Bull. Korean Math. Soc. **58** (2021), No. 1, pp. 133–146
https://doi.org/10.4134/BKMS.b200055
pISSN: 1015-8634 / eISSN: 2234-3016

CONTINUOUS ORBIT EQUIVALENCES ON SELF-SIMILAR GROUPS

Inhyeop Yi

Abstract. For pseudo-free and recurrent self-similar groups, we show that continuous orbit equivalence of inverse semigroup partial actions implies continuous orbit equivalence of group actions. Conversely, if group actions are continuous orbit equivalent, and the induced homeomorphism commutes with the shift maps on their groupoids, we obtain continuous orbit equivalence of inverse semigroup partial actions.

1. Introduction

The main purpose of this paper is to find relations of continuous orbit equivalences on self-similar groups. Continuous orbit equivalence of one-sided subshifts of finite type defined by Matsumoto [12] has had a major effect on the study of C^*-algebras and topological dynamics. The concept of continuous orbit equivalence has been generalized to many areas, including graph algebras [1, 3], group actions [10, 11], partial actions of inverse semigroups [2, 4–6], and asymptotic Ruelle algebras of Smale spaces [13]. In particular, for graphs, Cordeiro and Beuter [4] showed that two graphs are topologically orbit equivalent if and only if their graph semigroup partial actions on corresponding path boundaries are topologically orbit equivalent. Li [10] also determined that two graphs are topologically orbit equivalent if and only if partial actions of free groups generated by edge sets of graphs are topologically orbit equivalent.

Inspired by the results of [4, 10], we studied continuous orbit equivalences defined on self-similar groups. Introduced by Nekrashevych [15, 16], the class of self-similar groups has become an important example of C^*-dynamics. A self-similar group (G, X) has at least two naturally associated actions. One is G action on the infinite path space X^ω, and the other is the partial action of

Received January 18, 2020; Revised September 27, 2020; Accepted October 16, 2020.
2010 *Mathematics Subject Classification.* 37B05, 37A20, 37C85, 20M30, 46L35.
Key words and phrases. Self-similar group, continuous orbit equivalence, inverse semigroup, groupoid.
This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2018R1D1A1B07048313).

an inverse semigroup S_G on X^ω. Moreover, Nekrashevych defined the equivalence of self-similar groups to show conjugacy of group actions on the rooted trees. Then, it is a rational question to find interrelations among these three equivalences. The method we employ is groupoids of germs of these actions and their corresponding groupoid C^*-algebras. To use groupoids and their C^*-algebras effectively, we have to provide some restrictions on self-similar groups. When self-similar groups satisfy pseudo-free and recurrent conditions, we show that continuous orbit equivalence of inverse semigroup partial actions implies continuous orbit equivalence of group action (Theorem 4.7). The converse is also true if we assume an extra condition that the induced homeomorphism commutes with the shift maps on their groupoids, we obtain continuous orbit equivalence of inverse semigroup partial actions (Theorem 4.9).

2. Inverse semigroups

All of the material in this section is taken from [4] and [19]. We refer the reader to [9, 17] for inverse semigroups and their groupoid C^*-algebras.

Inverse semigroups

An inverse semigroup is a semigroup S such that for every $s \in S$, there is a unique element $s^* \in S$, called the inverse of s, satisfying

$$ss^*s = s \text{ and } s^*ss^* = s^*.$$

We assume that S has a unit element 1 and a zero element 0 with the property

$$1s = s1 = s \text{ and } 0s = s0 = 0 \text{ for every } s \in S.$$

An element $s \in S$ is called an idempotent if $s^2 = s$. We denote the set of all idempotents in S by $E(S)$.

Example 2.1 ([7, Definition 5.2])**.** Let X be a topological space and define

$$\mathrm{pHomeo}\,(X) = \{h\colon U \to V \mid U, V \subset X \text{ are open, and } h \text{ is a homeomorphism}\}.$$

Then $\mathrm{pHomeo}\,(X)$ is an inverse semigroup: its binary operation is given by composition, for $h_1, h_2 \in \mathrm{pHomeo}\,(X)$,

$$h_1h_2 = h_1 \circ h_2\colon h_2^{-1}(\mathrm{Dom}\, h_1 \cap \mathrm{Im}\, h_2) \to h_1(\mathrm{Dom}\, h_1 \cap \mathrm{Im}\, h_2).$$

The inverse is given by $h^* = h^{-1}$, the unit element is Id_X, the 0 element is the trivial map between empty sets, and h is an idempotent if and only if $h = \mathrm{Id}_U$ for some open subset U of X.

An element $h \in \mathrm{pHomeo}\,(X)$ is called a *partial homeomorphism* of X.

Inverse semigroup partial actions

For a topological space X and an inverse semigroup S, S is said to act on X if there is a semigroup homomorphism $\theta\colon S \to \mathrm{pHomeo}\,(X)$ that preserves the unit element and the zero element [2]. To describe that the partial action of S on X preserves the topological structure of X, we need the following definitions.

Definition 2.2 ([2,4])**.** A partial homeomorphism between inverse semigroups S and T is a map $\varphi\colon S \to T$ such that, for all $s_1, s_2 \in S$, one has

(1) $\varphi(s_1^*) = \varphi(s_1)^*$;
(2) $\varphi(s_1)\varphi(s_2) \leq \varphi(s_1 s_2)$;
(3) $\varphi(s_1) \leq \varphi(s_2)$ whenever $s_1 \leq s_2$.

Here, $s_1 \leq s_2$ if and only if $s_1 = s_1 s_2^* s_2$ holds.

Definition 2.3 ([4, Definition 2.4])**.** A (topological) partial action of an inverse semigroup S on a topological space X is a tuple $\theta = (\{X_s\}_{s\in S}, \{\theta(s)\}_{s\in S})$ such that:

(1) for every $s \in S$, X_s is an open subset of X and $\theta(s)\colon X_{s^*} \to X_s$ is a homeomorphism;
(2) the map $s \mapsto \theta(s)$ is a partial homomorphism of inverse semigroups;
(3) $X = \cup_{e\in E(S)} X_e$.

Suppose that $\theta = (\{X_s\}_{s\in S}, \{\theta(s)\}_{s\in S})$ is the partial action of an inverse semigroup S on a topological space X. To simplify notation, we denote $\theta(s)$ simply by s so that the S partial action on X will be written as $\theta = (\{X_s\}_{s\in S}, S)$. We consider S as a discrete topological space. A subset $S * X$ of $S \times X$ is defined by

$$S * X = \{(s, x) \in S \times X : x \in X_{s^*}\}.$$

For each $x \in X$, the subset $\{s \in S : x \in X_{s^*}\}$ of S is denoted by S_x.

Definition 2.4 ([4, Definition 7.2])**.** Suppose that $\theta = (\{X_s\}_{s\in S}, \{\theta(s)\}_{s\in S})$ is as stated previously. We say that the S partial action on X is *topologically free* if the following set is dense in X:

$$\{x \in X : \forall s \in S_x, \text{ if } s(x) = x, \text{ then there is an } e \in E(S) \text{ such that } e \leq s \text{ and } x \in X_e\}.$$

Groupoids of germs

Suppose that an inverse semigroup S acts on a locally compact Hausdorff space X. The *groupoid of germs* of S, denoted by $S \ltimes X$, is the set of equivalence classes of pairs (s, x) such that $s \in S_x$ and $x \in X_{s^*}$. Two pairs (s, x) and (t, y) are equivalent to each other if and only if $x = y$ and s and t coincide on a neighborhood of x. The equivalence class of (s, x) is denoted by $[s, x]$ and called the germ of s at x.

The domain and range maps of $S \ltimes X$ are

$$d([s, x]) = x \text{ and } r([s, x]) = s(x).$$

The set of composable pairs of $S \ltimes X$ is

$$S \ltimes X^{(2)} = \{([s, x], [t, y]) \in (S \ltimes X) \times (S \ltimes X) : t(y) = x\},$$

and the groupoid composition and inversion are given as

$$[s, x] \cdot [t, y] = [st, y] \text{ and } [s, x]^{-1} = [s^*, s(x)].$$

The unit space of $S \ltimes X$ is

$$S \ltimes X^{(0)} = \{[e,x] \colon e \in E(S), x \in X_{e^*}\},$$

which is identified with X via the bijective map $[e,x] \mapsto x$.

A topology on $S \ltimes X$ is given as follows: for an $s \in S$ and any open set $U \subseteq X_{s^*s}$, let

$$O(s,U) = \{[s,x] \colon x \in U\}.$$

Then the collection of all $O(s,U)$ forms a basis for a topology on $S \ltimes X$, which makes $S \ltimes X$ a locally compact étale groupoid [16].

Continuous orbit equivalence of inverse semigroup partial actions

Suppose that S and T are inverse semigroups and that X and Y are topological spaces such that S acts on X and T acts on Y. We denote the S partial action on X and T partial action on Y by $\theta = (\{X_s\}_{s\in S}, S)$ and $\eta = (\{Y_t\}_{t\in T}, T)$, respectively.

Definition 2.5 ([4, Definition 8.1])**.** Let θ and η be as stated previously. We say that θ is continuously orbit equivalent to η if there is a homeomorphism

$$f \colon X \to Y$$

and continuous maps

$$a \colon S * X \to T \text{ and } b \colon T * Y \to S$$

such that, for all $x \in X,\ s \in S_x,\ y \in Y,$ and $t \in T_y$:

(1) $f(s(x)) = a(s,x)(f(x))$; and
(2) $f^{-1}(t(y)) = b(t,y)(f^{-1}(y))$.

Implicitly, we require $a(s,x) \in T_{f(x)}$ and $b(t,y) \in S_{f^{-1}(y)}$.

Definition 2.6 ([4, Definition 8.7])**.** A topological partial action $\theta = (\{X_s\}_{s\in S}, \{\theta(s)\}_{s\in S})$ is called *almost ample* if X is a locally compact Hausdorff space and X_s is ultraparacompact for every $s \in S$.

Theorem 2.7 ([4, Theorem 8.15])**.** *Suppose that S and T are inverse semigroups and that X and Y are topological spaces such that the S partial action on X and T partial action on Y are denoted as θ and η, respectively. Assume that θ and η are almost ample and topologically free partial actions and that the corresponding groupoids of germs $S \ltimes X$ and $T \ltimes Y$ are Hausdorff groupoids. Then the following are equivalent.*

(1) *The partial actions θ and η are continuously orbit equivalent.*
(2) *The groupoids of germs $S \ltimes X$ and $T \ltimes Y$ are topologically isomorphic.*

Continuous orbit equivalence of groups actions

We briefly review the continuous orbit equivalence of group actions defined by Li [11].

Suppose that G is a discrete and countable group and that X is a compact Hausdorff space. We say that G acts on X if there is a group homomorphism $g \mapsto \lambda_g \in \mathrm{Homeo}\,(X)$. As in the case of inverse semigroup partial actions, we denote $\lambda(g)$ simply by g. The transformation groupoid $T(G, X)$ of the G action on X is given by the set $G \times X = \{(g, x)\}$ with the multiplication $(h, y)\cdot(g, x) = (hg, x)$ if $y = gx$. It is easy to observe that conditions for inverse semigroup partial actions are extended to group actions. See [4–7] for details.

Definition 2.8 ([11, Definition 2.5])**.** Suppose that G and H are discrete and countable groups and X and Y are compact Hausdorff spaces such that G acts on X and H acts on Y. Then G action on X and H action on Y are continuously orbit equivalent if there is a homeomorphism $f\colon X \to Y$ and continuous maps $a\colon G \times X \to H$ and $b\colon H \times Y \to G$ such that

$$f(g(x)) = a(g, x) \circ f(x),$$
$$f^{-1}(h(y)) = b(h, y) \circ f^{-1}(y)$$

for all $g \in G,\ h \in H,\ x \in X$, and $y \in Y$.

Definition 2.9 ([11, Definition 2.1])**.** Let G and X be as above. We say that the G action on X is *topologically free* if, for each nontrivial element $g \in G$, $\{x \in X : g(x) \neq x\}$ is dense in X.

Theorem 2.10 ([11, Theorem 1.2])**.** *Suppose that G action on X and H action on Y are topologically free actions. Then the following are equivalent.*

(1) *The G action on X and H action on Y are continuously orbit equivalent.*
(2) *The transformation groupoids $T(G, X)$ and $T(H, Y)$ are topologically isomorphic.*
(3) *There is a $*$-isomorphism $\phi\colon C_0(X) \rtimes_r G \to C_0(Y) \rtimes_r H$ with $\phi(C_0(X)) = C_0(Y)$.*

3. Self-similar groups

We review the properties of self-similar groups. All of the material in this section is taken from [15, 16].

Suppose that X is a finite alphabet. We denote by X^n the set of words of length n in X with $X^0 = \{\emptyset\}$, and let $X^* = \cup_{n=0}^{\infty} X^n$. We denote by X^ω the set of right-infinite paths of the form $x_1 x_2 \cdots$ where $x_i \in X$. The product topology of the discrete set X is given on X^ω. A cylinder set $Z(u)$ for each $u \in X^*$ is

$$Z(u) = \{\xi \in X^\omega : \xi = x_0 x_1 \cdots \text{ such that } x_0 \cdots x_{|u|-1} = u\}.$$

Then the collection of all such cylinder sets forms a basis for the product topology on X^ω. It is trivial that every cylinder set is a compact open set, and that X^ω is a compact metrizable space.

A *self-similar group* (G, X) consists of a finite set X and a *faithful action* of a group G on X^* such that, for all $g \in G$ and $x \in X$, there exist unique $y \in X$ and $h \in G$ such that

$$g(xu) = yh(u) \text{ for every } u \in X^*.$$

The unique element h is called the *restriction* of g at x and denoted by $g|_x$. The restriction extends to X^* via the inductive formula

$$g|_{xy} = (g|_x)\,|_y$$

so that for every $u, v \in X^*$ we have

$$g(uv) = g(u)g|_u(v).$$

The G-action extends to an action of G on X^ω given by

$$g(x_0x_1\cdots) = g(x_0\cdots x_{n-1})g|_{x_0\cdots x_{n-1}}(x_n\cdots).$$

Conditions on self-similar groups

A self-similar group (G, X) is called *contracting* if there is a finite subset N of G satisfying the following: for every $g \in G$, there is $n \geq 0$ such that $g|_v \in N$ for every $v \in X^*$ of length $|v| \geq n$. We say that (G, X) is *regular* if, for every $g \in G$ and every $\xi \in X^\omega$, either $g(\xi) \neq \xi$ or there is a neighborhood of ξ such that every point in the neighborhood is fixed by g. We say that (G, X) is *recurrent* if, for any two words a, b of equal length and every $h \in G$, there is a $g \in G$ such that $g(a) = b$ and $g|_a = h$ (see [14, p. 235]). We say that (G, X) is *pseudo-free* if $(g, x) \in G \times X$ is such that $g(x) = x$ and $g|_x = 1$, then $g = 1$ (see [8, Definition 5.4]).

Lemma 3.1. *If (G, X) is a pseudo-free self-similar group, then G action on X^ω is a topologically free action.*

Proof. Assume that the action is not topologically free. Then there is a non-trivial element $g \in G$ such that $\{\xi \in X^\omega \colon g(\xi) \neq \xi\}$ is not dense in X^ω. Thus, there is a $\mu \in X^\omega$ such that:

(1) $g(\mu) = \mu$; and
(2) there are $u \in X^*$ and $\nu \in X^\omega$ such that $\mu = u\nu$ and every element is $Z(u)$ is fixed by g.

As we have

$$g(\mu) = g(u\nu) = g(u)g|_u(\nu) = u\nu$$

for every $\alpha \in X^\omega$ so that $u\alpha \in Z(u)$,

$$g(u\alpha) = g(u)g|_u(\alpha) = u\alpha.$$

Hence, we obtain $g|_u = 1$, which implies $g = 1$ by the pseudo-free condition. This is a contradiction to the assumption of $g \neq 1$. Therefore, G action on X^ω is a topologically free action. $\square$

Recall that G action on X is free if $g(x) \neq x$ for every nontrivial $g \in G$ and every $x \in X$.

Lemma 3.2. *If (G, X) is a pseudo-free and regular self-similar group, then G action on X^ω is a free action.*

Proof. If there are $g \neq 1$ in G and ξ in X^ω such that $g(\xi) = \xi$, then the regular condition implies that there is a neighborhood U of ξ such that every point in U is fixed by g. Thus, the G action on X^ω is not topologically free, which is a contradiction to Lemma 3.1. Thus, G action on X^ω is a free action. $\square$

Inverse semigroups of self-similar groups

Suppose that (G, X) is a self-similar group. The Cuntz-Pimsner algebra $\mathcal{O}_G$ of a self-similar group (G, X) is the universal C^*-algebra generated by G and $\{s_x : x \in X\}$ satisfying the following relations (see [16, Definition 3.1]):

(1) all relations of G;
(2) $s_x^* s_x = 1$ for every $x \in X$ and $\sum_{x \in X} s_x s_x^* = 1$; and
(3) for all $g \in G$ and $x \in X$, $g \cdot s_x = s_{g(x)} \cdot g|_x$.

We denote by S_G the inverse semigroup generated by the elements s_x, s_x^*, and G in $\mathcal{O}_G$. For $u = u_1 \cdots u_n \in X^n$, we let $s_u = s_{u_1} \cdots s_{u_n}$ and $s_u^* = s_{u_n}^* \cdots s_{u_1}^*$. Likewise, we also let $s_\emptyset$ be the identity. Then every element of S_G is uniquely written in the form $s_u g s_v^*$ for some $u, v \in X^*$ and $g \in G$ [16, Proposition 3.2].

The inverse semigroup S_G acts on X^ω by the partial homeomorphism

$$s_u g s_v^*(v\xi) = ug(\xi)$$

with domain $Z(v)$ and range $Z(u)$. The groupoid of germs of S_G is denoted by C_G and called the *Cuntz-Pimsner groupoid* of (G, X).

For a self-similar group (G, X), we recall that the group G is a subset of the inverse semigroup S_G. Moreover, $v\xi$ is a fixed element of $s_u g s_v^* \in S_G$ if and only if $u = v$ and $g(\xi) = \xi$. Then the following property comes from [8, Corollary 14.14].

Lemma 3.3. *Suppose that (G, X) is a self-similar group with its inverse semigroup S_G. If (G, X) is pseudo-free, then S_G partial action on X^ω is topologically free.*

Remark 3.4. Let (G, X) be a pseudo-free self-similar group with its Cuntz-Pimsner groupoid C_G and inverse semigroup S_G.

(1) The notion of groupoid germs used in [4], due to Exel and Paterson [6,17], differs from the one we used. But both definitions coincide when an inverse semigroup partial action is topologically free [8].

(2) The S_G partial action on X^ω is almost ample because X^ω is compact, Hausdorff, and zero-dimensional.
(3) The Cuntz-Pimsner groupoid C_G is an étale, topologically principal, locally compact, and Hausdorff groupoid. The étale and locally compact properties come from the definition of groupoids of germs. The topologically principal property is by [8, Corollary 14.14], and the Hausdorff property is by [16, Lemma 5.4]. If (G, X) is contracting and recurrent, then C_G is amenable by [16, Theorem 5.6].
(4) The Cuntz-Pimsner algebra $\mathcal{O}_G$ is isomorphic to the convolution C^*-algebra of C_G [16, Theorem 5.1].

We refer the reader to [17, 18] for the definition and properties of groupoids of germs and groupoid algebras.

Theorem 3.5 ([20, Theorem 4.14])**.** *Suppose that (G, X) and (H, Y) are contracting, recurrent, and regular self-similar groups. Then the following are equivalent*:
(1) *S_G is isomorphic to S_H.*
(2) *C_G is isomorphic to C_H as topological groupoids.*
(3) *There is a $*$-isomorphism $\Phi\colon \mathcal{O}_G \to \mathcal{O}_H$ with $\Phi(C(X^\omega)) = C(Y^\omega)$.*

4. Continuous orbit equivalences on self-similar groups

Suppose that (G, X) and (H, Y) are self-similar groups with their inverse semigroups S_G and S_H, respectively. We show that continuous orbit equivalence of inverse semigroups with cocycle condition implies continuous orbit equivalence of group actions under pseudo-free and recurrent conditions and that the converse is also true under additional commuting condition of shift maps and homeomorphism.

It is a well-known fact that, for the transformation groupoid and the groupoid of germs from discrete group action on a topological space, the transformation groupoid coincides with the groupoid of germs if the group action is topologically free. So we have the following property from Lemma 3.1.

Proposition 4.1. *If (G, X) is a pseudo-free self-similar group, then the transformation groupoid $T(G, X^\omega)$ of G action on X^ω is isomorphic to the groupoid of germs $G \ltimes X^\omega$ of G action on X^ω.*

Note that $\{s_v g s_u^*\colon |u| = |v|, g \in G\}$ is a subsemigroup of S_G so that S_G partial action on X^ω induces an $\{s_v g s_u^*\colon |u| = |v|, g \in G\}$ partial action.

Proposition 4.2. *If (G, X) is a pseudo-free and recurrent self-similar group, then the transformation groupoid $T(G, X^\omega)$ is isomorphic to the groupoid of germs of $\{s_v g s_u^*\colon |u| = |v|, g \in G\}$ partial action on X^ω.*

Proof. Let us denote by D_G the groupoid of germs of $\{s_v g s_u^*\colon |u| = |v|, g \in G\}$ partial action on X^ω. For a $(g, \xi) \in T(G, X^\omega)$, let $a \in X^*$ be a prefix of ξ so that $\xi = a\alpha$ for some $\alpha \in X^\omega$. We define $\phi\colon T(G, X^\omega) \to D_G$ by

$$(g, \xi) \mapsto [s_{g(a)} g|_a s_a^*, \xi].$$

As $|a| = |g(a)|$, it is obvious that $[s_{g(a)} g|_a s_a^*, \xi] \in D_G$ for every $\xi \in Z(a)$ and

$$s_{g(a)} g|_a s_a^*(\xi) = s_{g(a)} g|_a s_a^*(a\alpha) = s_{g(a)} g|_a(\alpha) = g(a) g|_a(\alpha) = g(a\alpha) = g(\xi).$$

First, we show that ϕ is well-defined. Let $b \in X^*$ be another prefix of ξ so that $\xi = b\beta$ for some $\beta \in X^\omega$. Without loss of generality, we consider only the case $|a| > |b|$ so that there is a finite word w such that $a = bw$. Then we need to check that $[s_{g(a)} g|_a s_a^*, \xi] = [s_{g(b)} g|_b s_b^*, \xi]$, i.e., there is a neighborhood U of ξ such that

$$s_{g(a)} g|_a s_a^*(\eta) = s_{g(b)} g|_b s_b^*(\eta)$$

for every $\eta \in U$.

We let $U = Z(a)$. Then every $\eta \in U$ is given by $\eta = a\zeta = bw\zeta$ for some $\zeta \in Z^\omega$. Here, we recall that from the definition of self-similar groups, for $a = bw$,

$$g(a) = g(bw) = g(b) g|_b(w) \text{ and } (g|_b)|_w = g|_{bw} = g|_a$$

hold. Thus, for every $\eta = a\zeta = bw\zeta \in U$, we have

$$\begin{aligned}
s_{g(b)} g|_b s_b^*(\eta) &= s_{g(b)} g|_b \{s_b^*(bw\zeta)\} = s_{g(b)} g|_b(w\zeta) \\
&= g(b) g|_b(w\zeta) = g(b)\{g|_b(w)(g|_b)|_w(\zeta)\} \\
&= \{g(b) g|_b(w)\}\{(g|_b)|_w(\zeta)\} \\
&= g(a) g|_a(\zeta) \\
&= g(a) g|_a s_a^*(a\zeta) \\
&= s_{g(a)} g|_a s_a^*(\eta).
\end{aligned}$$

Hence, ϕ is a well-defined map.

If $\phi(g, \xi) = \phi(h, \eta)$, then we have $\xi = \eta = a\zeta$ for some $a \in X^*$ and $\zeta \in X^\omega$ so that

$$\phi(g, \xi) = [s_{g(a)} g|_a s_a^*, \xi] = [s_{h(a)} h|_a s_a^*, \xi] = \phi(h, \eta)$$

and $s_{g(a)} g|_a s_a^* = s_{h(a)} h|_a s_a^*$ on a neighborhood U of ξ. As

$$s_{g(a)} g|_a s_a^*(\xi) = s_{g(a)} g|_a s_a^*(a\zeta) = g(a) g|_a(\zeta) = h(a) h|_a(\zeta) = s_{h(a)} h|_a s_a^*(\xi),$$

we have $g(a) = h(a)$, which implies $g|_a = h|_a$ on X^ω. Thus, the pseudo-free condition induces $g = h$, and ϕ is an injective map.

By the recurrent condition, for any finite words u, v of equal length and $g \in G$, there is an $h \in G$ such that $h(u) = v$ and $h|_u = g$. Thus, we have

$$\phi(h, \xi) = [s_{h(u)} h|_u s_u^*, \xi] = [s_v g s_u^*, \xi]$$

for every $\xi \in Z(u)$, and ϕ is a surjective map.

It is routine to show that ϕ is a groupoid homomorphism. Therefore, ϕ is a groupoid isomorphism. $\square$

Before going further, we summarize some necessary properties of D_G.

Remark 4.3. Suppose that $\mathcal{O}_G$ is the Cuntz-Pimsner algebra of a self-similar group (G,X) and that $C^*(D_G)$ is the groupoid algebra of the groupoid of germs D_G. Then we have the following properties.

(1) By [16, Theorem 5.3], $C^*(D_G)$ is isomorphic to a universal C^*-algebra generated by partial isometries in D_G.
(2) By [16, Theorem 3.7], $C^*(D_G)$ is isomorphic to the gauge-invariant subalgebra of $\mathcal{O}_G$.

We recall that the Cuntz-Pimsner groupoid C_G of a self-similar group (G,X) is the groupoid of germs of S_G partial action on X^ω.

Lemma 4.4 ([8, Proposition 8.4 and Theorem 8.19])**.** *If (G,X) is a pseudo-free self-similar group, then the Cuntz-Pimsner groupoid C_G is isomorphic to*

$$E_G = \left\{ \begin{array}{rl} (\alpha; [\{g_{n+l}\}], l-k; \beta)\colon & \alpha, \beta \in X^\omega, g_i \in G, l, k \in \mathbb{N}, \\ & \alpha_{n+l} = g_{n+l}(\beta_{n+k}) \text{ and} \\ & g_{n+l+1} = g_{n+l}|_{\beta_{n+k}} \text{ for every } n \geq 1 \end{array} \right\},$$

where $\alpha \in X^\omega$ is given by $\alpha = \alpha_1\alpha_2\cdots$ and $[\{g_{l+n}\}]$ is explained in the following.

Remark 4.5. For $\{g_i\}, \{h_i\} \in G^\infty$, define $\{g_i\} \sim \{h_i\}$ if and only if there is a sufficiently large natural number N such that $g_i = h_i$ for every $i \geq N$. Then it is easy to check that $\sim$ is an equivalence relation on G^∞, and the equivalence class of $\{g_i\}$ is denoted by $[\{g_{n+l}\}]$. See [8, Section 7] for more details.

It is not difficult to see that the isomorphism $C_G \to E_G$ is given by

$$[s_u g s_v^*, \beta] \mapsto (s_u g s_v^*(\beta), [\{g_{n+|u|}\}], |u| - |v|, \beta)$$

where $g_{1+|u|} = g$ and $g_{n+|u|} = g|_{\beta_{1+k}\cdots\beta_{n-1+k}}$ for every $n \geq 2$. Then the following is trivial.

Lemma 4.6. *Let D_G be the groupoid of germs of $\{s_v g s_u^*\colon |u| = |v|, g \in G\}$ partial action on X^ω. Then D_G is isomorphic to*

$$F_G = \{(\alpha, [\{g_n\}], 0, \beta)\} \subset E_G,$$

where $g_1 = g$ and $g_n = g|_{\beta_1\cdots\beta_{n-1}}$ for every $n \geq 2$.

From now on, we consider E_G and F_G instead of C_G and D_G, respectively, of a self-similar group (G,X). For the groupoid E_G, we define a cocycle map $c_G\colon E_G \to \mathbb{Z}$ by

$$(\alpha, [\{g_{n+l}\}], l-k, \beta) \mapsto l-k.$$

Then it is trivial that $F_G = \ker c_G$.

We remind that, for self-similar groups (G,X) and (H,Y), continuously orbit equivalence of inverse semigroup partial actions implies that there is a groupoid isomorphism $\phi\colon E_G \to E_H$ by Theorem 2.7 and Lemma 4.4.

Theorem 4.7. *Suppose that (G, X) and (H, Y) are pseudo-free and recurrent self-similar groups. If S_G partial action on X^ω and S_H partial action on Y^ω are continuously orbit equivalent and that the cocylce maps and groupoid isomorphism satisfy $|c_G| = |c_H \circ \phi|$, then G action on X^ω and H action on Y^ω are continuously orbit equivalent.*

Proof. By Definition 2.10, Lemma 3.1, Proposition 4.2, and Lemma 4.6, it is enough to show that F_G is isomorphic to F_H.

Since we assumed $|c_G| = |c_H \circ \phi|$, it is easy to see

$$c_H \circ \phi(\alpha, [\{g_n\}], 0, \beta) = 0 \text{ and } c_G \circ \phi^{-1}(\gamma, [\{h_n\}], 0, \delta) = 0$$

for every $(\alpha, [\{g_n\}], 0, \beta) \in F_G$ and $(\gamma, [\{h_n\}], 0, \delta) \in F_H$. Then it is routine to check that $F_G = c_G^{-1}(0)$ is mapped to $F_H = c_H^{-1}(0)$ by ϕ. Therefore, F_G is isomorphic to F_H by ϕ restricted on F_G, and G action on X^ω and H action on Y^ω are continuously orbit equivalent. □

Remark 4.8. We define the shift map $\sigma\colon X^\omega \to X^\omega$ by $\beta_1\beta_2\cdots \mapsto \beta_2\beta_3\cdots$. Then

$$g_1(\beta) = g_1(\beta_1\sigma(\beta)) = g_1(\beta_1)g_1|_{\beta_1}(\sigma(\beta)) = g_1(\beta_1)g_2(\sigma(\beta)) = \alpha_1\sigma(\alpha)$$

implies that the shift map naturally extends to F_G by

$$(\alpha, [\{g_n\}], 0, \beta) \mapsto (\sigma(\alpha), [\{g_{n+1}\}], 0, \sigma(\beta)).$$

Let us denote this extended shift also σ.

Recall that, for self-similar groups (G, X) and (H, Y), continuous orbit equivalence of group actions induces that there is a homeomorphism $f\colon X^\omega \to Y^\omega$.

Theorem 4.9. *Suppose that (G, X) and (H, Y) are pseudo-free and recurrent self-similar groups. If G action on X^ω and H action on Y^ω are continuously orbit equivalent and the homeomorphism $f\colon X^\omega \to Y^\omega$ commutes with the shift maps on F_G and F_H, then S_G partial action on X^ω and S_H partial action on Y^ω are continuously orbit equivalent.*

Proof. Let $\varphi\colon F_G \to F_H$ be a groupoid isomorphism and show that φ induces a groupoid isomorphism $\phi\colon E_G \to E_H$. Before proving the statement, let us note the following.

(i) For any $(\alpha, [\{g_n\}], 0, \beta), (\beta, [\{g'_n\}], 0, \gamma) \in F_G$ with

$$\varphi(\alpha, [\{g_n\}], 0, \beta) = (\xi, [\{h_n\}], 0, \eta) \in F_H,$$

$$\begin{aligned}\varphi((\alpha, [\{g_n\}], 0, \beta) \cdot (\beta, [\{g'_n\}], 0, \gamma)) &= \varphi(\alpha, [\{g_n\}], 0, \beta) \cdot \varphi(\beta, [\{g'_n\}], 0, \gamma) \\ &= (\xi, [\{h_n\}], 0, \eta) \cdot \varphi(\beta, [\{g'_n\}], 0, \gamma)\end{aligned}$$

implies

$$\varphi(\beta, [\{g'_n\}], 0, \gamma) = (\eta, [\{h'_n\}], 0, \zeta) \in F_H \text{ and } \varphi(\alpha, [\{1_G\}], 0, \alpha) = (\xi, [\{1_H\}], 0, \xi).$$

(ii) For any $(\alpha, [\{g_n\}], 0, \beta) \in F_G$ and $u, v \in X^*$, we obtain

$$(u\alpha, [\{g_{n-|u|}\}_{n>|u|}], |u| - |v|, v\beta) \in E_G.$$

Conversely, for any $(u\alpha, [\{g_{n+|u|}\}], l-k, v\beta) \in E_G$ with the conditions $|u| \geq l, |v| \geq k$ and $|u| - |v| = l - k$, we also have

$$(\alpha, [\{g_{n+|u|}\}], 0, \beta) \in F_G.$$

(iii) For a finite word $u \in X^*$, denote the infinite circuit $uuu \cdots \in X^\omega$ by $\tilde{u}$. Then it is easy to see $(\tilde{u}, [\{1_G\}], 0, \tilde{u}) \in F_G$ and $\varphi(\tilde{u}, [\{1_G\}], 0, \tilde{u}) = (\mu, [\{1_H\}], 0, \mu)$ by (i). For $\mu = \mu_1\mu_2 \cdots \in Y^\omega$, we let $\mathbf{u} \in Y^{|u|}$ be

$$\mathbf{u} = \mu_1 \cdots \mu_{|u|}.$$

Recall that we assumed that the homeomorphism $f\colon X^\omega \to Y^\omega$ commutes with the shift maps on F_G and F_H, i.e.,

$$f \circ \sigma = \sigma \circ f$$

and that $\varphi\colon F_G \to F_H$ is determined by

$$(\alpha, [\{g_n\}], 0, \beta) \mapsto (f(\alpha), [\{h_n\}], 0, f(\beta)) = (\xi, [\{h_n\}], 0, \eta)$$

such that

$$f(g_n(\sigma^{n-1}(\beta))) = h_n(f \circ \sigma^{n-1}(\beta)).$$

For $\tilde{u} \in X^\omega$ and $\mu \in Y^\omega$ as above, $\mu = f(\tilde{u})$ and $\sigma^{|u|}(\tilde{u}) = \tilde{u}$ imply that

$$\mu = f(\tilde{u}) = f \circ \sigma^{|u|}(\tilde{u}) = \sigma^{|u|} \circ f(\tilde{u}) = \sigma^{|u|}(\mu).$$

Hence, μ is also an infinite circuit $\mathbf{uu} \cdots$ in Y^ω.

Now we define the induced map ϕ: for any $(u\alpha, [\{g_{n+|u|}\}], l-k, v\beta) \in E_G$ with $(\alpha, [\{g_{n+|u|}\}], 0, \beta) \in F_G$, $\varphi(\alpha, [\{g_{n+|u|}\}], 0, \beta) = (\xi, [\{h_{n+|u|}\}], 0, \eta) \in F_H$, $\mathbf{u} \in Y^{|u|}$ and $\mathbf{v} \in Y^{|v|}$, let

$$\phi(u\alpha, [\{g_{n+|u|}\}], l-k, v\beta) = (\mathbf{u}\xi, [\{h_{n+|u|}\}], l-k, \mathbf{v}\eta).$$

Then ϕ is one-to-one because f is a homeomorphism and (iii), and it is onto by (ii) and (iii). For the homeomorphism property, we consider

$$\begin{aligned}
\phi(u\alpha, [\{g_{n+|u|}\}], l-k, v\beta) &= (\mathbf{u}\xi, [\{h_{n+|u|}\}], l-k, \mathbf{v}\eta),\\
\phi(v\beta, [\{g'_{n+|v|}\}], j-i, w\gamma) &= (\mathbf{v}\eta, [\{h'_{n+|v|}\}], j-i, \mathbf{w}\zeta).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\phi(u\alpha, [\{g_{n+|u|}\}], l-k, v\beta) \cdot \phi(v\beta, [\{g'_{n+|v|}\}], j-i, w\gamma)\\
&= (\mathbf{u}\xi, [\{h_{n+|u|}\}], l-k, \mathbf{v}\eta) \cdot (\mathbf{v}\eta, [\{h'_{n+|v|}\}], j-i, \mathbf{w}\zeta)\\
&= (\mathbf{u}\xi, [\{h_{n+|u|}h'_{n+|v|}\}], l+j-(k+i), \mathbf{w}\zeta).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(u\alpha, [\{g_{n+|u|}\}], l-k, v\beta) \cdot (v\beta, [\{g'_{n+|v|}\}], j-i, w\gamma)\\
&= (u\alpha, [\{g_{n+|u|}g'_{n+|v|}\}], l+j-(k+i), w\gamma)
\end{aligned}$$

implies

$$\begin{aligned}&\phi((u\alpha,[\{g_{n+|u|}\}],l-k,v\beta)\cdot(v\beta,[\{g'_{n+|v|}\}],j-i,w\gamma))\\&=(\mathbf{u}\xi,[\{*\}],l+j-(k+i),\mathbf{w}\zeta).\end{aligned}$$

We have to decide the $*$ part.

Combining

$$\begin{aligned}f(\sigma^{n-1}(\alpha))&=f(g_{n+|u|}(\sigma^{n-1}(\beta)))=h_{n+|u|}(f\circ\sigma^{n-1}(\beta))\text{ and}\\f(\sigma^{n-1}(\beta))&=f(g'_{n+|v|}(\sigma^{n-1}(\gamma)))=h'_{n+|v|}(f\circ\sigma^{n-1}(\gamma)),\end{aligned}$$

we have

$$f(\sigma^{n-1}(\alpha))=h_{n+|u|}(f\circ\sigma^{n-1}(\beta))=h_{n+|u|}h'_{n+|v|}(f\circ\sigma^{n-1}(\gamma)).$$

Hence, we obtain

$$\begin{aligned}&\phi((u\alpha,[\{g_{n+|u|}\}],l-k,v\beta)\cdot(v\beta,[\{g'_{n+|v|}\}],j-i,w\gamma))\\&=(\mathbf{u}\xi,[\{h_{n+|u|}h'_{n+|v|}\}],l+j-(k+i),\mathbf{w}\zeta)\\&=\phi(u\alpha,[\{g_{n+|u|}\}],l-k,v\beta)\cdot\phi(v\beta,[\{g'_{n+|v|}\}],j-i,w\gamma),\end{aligned}$$

and $\phi\colon E_G\to E_H$ is a groupoid isomorphism. Therefore, G action on X^ω and H action on Y^ω are continuously orbit equivalent by Theorem 2.7 and Lemma 4.4. □

References

[1] N. Brownlowe, T. M. Carlsen, and M. F. Whittaker, *Graph algebras and orbit equivalence*, Ergodic Theory Dynam. Systems **37** (2017), no. 2, 389–417. https://doi.org/10.1017/etds.2015.52

[2] A. Buss and R. Exel, *Inverse semigroup expansions and their actions on C^*-algebras*, Illinois J. Math. **56** (2012), no. 4, 1185–1212. http://projecteuclid.org/euclid.ijm/1399395828

[3] T. M. Carlsen and M. L. Winger, *Orbit equivalence of graphs and isomorphism of graph groupoids*, Math. Scand. **123** (2018), no. 2, 239–248. https://doi.org/10.7146/math.scand.a-105087

[4] L. G. Cordeiro and V. Beuter, *The dynamics of partial inverse semigroup actions*, J. Pure Appl. Algebra **224** (2020), no. 3, 917–957. https://doi.org/10.1016/j.jpaa.2019.06.001

[5] M. Dokuchaev and R. Exel, *Partial actions and subshifts*, J. Funct. Anal. **272** (2017), no. 12, 5038–5106. https://doi.org/10.1016/j.jfa.2017.02.020

[6] R. Exel, *Inverse semigroups and combinatorial C^*-algebras*, Bull. Braz. Math. Soc. (N.S.) **39** (2008), no. 2, 191–313. https://doi.org/10.1007/s00574-008-0080-7

[7] ———, *Partial dynamical systems, Fell bundles and applications*, Mathematical Surveys and Monographs, 224, American Mathematical Society, Providence, RI, 2017. https://doi.org/10.1090/surv/224

[8] ———, *Self-similar graphs, a unified treatment of Katsura and Nekrashevych* C*-*algebras*, Adv. Math. **306** (2017), 1046–1129. https://doi.org/10.1016/j.aim.2016.10.030

[9] M. V. Lawson, *Inverse Semigroups, the Theory of Partial Symmetries*, World Scientific, 1998.

[10] X. Li, *Partial transformation groupoids attached to graphs and semigroups*, Int. Math. Res. Not. IMRN **2017** (2017), no. 17, 5233–5259. https://doi.org/10.1093/imrn/rnw166

[11] ———, *Continuous orbit equivalence rigidity*, Ergodic Theory Dynam. Systems **38** (2018), no. 4, 1543–1563. https://doi.org/10.1017/etds.2016.98

[12] K. Matsumoto, *Orbit equivalence of topological Markov shifts and Cuntz-Krieger algebras*, Pacific J. Math. **246** (2010), no. 1, 199–225. https://doi.org/10.2140/pjm.2010.246.199

[13] ———, *Asymptotic continuous orbit equivalence of smale spaces and Ruelle algebras*, Canad. J. Math. **71** (2019), no. 5, 1243–1296. https://doi.org/10.4153/cjm-2018-012-x

[14] V. V. Nekrashevych, *Cuntz-Pimsner algebras of group actions*, J. Operator Theory **52** (2004), no. 2, 223–249.

[15] ———, *Self-similar groups*, Mathematical Surveys and Monographs, 117, American Mathematical Society, Providence, RI, 2005. https://doi.org/10.1090/surv/117

[16] ———, *C^*-algebras and self-similar groups*, J. Reine Angew. Math. **630** (2009), 59–123. https://doi.org/10.1515/CRELLE.2009.035

[17] A. L. T. Paterson, *Groupoids, Inverse Semigroups, and their Operator Algebras*, Progress in Mathematics, 170, Birkhäuser Boston, Inc., Boston, MA, 1999. https://doi.org/10.1007/978-1-4612-1774-9

[18] J. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Mathematics, 793, Springer, Berlin, 1980.

[19] C. Starling, *Inverse semigroups associated to subshifts*, J. Algebra **463** (2016), 211–233. https://doi.org/10.1016/j.jalgebra.2016.06.014

[20] I. Yi, *Orbit equivalence on self-similar groups and their C^*-algebras*, J. Korean Math. Soc. **57** (2020), no. 2, 383–399. https://doi.org/10.4134/JKMS.j190090

Inhyeop Yi
Department of Mathematics Education
Ewha Womans University
Seoul 03760, Korea
Email address: yih@ewha.ac.kr