right C ring, then the class of sa-supplementing modules is closed under sa-supplement quotients.*

Proof. If R is a right C ring, then every singular module is semiartinian, i.e., every socle-free module is nonsingular. Consider the sequence $0 \to sa(E) \to E \to E/sa(E) \to 0$ for an injective module E. Note that $E/sa(E)$ is socle-free, and it is nonsingular by our hypothesis. Then, by [20, Lemma 2.3], $sa(E)$ is closed in E, and hence $E \cong sa(E) \oplus E/sa(E)$. So, $E/sa(E)$ is injective. Now, the claim follows by Corollary 1.10. □

Lemma 1.12. *The following are equivalent for a projective module P.*

(1) *P is sa-supplementing.*
(2) *$P/sa(P)$ is a homomorphic image of an injective module.*
(3) *For some semiartinian submodule N of P, P/N is a homomorphic image of an injective module.*

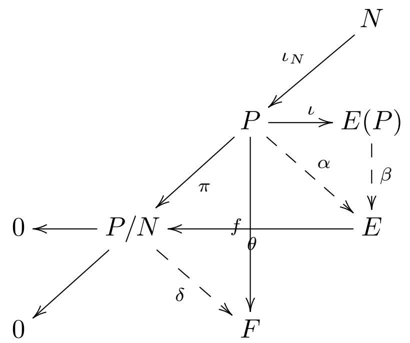
Proof. (1) ⇒ (2) Consider the following diagram:



Since P is sa-supplementing and $P/sa(P)$ is socle-free, by Proposition 1.1, there is a homomorphism $f : E(P) \to P/sa(P)$ such that $f\iota = \pi$. Since π is an epimorphism, f is also an epimorphism. Therefore, $P/sa(P)$ is a quotient of the injective module $E(P)$, as claimed.

(2) ⇒ (3) is clear.

(3) ⇒ (1) Assume that, for some semiartinian submodule N of P, there is an epimorphism $\theta : E \to P/N$ where E is an injective module. Let $f : P \to F$ be a homomorphism with F socle-free. Without loss of generality, since socle-free modules are closed under submodules, we may assume f is an epimorphism. Consider the following diagram



where $\iota_N : N \to P$ and $\iota : P \to E(P)$ are inclusion homomorphisms and $\pi : P \to P/N$ is the canonical epimorphism. Since N is semiartinian and F is socle-free, $f\iota_N(N) = 0$, and hence there exists a unique homomorphism $\delta : P/N \to F$ such that $\delta\pi = f$. By projectivity of P, there exists a homomorphism $\alpha : P \to$

E such that $\theta\alpha = \pi$. Then since E is injective, there is a homomorphism $\beta : E(P) \to E$ such that $\beta\iota = \alpha$. Then $\delta\theta\beta : E(P) \to F$ and $(\delta\theta\beta)\iota = (\delta\theta)\beta\iota = \delta(\theta\alpha) = \delta\pi = f$. By Proposition 1.1, P is sa-supplementing. □

It is well-known that all projective right R-modules are injective (i.e., R is a right QF-ring) if and only if every right R-module has an epic injective cover.

Corollary 1.13. *All projective modules are sa-supplementing if and only if every socle-free module is a homomorphic image of an injective module.*

Proof. Let M be a socle-free module. There is an epimorphism $\pi : P \to M$ where P is free. Since P is sa-supplementing, M is socle-free and π is an epimorphism, there is an homomorphism $f : E(P) \to M$ by Proposition 1.1. The converse follows by Lemma 1.12. □

Corollary 1.14. *If R is a right C ring, then a projective module P is sa-supplementing if and only if $P/sa(P)$ is injective.*

Proof. Assume that a projective module P is sa-supplementing. Then, by Lemma 1.12, there is an epimorphism $\theta : E \to P/sa(P)$ where E is an injective module. Since R is a right C ring and $P/sa(P)$ is socle-free, $P/sa(P)$ is nonsingular. By [20, Lemma 2.3], $ker(\theta)$ is closed in E, and hence $ker(\theta) \oplus P/sa(P) \cong E$. Therefore, $P/sa(P)$ is injective. The converse follows by the fact that sa-supplementing modules are closed under extensions. □

A ring R is right perfect if and only if every right R-module is supplementing ([7, 20.39(14)]). Analogously we have:

Proposition 1.15. *The following are equivalent for a ring R.*

(1) *Every right R-module is sa-supplementing.*
(2) *R is right semiartinian.*

Proof. (2) ⇒ (1) is clear.

(1) ⇒ (2) Let M be a socle-free right module and K a submodule of M. Then K is a socle-free module. By our hypothesis and by Proposition 1.1, K is injective. Then it is a direct summand of M. This implies that M is semisimple, but it is socle-free, and hence $M = 0$. Therefore, every right module is semiartinian, i.e., R is right semiartinian. □

It is well-known that a right noetherian right semiartinian ring is artinian. Moreover, a noetherian ring R is right artinian if and only if every free right R-module is supplementing ([7, 20.39(14)]). Analogously, we have the following for sa-supplementing modules by Proposition 1.15.

Corollary 1.16. *A right noetherian ring R is right artinian if and only if every free right R-module is sa-supplementing.*

A ring R is called a right SSI-ring if every semisimple right R-module is injective. Then R is a right SSI-ring if and only if R is a right noetherian right V-ring (see [6]).

Theorem 1.17. *The following are equivalent.*

(1) *All sa-supplementing right R-modules are injective.*
(2) *All semiartinian right R-modules are injective.*
(3) *R is right SSI ring.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1) Let M be an sa-supplementing right R-module. Then there is a submodule K in $E(M)$ such that $M + K = E(M)$ and $M \cap K$ is semiartinan. Recall that $soc(M \cap K)$ is essential in $M \cap K$. Then since R is a right SSI-ring, $soc(M \cap K) = M \cap K$ is injective. So, $E(M) = X \oplus (M \cap K)$ for some $X \subseteq E(M)$. By the modular law, $K = (K \cap X) \oplus (M \cap K)$, and so $M \oplus (K \cap X) = M + K = E(M)$. Therefore, M is injective. □

2. Sa-supplemented modules

Recall that a module M is called *supplemented* if every submodule of M has a supplement in M. A module M is said to be a *sa-supplemented* module if all its submodules have sa-supplements in M. In this section, we show that sa-supplemented modules are exactly semiartinian modules.

An R-module M is injective relative to an R-module N (or, M is N-injective) if, for any submodule K of N, any R-homomorphism $f : K \to M$ extends to some member of $\mathrm{Hom}_R(N, M)$. It is evident that every module is injective relative to semisimple modules. The following result follows directly from Proposition 1.1.

Proposition 2.1. *An R-module N is sa-supplemented if and only if every socle-free module M is N-injective.*

Semiartinian modules are obvious examples of sa-supplemented modules.

Proposition 2.2. *Any submodule or quotient of an sa-supplemented module is an sa-supplemented module.*

Proof. Let M be an sa-supplemented module and H any submodule of M. We will show that H and M/H are sa-supplemented modules. To show that H is an sa-supplemented module, let $T \leq H$. Since M is an sa-supplemented module, there is an $N \leq M$ such that $N + T = M$ and $N \cap T$ is semiartinian. By the modular law, $H = T + (N \cap H)$. Moreover, $T \cap N \cap H$ is semiartinian as a submodule of a semiartinian module $N \cap T$. This shows that H is an sa-supplemented module.

Let $A/H \leq M/H$. Since M is an sa-supplemented module, there is an $N \leq M$ such that $N + A = M$ and $N \cap A$ is semiartinian. Then $(A/H) + ((N + H)/H) = M/H$. Since semiartinian modules are closed under homomorphic images, $(A/H) \cap ((H + N)/H) = ((A \cap N) + H)/H \cong (A \cap N)/(N \cap H)$ is semiartinian. Thus $(N + H)/H$ is an sa-supplement of A/H in M/H. So M/H is an sa-supplemented module. □

Proposition 2.3. *A socle-free sa-supplemented module is the zero module.*

Proof. Let Q be a socle-free sa-supplemented module and $N \leq Q$. By our hypothesis, there exists $S \leq Q$ such that $N + S = Q$ and $N \cap S$ is semiartinian. But Q is socle-free, and hence $N \cap S = 0$. Therefore, Q is semisimple, i.e., all submodule of Q are direct summands. Then $Q = soc(Q) = 0$, as claimed. □

Corollary 2.4. *A module N is an sa-supplemented module if and only if it is semiartinian.*

Proof. Let N be an sa-supplemented module. There exists an exact sequence $0 \to sa(N) \to N \to N/sa(N) \to 0$. Note that $N/sa(N)$ is a socle-free module. Then, by Proposition 2.3, $N = sa(N)$, as desired. The sufficiency is clear. □

Corollary 2.5. *The following statements are equivalent.*

(1) *All modules are sa-supplemented.*
(2) *All injective modules are sa-supplemented.*
(3) *R is a right sa-supplemented module.*
(4) *All projective modules are sa-supplemented.*
(5) *R is a right semiartinian.*

3. $\oplus$-sa-supplemented modules

A module M is called *$\oplus$-supplemented* if every supplement submodule of M is a direct summand ([7]). In this section, a module is called *$\oplus$-sa-supplemented* if its sa-supplement submodules are direct summands. Note that a semiartinian module is $\oplus$-sa-supplemented if and only if it is semisimple.

Proposition 3.1. *Socle-free modules are $\oplus$-sa-supplemented.*

Proof. Let Q be a socle-free module and Z an sa-supplement in Q. Then there exists $S \leq Q$ such that $S + Z = Q$ and $S \cap Z$ is semiartinian. But Q is socle-free, and so $S \cap Z = 0$. Then, $S \oplus Z = Q$, as desired. □

Proposition 3.2. *The class of $\oplus$-sa-supplemented modules is closed under submodules.*

Proof. Let W be a $\oplus$-sa-supplemented module and Z a submodule of W. Let T be an sa-supplement submodule of Z. Then there exists $H \leq Z$ such that $T + H = Z$ and $T \cap H$ is semiartinian. Moreover, since W is a $\oplus$-sa-supplemented module and $T \cap H$ is an sa-supplement in W, there exists $K \leq W$ such that $K \oplus (T \cap H) = W$. Then, by the modular law, $H = (H \cap K) \oplus (T \cap H)$. Let us point out that $0 = (H \cap K) \cap (T \cap H) = (H \cap K) \cap T$. Now, $Z = T + H = (H \cap K) \oplus T$. This proves that Z is a $\oplus$-sa-supplemented module. □

Proposition 3.3. *The class of $\oplus$-sa-supplemented modules is closed under sa-supplement quotients.*

Proof. Let W be a $\oplus$-sa-supplemented module and Z an sa-supplement submodule of W. Then since W is a $\oplus$-sa-supplemented module, there exists $K \leq W$ such that $K \oplus Z = W$. To show that W/Z is a $\oplus$-sa-supplemented module, let Y/Z be an sa-supplement submodule of W/Z. Then there exists $Z \leq L \leq W$ such that $(L/Z)+(Y/Z) = W/Z$ and $(L\cap Y)/Z$ is semiartinian. By the modular law, $(L\cap K)\oplus Z = L$, and hence $Y\cap L = (Y\cap L\cap K)\oplus Z$. Then since $(Y\cap L)/Z$ is semiartinian and $(Y\cap L)/Z \cong Y\cap L\cap K$, it follows that $Y\cap L\cap K$ is semiartinian. Then since $W = Y + L = Y + ((L\cap K)\oplus Z) = Y + (L\cap K)$ and $Y\cap L\cap K$ is semiartinian, Y is an sa-supplement in W. But W is $\oplus$-sa-supplemented, and so there exists $A \leq W$ such that $A\oplus Y = W$. Then $((A+Z)/Z)\oplus(Y/Z) = W/Z$, which implies that W/Z is a $\oplus$-sa-supplemented module. □

Corollary 3.4. *$\oplus$-sa-supplemented modules are closed under direct summands.*

Lemma 3.5. *A module W is a $\oplus$-sa-supplemented module if and only if $W = soc(W)\oplus X$ for some submodule X of W.*

Proof. ($\Rightarrow$) Assume that a module W is a $\oplus$-sa-supplemented module. Since $sa(W)$ is an sa-supplement in every extension and W is a $\oplus$-sa-supplemented module, there exists $X \leq W$ such that $W = sa(W)\oplus X$. Recall that $sa(W)$ is the largest semiartinian submodule of W, and so X is socle-free. By Proposition 3.2, $sa(W)$ is a $\oplus$-sa-supplemented module, and so it is semisimple. Therefore $sa(W) = soc(W)$, as desired.

($\Leftarrow$) Note that, by our hypothesis, $sa(X) = 0$, and hence

$$sa(W) = sa(soc(W))\oplus sa(X) = soc(W).$$

Let P be an sa-supplement in W. Then there exists $F \leq W$ such that $F + P = W$ and $F\cap P$ is semiartinian. Since $F\cap P$ is semiartinian and $sa(W)$ is defined as sum of all semiartinian submodules of W, $F\cap P \leq sa(W)$. Then since $sa(W)$ is semisimple, there exists $U \leq sa(W)$ such that $U\oplus(F\cap P) = sa(W)$. Therefore $W = sa(W)\oplus X = U\oplus(F\cap P)\oplus X$. By the modular law, we have $F = F\cap(U\oplus X)\oplus(F\cap P)$. Now $W = F + P = P\oplus(F\cap(U\oplus X))$, as desired. □

Note that semiartinian modules, semisimple modules and socle-free modules are closed under direct sums. Therefore, we have the following by Lemma 3.5.

Corollary 3.6. *The class of $\oplus$-sa-supplemented modules is closed under direct sums.*

Theorem 3.7. *The following statements are equivalent.*

(1) *All modules are $\oplus$-sa-supplemented.*
(2) *All injective modules are $\oplus$-sa-supplemented.*
(3) *R is a right SSI-ring*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) Let Z be a semisimple module. Since Z is an sa-supplement in $E(Z)$ and $E(Z)$ is a $\oplus$-sa-supplemented module, Z is a direct summand of $E(Z)$, and hence it is injective.

(3) ⇒ (1) Let W be any module and $U \leq W$. If U is an sa-supplement in W, then there exists $Z \leq W$ such that $Z + U = W$ and $Z \cap U$ is semiartinian. By Theorem 1.17, $Z \cap U$ is injective, and hence there exists $Z_1 \leq Z$ such that $(Z \cap U) \oplus Z_1 = X$. Then $W = U + Z = U \oplus Z_1$, as claimed. □

Theorem 3.8. *The following statements are equivalent.*

(1) *All projective right R-modules are $\oplus$-sa-supplemented module.*
(2) *For any projective module P, $P = A \oplus B$ where A is semisimple and B is socle-free.*
(3) *There is a ring decomposition $R \cong A \times B$, where A is semisimple and B is right socle-free.*

Proof. (1) ⇔ (2) follows by Lemma 3.5.

(2) ⇒ (3) By our assumption, $R = A \oplus B$ where A is semisimple and B is socle-free. Then A is projective. Assume that $\mathrm{Hom}(B, A) \neq 0$. Then there is a non-zero homomorphism $f : B \to A$. So $f(B)$ is semisimple and projective as a direct summand of A. Then $B \cong ker(f) \oplus f(B)$, but B is socle-free, and hence $f(B) = 0$. Then, we have a ring direct sum $R \cong A \times B$, where A is semisimple and B is right socle-free. (3) ⇒ (2) is clear. □

We close the paper with some results on $\mathcal{SAS}$-projective modules. A module is called $\mathcal{SAS}$-*projective* if it is projective with respect to all sa-supplement exact sequences. A module M is $\mathcal{SAS}$-projective if and only if $\mathrm{Ext}^1_R(M, S) = 0$ for each semiartinian R-module S by [15, Theorem 3.1(ix)]. Note that if R is a right semiartinian ring, then all short exact sequences are in $\mathcal{SAS}$ by Proposition 1.15. Thus, $\mathcal{SAS}$-projective right modules are only projective modules over right semiartinian rings.

Theorem 3.9. *Let R be a commutative C ring. Then $\mathcal{SAS}$-projective modules are flat.*

Proof. Let M be an $\mathcal{SAS}$-projective R-module. In particular, since every simple R-module is semiartinian, $\mathrm{Ext}^1_R(M, S) = 0$ for each simple R-module S by [15, Theorem 3.1(ix)]. Note that if R is commutative and E is an injective cogenerator, then $\mathrm{Hom}(S, E) \cong S$ for each simple R-module S. Then, for any simple R-module S, $\mathrm{Ext}^1_R(M, S^+) = 0$. By the standard adjoint isomorphism $\mathrm{Ext}^1_R(M, S^+) \cong (\mathrm{Tor}^R_1(M, S))^+ \cong \mathrm{Ext}^1_R(S, M^+) = 0$. We claim to show that M^+ is injective. Consider the exact sequence $0 \to M^+ \to E(M^+) \to E(M^+)/M^+ \to 0$. Then $soc(E(M^+)/M^+)$ is projective, otherwise there exists a submodule K in $E(M^+)$ such that M^+ is an essential and maximal submodule of K. But this is not possible since $\mathrm{Ext}^1_R(K/M^+, M^+) = 0$. Then since R is a C ring, $E(M^+)/M^+$ is nonsingular. But, this would mean that M^+ is closed in $E(M^+)$, and hence it is injective. Then M is flat by [19, Proposition 3.54]. □

Acknowledgments. I would like to thank the referee for his/her careful reading of the paper.

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