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ON DIVISORS COMPUTING MLD'S AND LCT'S

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Abstract. We show that if a divisor centered over a point on a smooth surface computes a minimal log discrepancy, then the divisor also computes a log canonical threshold. To prove the result, we study the asymptotic log canonical threshold of the graded sequence of ideals associated to a divisor over a variety. We systematically study this invariant and prove a result describing which divisors compute asymptotic log canonical thresholds.

1. Introduction

The *log canonical threshold* and *minimal log discrepancy* are two invariants of singularities that arise naturally in the study of birational geometry. Minimal log discrepancies are of particular interest due to work of Shokoruv [23] in which he proved that two conjectures on minimal log discrepancies (semicontinuity and the ascending chain condition (ACC)) imply the termination of flips, a result needed to complete the minimal model program in full generality.

Shokurov originally conjectured that both the set of minimal log discrepancies and log canonical thresholds in fixed dimension should satisfy the ACC. The conjecture was proven for log canonical thresholds on smooth varieties in [4] and later in full generality [9]. The general form of the ACC conjecture for minimal log discrepancies remains open. In this way, as well as others, minimal log discrepancies are less well understood than log canonical thresholds.

In order to define these two invariants, we recall the following notions. Let X be a normal variety such that K_X is $\mathbf{Q}$-Cartier. We call E a *divisor over*[1] X if E is a prime divisor on a normal variety Y, proper and birational over X, and $(X, \mathfrak{a}^\lambda)$ a *pair* if $\mathfrak{a} \subseteq \mathcal{O}_X$ a nonzero ideal and $\lambda \in \mathbf{R}_{\geq 0}$. The *log discrepancy* of a pair $(X, \mathfrak{a}^\lambda)$ along E is defined as

$$a\big(E; X, \mathfrak{a}^\lambda\big) := k_E + 1 - \lambda \operatorname{ord}_E(\mathfrak{a}),$$

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[1]In the literature, this is sometimes referred to as a *prime divisor over X*.

where k_E is the coefficient of E in the relative canonical divisor and ord_E is the valuation given by order of vanishing along E. A pair $(X, \mathfrak{a}^\lambda)$ is klt (resp., lc) if for all divisors E over X, $a(E; X, \mathfrak{a}^\lambda) > 0$ (resp., ≥ 0).

Arising from these definitions are two invariants that measure the "nastiness" of a singularity. The *log canonical threshold* of a nonzero ideal $\mathfrak{a}$ on X is defined as

$$\mathrm{lct}(\mathfrak{a}) := \sup\{\lambda \in \mathbf{R}_{\geq 0} \mid (X, \mathfrak{a}^\lambda) \text{ is lc}\}.$$

Given a klt pair $(X, \mathfrak{a}^\lambda)$ and a (not necessarily closed) point $\eta \in X$, the *minimal log discrepancy* of $(X, \mathfrak{a}^\lambda)$ at η is defined as

$$\mathrm{mld}_\eta(X, \mathfrak{a}^\lambda) = \min\{a(E; X, \mathfrak{a}^\lambda) \mid E \text{ is a divisor over } X \text{ with } c_X(E) = \overline{\eta}\}.$$

See Section 2 for further details on these definitions.

In understanding these two invariants it is natural to make the following definition. Given a divisor E over X, we say that E *computes a log canonical threshold* if there exists a non-zero ideal $\mathfrak{a}$ on X such that

$$a\big(E; X, \mathfrak{a}^{\mathrm{lct}(\mathfrak{a})}\big) = 0.$$

Similarly, we say that E *computes a minimal log discrepancy* if there exists an lc pair $(X, \mathfrak{a}^\lambda)$ such that

$$\mathrm{mld}_\eta(X, \mathfrak{a}^\lambda) = a\big(E; X, \mathfrak{a}^\lambda\big)$$

with $\overline{\eta} = c_X(E)$.

Question 1.1. Which divisors over a variety compute log canonical thresholds (resp., minimal log discrepancies)?

Divisors computing log canonical thresholds satisfy special properties. As we will explain shortly, it is well known that divisors computing log canonical thresholds have finitely generated graded sequences of ideals. It is not known if the same can be said for divisors computing minimal log discrepancies. While it is clear from the above definition that divisors computing log canonical thresholds also compute minimal log discrepancies, the reverse statement is not known.

The goal of this paper is to study an invariant that provides information on whether a divisor computes a log canonical threshold or minimal log discrepancy. As a consequence of this analysis, we prove the following result for surfaces.[2]

Theorem 1.2. *If X is a smooth surface, then every divisor over X centered at a point that computes a minimal log discrepancy also computes a log canonical threshold.*

[2] Shortly after this paper was originally posted to the arXiv, Kawakita posted a related result which he proved independently [14]. Kawakita showed that if E is a divisor over a smooth surface X with $c_X(E) = \{x\}$ such that E computes a minimal log discrepancy, then ord_E is a monomial valuation in some local coordinates at x.

It has long been understood which divisors compute log canonical thresholds on smooth surfaces. For example, see [6], [18], [24], and [25]. In particular, [6, Lemma 2.11] implies that if E is a divisor over a smooth surface X with $c_X(E) = \{x\}$ such that E computes a log canonical threshold, then ord_E is a monomial valuation in some analytic coordinates at x.

In proving Theorem 1.2, we make use of the following object associated to a divisor over a variety. Let $f : Y \to X$ be a proper birational morphism of normal varieties and E a prime divisor on Y. Associated to E, there is a corresponding graded sequence of ideals $\mathfrak{a}_\bullet^E = \{\mathfrak{a}_m^E\}_{m\in\mathbf{N}}$ on X defined as

$$\mathfrak{a}_m^E := f_*\mathcal{O}_Y(-mE).$$

Recall that a graded sequence of ideals $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_{m\in\mathbf{N}}$ on X is a sequence of ideals on X such that $\mathfrak{a}_m \cdot \mathfrak{a}_n \subseteq \mathfrak{a}_{m+n}$ for all $m, n \in \mathbf{N}$. We say that a graded sequence of ideals $\mathfrak{a}_\bullet$ is finitely generated if the graded $\mathcal{O}_X$-algebra $R(\mathfrak{a}_\bullet) := \bigoplus_{m\in\mathbf{N}} \mathfrak{a}_\bullet$ is of finite type.

Using this formalism, we find a criterion to classify which divisors over X compute log canonical thresholds. As utilized in [13], if E computes $\mathrm{lct}(\mathfrak{b})$, then E computes $\mathrm{lct}(\mathfrak{a}_m^E)$ for all m divisible by $\mathrm{ord}_E(\mathfrak{b})$. Similarly, if E computes $\mathrm{mld}_\eta(X, \mathfrak{b}^\lambda)$, then E computes $\mathrm{mld}_\eta(X, (\mathfrak{a}_m^E)^{\lambda/m})$ for all $m \in \mathbf{Z}_{>0}$ divisible by $\mathrm{ord}_E(\mathfrak{b})$.

In order to prove Theorem 1.2, we show that if X is a smooth surface and E is a divisor centered over a closed point that computes a minimal log discrepancy, then E computes the asymptotic log canonical threshold $\mathrm{lct}(\mathfrak{a}_\bullet^E)$. (See Section 2.6 for the definition of a the log canonical threshold of a graded a graded sequence of ideals). After proving the previous statement, the following theorem completes the proof of Theorem 1.2.

Theorem 1.3. *Let E be a divisor over a klt variety X. The following conditions are equivalent.*

(1) *The divisor E computes a log canonical threshold.*
(2) *The divisor E computes an asymptotic log canonical threshold.*
(3) *The equality* $\mathrm{lct}(\mathfrak{a}_\bullet^E) = k_E + 1$ *holds.*

The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) appear in [13] and follow from definitions. The remaining implication requires a finite generation statement arising from the MMP.[3] We find the equivalence between (1) and (2) to be somewhat surprising.

As we will see, the value $\mathrm{lct}(\mathfrak{a}_\bullet^E)$ is an important invariant with geometric ramifications. First, we note the following interpretation of the finite generation of $\mathfrak{a}_\bullet^E$.

[3]When X is a surface, this implication does not require the machinery of the MMP and has an elementary proof (see Remark 5.1). Therefore, if one aims at proving Theorem 1.2 alone, our argument does not require results from the MMP.

Theorem 1.4 ([11, Corollary 3.3]). *Let X be a normal variety and E a divisor over X. If $\mathfrak{a}_\bullet^E$ is finitely generated, then*

$$Y := \underline{\mathrm{Proj}}_X\Big(\bigoplus_{m\geq 0}\mathfrak{a}_m^E\Big)\to X$$

is a proper birational morphism of normal varieties and there exists a prime divisor E_Y on Y that is the birational transform of E. Furthermore, E_Y is $\mathbf{Q}$-Cartier and $-E_Y$ is relatively ample over X.

In the case when $\mathrm{codim}_X(c_X(E))\geq 2$, the above statement implies that E_Y is the only exceptional divisor of X. Such birational morphisms with exactly one exceptional divisor were studied in [11] and referred to as *prime blowups*. In [22], the author looked at *plt-blowups* which can be thought of as prime blowups with mild singularities (see Definition 4). The following proposition relates the value of $\mathrm{lct}(\mathfrak{a}_\bullet^E)$ to the model $Y := \underline{\mathrm{Proj}}_X(\oplus_{m\geq 0}\mathfrak{a}_m^E)$.

Theorem 1.5. *If X is a klt variety and E a divisor over X with $k_E < \mathrm{lct}(\mathfrak{a}_\bullet^E)$, then the following hold.*

1. *The graded sequence $\mathfrak{a}_\bullet^E$ is finitely generated.*
2. *The variety Y is klt, and $\mathrm{lct}(E_Y) = \mathrm{lct}(\mathfrak{a}_\bullet^E) - k_E$ (where Y and E_Y are defined in Theorem 1.4).*

The first assertion of the above proposition is a consequence of the well known fact that if a divisor E has log discrepancy in the interval [0,1) along a klt pair, then $\mathfrak{a}_\bullet^E$ is finitely generated (see [16, Corollary 1.39]). Note that there are examples of divisors over smooth varieties with non-finitely generated graded sequences of ideals [3, 19].

Most of this paper is spent systematically studying the graded sequence of ideals associated to a divisor over a variety. We emphasize that many of the results in this paper are not needed in the proof of Theorem 1.2.

Structure of the Paper: In Section 2, we provide preliminary information on log discrepancies, log canonical thresholds, and graded sequence of ideals. Section 3 provides a proof of Theorem 1.4 and related results. In Section 4, we prove Theorem 1.5. Section 5 gives a proof of Theorem 1.3 and a condition on which divisors compute minimal log discrepancies. Section 6 concerns minimal log discrepancies on surfaces and gives a proof of Theorem 1.2. Lastly, Section 7 provides computations of $\mathrm{lct}(\mathfrak{a}_\bullet^E)$ in a couple examples.

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2. Preliminaries

Conventions: Throughout this paper, a *variety* is an integral separated scheme of finite type over a field k. Furthermore, we will assume that k is of characteristic 0 and algebraically closed.

2.1. Log resolutions

Let X be a variety and $\mathfrak{a} \subseteq \mathcal{O}_X$ a nonzero ideal. A morphism $f : Y \to X$ is a *log resolution* of $(X, \mathfrak{a})$ if f is a projective birational morphism, Y is smooth, $\operatorname{Exc}(f)$ has pure codimension one, $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D)$ for some effective divisor D on Y, and $\operatorname{Exc}(f) + D$ is a simple normal crossing divisor.

2.2. Divisors over a variety

Let $f : Y \to X$ be a proper birational morphism of normal varieties and E a prime divisor on Y. We define the *center* of E on X to be $c_X(E) := f(E)$. Arising from E is a discrete valuation of $K(X)$ that we denote by ord_E. The valuation corresponds to the DVR $\mathcal{O}_{Y,E} \subseteq K(Y) \simeq K(X)$.

Given $Y \to X$ and $Y' \to X$ as above, we identify a prime divisor E on Y with a prime divisor E' on Y' if they induce the same valuation of $K(X)$. This is equivalent to the condition that E' is the birational transform of E.[4] A *divisor over* X is an equivalence class given by this relation.

Given a nonzero ideal $\mathfrak{a} \subseteq \mathcal{O}_X$, we set

$$\operatorname{ord}_E(\mathfrak{a}) := \min\{\operatorname{ord}_E(f) \mid f \in \mathfrak{a} \cdot \mathcal{O}_{X,c_X(E)}\}.$$

We use the convention that $\operatorname{ord}_E(\mathbf{0}) = +\infty$, where $\mathbf{0}$ denotes the zero ideal.

For an effective $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor D on X, we write $\operatorname{ord}_E(D)$ for the coefficient of E along f^*D. Note that if $m \in \mathbf{Z}_{>0}$ is chosen so that mD is Cartier, then $\operatorname{ord}_E(D) = m^{-1} \operatorname{ord}_E(\mathcal{O}_X(-mD))$.

2.3. Rees valuations

Let X be a normal variety and $\mathfrak{a} \subseteq \mathcal{O}_X$ a nonzero ideal. Write $\pi : Z \to X$ for the normalized blowup of X along $\mathfrak{a}$ and D for the effective Cartier divisor on Z such that $\mathfrak{a} \cdot \mathcal{O}_Z = \mathcal{O}_Z(-D)$. The *integral closure of* $\mathfrak{a}$ is the ideal

$$\overline{\mathfrak{a}} := \pi_*\mathcal{O}_Z(-D)\,.$$

It is clear that $\mathfrak{a} \subseteq \overline{\mathfrak{a}}$. We say that $\mathfrak{a}$ is *integrally closed* if $\mathfrak{a} = \overline{\mathfrak{a}}$. The *Rees valuations of* $\mathfrak{a}$ are the valuations of $K(X)$ corresponding to prime divisors in the support of D. The following propositions provide information on Rees valuations and integral closures of ideals.

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[4]By the *birational transform* of E on Y', we mean the closure of $g(E\cap U)$, where g denotes the birational map $Y \dashrightarrow Y'$ and $U \subset Y$ is the open set on which the map is regular.

Proposition 2.1. *Let $\mathfrak{a}$ be a nonzero ideal on a normal variety X. The set of Rees valuations of $\mathfrak{a}$ is the valuations corresponding to the smallest set of divisors $\{E_1, \dots, E_r\}$ over X such that*

$$\overline{\mathfrak{a}^m} = \bigcap_{i=1}^{r} \{f \in \mathcal{O}_X \mid \operatorname{ord}_{E_i}(f) \geq m \cdot \operatorname{ord}_{E_i}(\mathfrak{a})\}$$

for all $m \in \mathbf{Z}_{>0}$.

Proof. See [10, Section 10.2]. □

Proposition 2.2. *Let $\mathfrak{a}$ be a nonzero ideal on a normal variety X. The ideal $\mathfrak{a}$ is integrally closed if and only if there exists a proper birational morphism $f : Y \to X$ with Y normal and D an effective Cartier divisor on Y such that*

$$\mathfrak{a} = f_* \mathcal{O}_Y(-D).$$

Proof. The forward direction follows immediately from the definition of a Rees valuation. See [20, Proposition 9.6.11] for the reverse direction. □

Proposition 2.3. *If X is a normal variety and $\mathfrak{a}$ a nonzero ideal on X such that $\mathfrak{a}^n$ is integrally closed for all $n \in \mathbf{Z}_{>0}$, then the blowup of X along $\mathfrak{a}$ is normal.*

Proof. This follows from [10, Proposition 5.2.1] □

2.4. Log discrepancies

Throughout, let X be a normal variety with K_X $\mathbf{Q}$-Cartier. If $f : Y \to X$ is a proper birational morphism of normal varieties, the *relative canonical divisor* of the morphism is defined as $K_{Y/X} := K_Y - f^* K_X$ where K_Y and K_X are chosen so that $f_* K_Y = K_X$. If E is a prime divisor on Y, we set $k_E := \operatorname{coeff}_E(K_{Y/X})$. When the base variety is unclear, we will use the notation $k_{E,X}$. The value k_E is not dependent on the model Y but on the valuation ord_E.

If $\mathfrak{a} \subseteq \mathcal{O}_X$ is a nonzero ideal and $\lambda \in \mathbf{R}_{\geq 0}$, we refer to $(X, \mathfrak{a}^\lambda)$ as a *pair* and define the *log discrepancy* of $(X, \mathfrak{a}^\lambda)$ along E as

$$a\big(E; X, \mathfrak{a}^\lambda\big) := k_E + 1 - \lambda \operatorname{ord}_E(\mathfrak{a}).$$

We say that the pair $(X, \mathfrak{a}^\lambda)$ is klt (resp. lc) if $a\big(E; X, \mathfrak{a}^\lambda\big) > 0$ (resp. ≥ 0) for all prime divisors E over X.

If Δ is a $\mathbf{Q}$-Cartier divisor, the log discrepancy of (X, Δ) along E is

$$a\big(E; X, \Delta\big) := k_E + 1 - \operatorname{ord}_E(\Delta).$$

We say that (X, Δ) is a klt (resp. lc) pair if $a\big(E; X, \Delta\big) > 0$ (resp. ≥ 0) for all divisors E over X. We say that (X, Δ) is plt if $a\big(E; X, \Delta\big) > 0$ for all divisors E over X with $\operatorname{codim} c_X(E) \geq 2$. Lastly, we say that X is klt (resp. log canonical) if $(X, 0)$ is klt (resp. log canonical).

2.5. Minimal log discrepancies

Let X be a klt variety. Given a pair $(X, \mathfrak{a}^\lambda)$ and a (not necessarily closed) point $\eta \in X$, we define the *minimal log discrepancy* of $(X, \mathfrak{a}^\lambda)$ at η to be

$$\mathrm{mld}_\eta(X, \mathfrak{a}^\lambda) = \inf \left\{ a\big(E; X, \mathfrak{a}^\lambda\big) \mid E \text{ is a divisor over } X \text{ with } c_X(E) = \overline{\eta} \right\}.$$

Assuming $\mathrm{codim}_X(\overline{\eta}) \geq 2$, the value of $\mathrm{mld}_\eta(\mathfrak{a}^\lambda)$ is either ≥ 0 or $= -\infty$. If the above infimum is ≥ 0, it is necessarily a minimum. See [1] for further details on minimal log discrepancies.

We say a divisor E over X computes a minimal log discrepancy, if there exists a pair $(X, \mathfrak{a}^\lambda)$ such that $\mathrm{mld}_\eta(X, \mathfrak{a}^\lambda) = a\big(E; X, \mathfrak{a}^\lambda\big)$, where $\overline{\eta} = c_X(E)$.

2.6. Log canonical thresholds

For a nonzero ideal $\mathfrak{a}$ on a klt variety X, the *log canonical threshold* of the ideal is defined as

$$\mathrm{lct}(\mathfrak{a}) := \sup\{\lambda \in \mathbf{Q}_{\geq 0} \mid (X, \mathfrak{a}^\lambda) \text{ is lc}\}.$$

From this definition, it follows that $\mathrm{lct}(\mathcal{O}_X) = +\infty$. We define $\mathrm{lct}(\mathbf{0}) = 0$.

Since $(X, \mathfrak{a}^\lambda)$ is lc if and only if $k_E + 1 - \lambda \,\mathrm{ord}_E(\mathfrak{a}) \geq 0$ for all divisors E over X, it follows that

$$\mathrm{lct}(\mathfrak{a}) = \inf \left\{ \frac{k_E + 1}{\mathrm{ord}_E(\mathfrak{a})} \;\middle|\; E \text{ is a divisor over } X \text{ with } \mathrm{ord}_E(\mathfrak{a}) > 0 \right\}.^5$$

We say that a divisor E over X *computes* $\mathrm{lct}(\mathfrak{a})$ if it achieves the above infimum. We say E *computes a log canonical threshold* if there exists a nonzero ideal $\mathfrak{a}$ on X such that E computes $\mathrm{lct}(\mathfrak{a})$.

Similarly, if Δ is an effective $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor on X, then the log canonical threshold of Δ is

$$\mathrm{lct}(X, \Delta) := \sup\{\lambda \in \mathbf{R}_{\geq 0} \mid (X, \lambda\Delta) \text{ is lc}\}.$$

The following elementary properties of log canonical thresholds give rise to the definition of the log canonical threshold for a graded sequence of ideals.

Lemma 2.4. *Let $\mathfrak{a}, \mathfrak{b}$ be nonzero ideals on a klt variety X. The following hold*:

(1) $\mathrm{lct}(\mathfrak{a}) = m \cdot \mathrm{lct}(\mathfrak{a}^m)$.
(2) $\mathrm{lct}(\mathfrak{a}) \leq \mathrm{lct}(\mathfrak{b})$ *if* $\mathfrak{a} \subseteq \mathfrak{b}$.

[5] Note that if $Y \to X$ is a log resolution of $(X, \mathfrak{a})$ and $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D)$, it is sufficient to take the above infimum over all prime divisors contained in $\mathrm{Supp}(D)$ (see [20, Section 9.3]). Since $\mathrm{Supp}(D)$ has finitely many components, the above infimum is necessarily a minimum.

2.7. Graded sequences of ideals

A *graded sequence of ideals* on a variety X is a sequence of ideals $\mathfrak{a}_\bullet = (\mathfrak{a}_m)_{m\in\mathbf{Z}_{>0}}$ on X such that $\mathfrak{a}_m \cdot \mathfrak{a}_n \subseteq \mathfrak{a}_{m+n}$ for all $m, n \in \mathbf{Z}_{>0}$. By convention, we set $\mathfrak{a}_0 := \mathcal{O}_X$ and will always assume $\mathfrak{a}_m$ is nonzero for some positive integer m.

The *Rees algebra* of $\mathfrak{a}_\bullet$ is the $\mathbf{N}$-graded $\mathcal{O}_X$-algebra

$$R(\mathfrak{a}_\bullet) = \bigoplus_{m\in\mathbf{N}} \mathfrak{a}_m.$$

We say that $\mathfrak{a}_\bullet$ is *finitely generated* if $R(\mathfrak{a}_\bullet)$ is a finitely generated $\mathcal{O}_X$-algebra. We say that $\mathfrak{a}_\bullet$ is *finitely generated in degree* n if $\mathfrak{a}_{nm} = (\mathfrak{a}_n)^m$ for all $m \in \mathbf{N}$. As a consequence of [8, Lemma 2.1.6.v], a graded sequence of ideals $\mathfrak{a}_\bullet$ is finitely generated if and only there exists $n \in \mathbf{Z}_{>0}$ such that $\mathfrak{a}_\bullet$ is finitely generated in degree n.

We list three examples of graded sequences of ideals that arise in algebraic geometry.

- For a trivial example, let $\mathfrak{b}$ be an ideal on X and define $\mathfrak{a}_m := \mathfrak{b}^m$.
- Let $f : Y \to X$ be a proper birational morphism of normal varieties and E be a prime divisor on Y. The divisor E gives rise to a graded sequence of ideals, denoted by $\mathfrak{a}_\bullet^E$, defined by

$$\mathfrak{a}_m^E := f_*\mathcal{O}_Y(-mE).$$

  Note that this only depends on E and not on the model Y.
- Let $\mathcal{L}$ be a line bundle on a variety X, having nonnegative Kodaira dimension and $\mathfrak{b}_m(\mathcal{L})$ denote the base locus of $|\mathcal{L}^m|$. This example was studied in [5].

For a graded sequence $\mathfrak{a}_\bullet$, let $S(\mathfrak{a}_\bullet) = \{m \in \mathbf{N} \mid \mathfrak{a}_m \neq \mathbf{0}\}$. We define

$$\operatorname{ord}_E(\mathfrak{a}_\bullet) := \lim_{m\to\infty, m\in S(\mathfrak{a}_\bullet)} \frac{1}{m}\operatorname{ord}_E(\mathfrak{a}_m) = \inf_{m\geq 1} \frac{1}{m}\operatorname{ord}_E(\mathfrak{a}_m).$$

See [13, Section 2] for further details.

Lemma 2.5. *Let E be a divisor over a normal variety X. We have that*

(1) $\operatorname{ord}_E(\mathfrak{a}_m^E) = m$ *for all* $m \in \mathbf{Z}_{>0}$ *sufficiently divisible and*
(2) $\operatorname{ord}_E(\mathfrak{a}_\bullet^E) = 1$.

Proof. Since elements of $\mathfrak{a}_m^E$ vanish to at least order m along E, we see that $\operatorname{ord}_E(\mathfrak{a}_m^E) \geq m$. Set $n := \operatorname{ord}_E(\mathfrak{a}_1^E)$. Since $(\mathfrak{a}_1^E)^m \subset \mathfrak{a}_{n\cdot m}^E$, we see that $\operatorname{ord}_E(\mathfrak{a}_{n\cdot m}) \leq n \cdot m$ and (1) holds. Statement (2) follows from (1). □

2.8. Asymptotic log canonical thresholds.

Let X be a graded sequence of ideals on a klt variety X. Since $m \cdot \operatorname{lct}(\mathfrak{a}_m) \leq mn \cdot \operatorname{lct}(\mathfrak{a}_{mn})$ by Lemma 2.4, the *asymptotic log canonical threshold* of $\mathfrak{a}_\bullet$ is defined by

$$\operatorname{lct}(\mathfrak{a}_\bullet) := \sup_{m\geq 1} m\cdot \operatorname{lct}(\mathfrak{a}_m) = \lim_{m\to\infty, m\in S(\mathfrak{a}_\bullet)} m\cdot \operatorname{lct}(\mathfrak{a}_m).$$

For a proof of the second equality, see [13, Lemma 2.3].

In the following statements we collect some basic information on this asymptotic invariant.

Proposition 2.6. *Let X be a klt variety and $\mathfrak{a}_\bullet$ a graded sequence of ideals on X. The following hold*:

(1) $\operatorname{lct}(\mathfrak{a}_\bullet) = \inf_F \frac{k_F+1}{\operatorname{ord}_F(\mathfrak{a}_\bullet)}$, *where the infimum runs over all divisors F over X.*
(2) *If $\mathfrak{a}_\bullet$ is finitely generated in degree n, then* $\operatorname{lct}(\mathfrak{a}_\bullet) = n\cdot\operatorname{lct}(\mathfrak{a}_n)$.

Proof. See [13, Corollary 2.16] for (1). Lemma 2.4.1 implies (2) holds. □

In light of the previous statement, we say E *computes* $\operatorname{lct}(\mathfrak{a}_\bullet)$ if $\operatorname{lct}(\mathfrak{a}_\bullet) = \frac{k_E+1}{\operatorname{ord}_E(\mathfrak{a}_\bullet)}$. We say E *computes an asymptotic log canonical threshold* if there exists a graded sequence of ideals $\mathfrak{a}_\bullet$ such that E computes $\operatorname{lct}(\mathfrak{a}_\bullet)$.

Lemma 2.7. *If E is a divisor over a klt variety X, then*

$$\operatorname{lct}(\mathfrak{a}^E_\bullet) \leq k_E + 1$$

and equality holds if and only if E computes $\operatorname{lct}(\mathfrak{a}^E_\bullet)$.

Proof. The statement follows from Lemma 2.5.2 and Proposition 2.6.1. □

The following proposition explains the difference between $\operatorname{lct}(\mathfrak{a}^E_\bullet)$ and k_E+1.

Proposition 2.8. *If X is a klt variety and E a divisor over X, then*

$$k_E + 1 - \operatorname{lct}(\mathfrak{a}^E_\bullet) = \inf\left\{a(E;X,\mathfrak{a}^\lambda) \mid (X,\mathfrak{a}^\lambda) \text{ is an lc pair}\right\}.$$

Proof. We first note that

$$\begin{aligned} k_E + 1 - \operatorname{lct}(\mathfrak{a}^E_\bullet) &= \inf_{m\geq 1}\{k_E + 1 - m\cdot\operatorname{lct}(\mathfrak{a}^E_m)\} \\ &= \lim_{m\to\infty}\left(k_E+1-m\cdot\operatorname{lct}(\mathfrak{a}^E_m)\right). \end{aligned} \tag{1}$$

By Lemma 2.5, for m divisible enough $\operatorname{ord}_E(\mathfrak{a}^E_m) = m$, and, thus,

$$k_E + 1 - m\cdot\operatorname{lct}(\mathfrak{a}^E_m) = a\big(E;X,(\mathfrak{a}^E_m)^{\operatorname{lct}(\mathfrak{a}^E_m)}\big).$$

Since $(X,(\mathfrak{a}^E_m)^{\operatorname{lct}(\mathfrak{a}^E_m)})$ is lc, the relation "$\geq$" of the desired equality follows.

To show "$\leq$" holds, fix an lc pair $(X,\mathfrak{b}^\lambda)$ and set $m' = \operatorname{ord}_E(\mathfrak{b})$. Since $\mathfrak{b}\subseteq\mathfrak{a}^E_{m'}$, we have $\lambda\leq\operatorname{lct}(\mathfrak{b})\leq\operatorname{lct}(\mathfrak{a}^E_{m'})$. Therefore,

$$a\big(E;X,\mathfrak{b}^\lambda\big) := k_E+1-\lambda\operatorname{ord}_E(\mathfrak{b}) \geq k_E+1-m'\operatorname{lct}(\mathfrak{a}^E_{m'}).$$

Applying (1) now shows $a\big(E;X,\mathfrak{b}^\lambda\big) \geq k_E+1-\operatorname{lct}(\mathfrak{a}^E_\bullet)$, which completes the proof. □

3. Rees valuations and graded sequences

In this section, we prove Theorem 1.4. While the statement was proven in [11], we give a short proof for the benefit of the reader.

Lemma 3.1. *Let E be a divisor over a normal variety X. If $\mathfrak{a}_\bullet^E$ is finitely generated in degree n, then the following hold.*

(1) *The powers of the ideal $\mathfrak{a}_n^E$ are integrally closed.*
(2) *The ideal $\mathfrak{a}_n^E$ has exactly one Rees valuation, namely* ord_E.

Proof. Let $f : Y \to X$ be a projective birational morphism such that Y is normal and E appears as a prime divisor on Y. For each positive integer m, we have

$$f_*\mathcal{O}_Y(-(n\cdot m)E) = \mathfrak{a}_{m\cdot n}^E = (\mathfrak{a}_n^E)^m,$$

where the first equality comes from the definition of $\mathfrak{a}_\bullet^E$ and the second from our assumption that $\mathfrak{a}_\bullet^E$ is finitely generated in degree n. Proposition 2.2 now implies $(\mathfrak{a}_n^E)^m$ is integrally closed, while Proposition 2.1 implies E is the unique Rees valuation of $\mathfrak{a}_n^E$. □

Theorem 1.4 is an easy consequence of the previous lemma.

Proof of Theorem 1.4. Fix $n \in \mathbf{Z}_{>0}$ so that $\mathfrak{a}_\bullet^E$ is finitely generated in degree n. Note that $Y := \underline{\mathrm{Proj}}_X\left(\bigoplus_{m\geq 0}\mathfrak{a}_m^E\right)$ is isomorphic to the blowup of X along $\mathfrak{a}_n^E$. Indeed, $Y \simeq \underline{\mathrm{Proj}}\left(\bigoplus_{m\geq 0}\mathfrak{a}_{nm}^E)\right)$ by the Veronese isomorphism and $\mathfrak{a}_{nm}^E = (\mathfrak{a}_n^E)^m$ for all $m \in \mathbf{N}$.

Lemma 3.1.1 combined with Proposition 2.3 now implies Y is normal. By Lemma 3.1.2, we know $\mathfrak{a}_n^E \cdot \mathcal{O}_Y = \mathcal{O}_Y(-nE_Y)$, where E_Y is a prime divisor on Y satisfying $\mathrm{ord}_{E_Y} = \mathrm{ord}_E$. Since Y is isomorphic to the blowup of X along $\mathfrak{a}_n^E$, we conclude $-nE_Y$ is Cartier and f-ample. □

4. Finite generation using MMP

In this section we prove Theorem 1.5, which gives a sufficient condition for the finite generation of $\mathfrak{a}_\bullet^E$. One of the key ingredients of the proof is the following proposition that is well known to experts and follows from [2].

Proposition 4.1. *Let (X,Δ) be a klt pair. If E is a divisor over X such that $a(E,X,\Delta) < 1$, then $\mathfrak{a}_\bullet^E$ is finitely generated.*

Proof. Set $a := a\big(E;X,\Delta\big) < 1$ and fix a log resolution $f : Y \to X$ of the pair (X,Δ) such that E appears on Y. We have

$$K_Y + f_*^{-1}\Delta \sim_{\mathbf{Q}} f^*(K_X+\Delta) + (a-1)E + \sum_{i=1}^{r}(a\big(E_i;X,\Delta\big)-1)F_i,$$

where $f_*^{-1}\Delta$ denotes the strict transform of Δ and $E_1, \ldots, E_r$ are the exceptional divisors of f not equal to E. Since (X, Δ) is klt, $a\big(E_i; X, \Delta\big) > 0$ for all i. Thus, we may choose $0 < \epsilon \ll 1$ so that $a\big(E_i; X, \Delta\big) - \epsilon > 0$ for all i. Hence,

$$K_Y + f_*^{-1}\Delta + (1-\epsilon)\sum E_i \sim_{\mathbf{Q},f} (a-1)E_Y + \sum_{i=1}^{r}(a\big(E_i; X, \Delta\big) - \epsilon)E_i.$$

Now, $(Y, f_*^{-1}\Delta + (1-\epsilon)\sum E_i)$ is a klt pair, since Y is smooth and $f_*^{-1}\Delta + (1-\epsilon)\sum E_i$ is a simple normal crossing divisor with coefficients in $[0,1)$. Additionally, $K_Y + f_*^{-1}\Delta + (1-\epsilon)\sum E_i$ is f-big, since f is a birational morphism. Therefore, [2, Theorem 1.2] implies

$$\bigoplus_{m \geq 0} f_*\mathcal{O}_Y(\lfloor m(K_Y + f_*^{-1}\Delta + (1-\epsilon)\sum E_i)\rfloor)$$

is a finitely generated $\mathcal{O}_X$-algebra. Additionally, so is

$$\bigoplus_{m \geq 0} f_*\mathcal{O}_Y(\lfloor m((a-1)E_Y + \sum_{i=1}^{r}(a\big(E_i; X, \Delta\big) - \epsilon)E_i)\rfloor).$$

Since $a\big(E_i; X, \Delta\big) - \epsilon > 0$ for all i, it follows

$$f_*\mathcal{O}_Y(\lfloor m((a-1)E_W + \sum_{i=1}^{r}(a\big(E_i; X, \Delta\big) - \epsilon)E_i)\rfloor) = f_*\mathcal{O}_Y(\lfloor m(a-1)E_W\rfloor).$$

After taking a proper Veronese of the previous graded $\mathcal{O}_X$-algebra, the result follows. □

The following lemma will give a condition for when the hypotheses in Proposition 4.1 hold.

Lemma 4.2. *Let E be a divisor over an affine klt variety X. If $k_E < \operatorname{lct}(\mathfrak{a}_\bullet^E)$, then there exists an effective $\mathbf{Q}$-divisor Δ such that (X, Δ) is a klt pair and $a\big(E; X, \Delta\big) < 1$.*

Proof. Choose $m \in \mathbf{Z}_{>0}$ sufficiently divisible so that

$$k_E < m \cdot \operatorname{lct}(\mathfrak{a}_m^E), \quad \operatorname{lct}(\mathfrak{a}_m^E) < 1, \quad \text{and} \quad \operatorname{ord}_E(\mathfrak{a}_m^E) = m.$$

As described in [20, Proposition 9.2.28], for a general element $f \in \mathfrak{a}_m^E$,

$$\operatorname{lct}(f) = \operatorname{lct}(\mathfrak{a}_m^E) \quad \text{and} \quad \operatorname{ord}_E(f) = m.$$

Now, set $\Delta = \operatorname{lct}(f)\{f = 0\}$. Note that (X, Δ) is log canonical by construction and satisfies

$$a\big(E; X, \Delta\big) = k_E + 1 - \operatorname{lct}(f)\operatorname{ord}_E(f) = k_E + 1 - m \cdot \operatorname{lct}(\mathfrak{a}_m^E) < 1.$$

While (X, Δ) is lc (but not klt), $\Delta' = (1-\epsilon)\Delta$ will satisfy the conclusion of the lemma for $0 < \epsilon \ll 1$. □

The following proposition is the last ingredient needed to prove Theorem 1.5.

Proposition 4.3. *Let X be a klt variety and E a divisor over X with $\mathfrak{a}_\bullet^E$ finitely generated. Let $f : Y \to X$ and E_Y be as defined in Theorem 1.4. The following hold*:

(i) *The divisor K_Y is $\mathbf{Q}$-Cartier.*
(ii) *For a divisor F over Y and $\lambda \in \mathbf{R}_{\geq 0}$, we have*

$$a\big(F; Y, \lambda E_Y\big) = k_{F,X} + 1 - (k_{E,X} + \lambda)\operatorname{ord}_F(\mathfrak{a}_\bullet^E).$$

Proof. Fix $n \in \mathbf{Z}_{>0}$ so that $\mathfrak{a}_\bullet^E$ is finitely generated in degree n. As described in the proof of Theorem 1.4, $f : Y \to X$ is isomorphic to the blowup of X along $\mathfrak{a}_n^E$ and $\mathfrak{a}_n^E \cdot \mathcal{O}_Y = \mathcal{O}_Y(-nE_Y)$. Hence, $\operatorname{Exc}(f) \subseteq E_Y$ and $K_{Y/X} = k_E E_Y$.

To see (i) holds, note that $K_Y = K_{Y/X} + f^*K_X$. Now, f^*K_X is $\mathbf{Q}$-Cartier, since K_X is $\mathbf{Q}$-Cartier by assumption. Additionally, $K_{Y/X}$ is $\mathbf{Q}$-Cartier, since E_Y is $\mathbf{Q}$-Cartier by Theorem 1.4. Therefore, (i) holds.

For (ii), we first compute $k_{F,Y}$ in terms of $k_{F,X}$. Choose a projective birational morphism $g : Z \to Y$ such that Z is normal and F arises a prime divisor on Z. Since $K_{Z/X} = K_{Z/Y} + g^*K_{Y/X}$ and $K_{Y/X} = k_{E,X}E_Y$, we see

$$k_{F,X} = k_{F,Y} + k_{E,X}\operatorname{ord}_F(E_Y). \tag{2}$$

Additionally,

$$\operatorname{ord}_F(E_Y) = (1/n)\operatorname{ord}_F(\mathfrak{a}_n^E) = \operatorname{ord}_F(\mathfrak{a}_\bullet^E), \tag{3}$$

where the first equality arises from the fact that $\mathfrak{a}_n^E \cdot \mathcal{O}_Y = \mathcal{O}_Y(-nE_Y)$ and the second from the finite generation of $\mathfrak{a}_\bullet^E$ in degree n. Statement (ii) now follows from (2) and (3). □

Proof of Theorem 1.5. We first prove (i). Since the condition that $\mathfrak{a}_\bullet^E$ is finitely generated is local on X, we may assume X is affine. It follows from Proposition 4.1 and Lemma 4.2 that $\mathfrak{a}_\bullet^E$ is finitely generated.

We move on to (ii). By Proposition 4.3, K_Y is $\mathbf{Q}$-Cartier. Additionally, if F is a divisor over Y and $\lambda \in \mathbf{R}_{\geq 0}$, then

$$a\big(F; Y, \lambda E_Y\big) = k_{F,X} + 1 - (k_{E,X} + \lambda)\operatorname{ord}_F(\mathfrak{a}_\bullet^E). \tag{4}$$

Hence, $(Y, \lambda E_Y)$ is lc if and only if

$$\frac{k_{F,X} + 1}{\operatorname{ord}_F(\mathfrak{a}_\bullet^E)} \geq k_{E,X} + \lambda$$

for all divisor F over X with $\operatorname{ord}_F(\mathfrak{a}_\bullet^E) > 0$. Since the latter condition is equivalent to the inequality $\operatorname{lct}(\mathfrak{a}_\bullet^E) \geq k_{E,X} + \lambda$, (ii) holds. □

We now discuss the relation between the above result and plt blowups.

Definition. Let $f : Y \to X$ be a projective birational morphism with exactly one irreducible exceptional divisor E. We say $f : (Y, E) \to X$ is a *plt blow-up* if (Y, E) is plt and $-(K_Y + E)$ is f-ample [22]. When $f(E)$ is a closed point, E is called a *Kollár component* [26].

We note that such plt blowups were constructed in [26, Lemma 1]. Inspired by the work of Xu and others, we interpret these plt blowups in our framework.

Proposition 4.4. *If E is a divisor over a klt variety X with*

$$k_E + 1 < \frac{k_F + 1}{\operatorname{ord}_F(\mathfrak{a}_\bullet^E)}$$

for all divisors F over X not equal to E, then $\mathfrak{a}_\bullet^E$ is finitely generated and the pairs (Y, E_Y) from Theorem 1.4 is plt and $-(K_Y + E_Y)$ is relatively ample over X.

Remark 4.5. We make a couple remarks regarding Proposition 4.4.

(1) The condition that $k_E + 1 < \frac{k_F+1}{\operatorname{ord}_F(\mathfrak{a}_\bullet^E)}$ for all $F \neq E$ implies that $\operatorname{lct}(\mathfrak{a}_\bullet^E) = k_E + 1$. The converse does not hold in general.
(2) In the case when $\operatorname{codim}_X(c_X(E)) \geq 2$, then the projective birational morphism $Y \to X$ satisfying the conclusion of the proposition is indeed a plt blowup.

Proof. Since $\operatorname{lct}(\mathfrak{a}_\bullet^E) = k_E + 1$, Theorem 1.5 says (Y, E_Y) is lc. For a divisor F over Y, we have

$$a\big(F; Y, E_Y\big) = k_{F,X} + 1 - (k_{E,X} + 1)\operatorname{ord}_F(\mathfrak{a}_\bullet^E)$$

by Proposition 4.3.2. By our assumption, the latter value is > 0 when F is not equal to E. Thus, (Y, E_Y) is plt. To see $-(K_Y + E_Y)$ is f-ample, note that

$$-(K_Y + E_Y) \sim_{\mathbf{Q},f} -(K_{Y/X} + E_Y) = -(k_E + 1)E_Y$$

and $-E_Y$ is f-ample by Theorem 1.4. □

The following lemma gives a criterion for when the hypotheses of the above proposition hold.

Lemma 4.6. *Let X be a klt variety and $(X, \mathfrak{a}^\lambda)$ a pair with log resolution $f : Y \to X$. Let Δ_Y be the divisor on Y such that $K_Y + \Delta_Y = f^*(K_X) + \lambda D$ where $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D)$. If E is an exceptional divisor of f and the coefficients of Δ_Y are ≤ 1 with equality precisely along E, then*

$$k_E + 1 < \frac{k_F + 1}{\operatorname{ord}_F(\mathfrak{a}_\bullet^E)}$$

for all divisors F over X not equal to E.

Proof. Let F be a divisor over X. Note that the proof of [17, Lemma 2.30] gives that $a\big(F; X, \mathfrak{a}^\lambda\big) = a\big(F; Y, \Delta_Y\big)$. By [17, Lemma 2.31] and our hypotheses on the coefficients of Δ_Y, we see $a\big(F; X, \mathfrak{a}^\lambda\big) = a\big(F; Y, \Delta_Y\big) > 0$ when F is not equal to E. Additionally, $a\big(E; X, \mathfrak{a}^\lambda\big) = a\big(E; Y, \Delta_Y\big) = 1$. Thus,

$$k_F + 1 > \lambda \operatorname{ord}_F(\mathfrak{a}) \quad \text{and} \quad k_E + 1 = \lambda \operatorname{ord}_E(\mathfrak{a}). \tag{5}$$

Next, note

$$\operatorname{ord}_F(\mathfrak{a}) \geq \operatorname{ord}_F(\mathfrak{a}^E_{\operatorname{ord}_E(\mathfrak{a})}) \geq \operatorname{ord}_E(\mathfrak{a}) \operatorname{ord}_E(\mathfrak{a}_\bullet^E), \tag{6}$$

where the first inequality comes from the inclusion $\mathfrak{a} \subseteq \mathfrak{a}^E_{\mathrm{ord}_E(\mathfrak{a})}$ and the second from the definition of $\mathrm{ord}_E(\mathfrak{a}^E_\bullet)$ as an infimum. The desired inequality now follows from (5) and (6). □

Remark 4.7. The previous lemma gives a condition for when the hypotheses of Proposition 4.4 hold. By "tie breaking" [15, Proposition 8.7.1], if $\mathfrak{a} \subseteq \mathcal{O}_X$ is a non-zero ideal on klt variety, then at least one of the divisors computing $\mathrm{lct}(\mathfrak{a})$ satisfy the hypotheses of Proposition 4.4. Therefore, one of the divisors computing $\mathrm{lct}(\mathfrak{a})$ satisfy the conclusion of Proposition 4.4. We note that a similar result was independently obtained by Kento Fujita in [7].

5. Connection with divisors that compute lct's and mld's

We proceed to prove Theorem 1.3. Recall that the theorem says that the value of $\mathrm{lct}(\mathfrak{a}^E_\bullet)$ determines whether E computes a log canonical threshold.

Proof of Theorem 1.3. The implication (1) implies (2) is trivial. Indeed, if E compute $\mathrm{lct}(\mathfrak{b})$, then E computes $\mathrm{lct}(\mathfrak{b}_\bullet)$, where $\mathfrak{b}_\bullet$ is the graded sequence defined by $\mathfrak{b}_m := \mathfrak{b}^m$ for all $m \in \mathbf{N}$.

To show (2) implies (3), we follow an argument in [13]. Assume E computes $\mathrm{lct}(\mathfrak{a}_\bullet)$, where $\mathfrak{a}_\bullet$ is a graded sequence of ideals on X. Therefore,

$$\frac{k_E+1}{\mathrm{ord}_E(\mathfrak{a}_\bullet)} = \mathrm{lct}(\mathfrak{a}_\bullet) \leq \frac{k_F+1}{\mathrm{ord}_F(\mathfrak{a}_\bullet)}$$

for any divisor F over X. If we can show $\mathrm{ord}_E(\mathfrak{a}_\bullet)\cdot \mathrm{ord}_F(\mathfrak{a}^E_\bullet) \leq \mathrm{ord}_F(\mathfrak{a}_\bullet)$ for any divisor F over X, it will follow that

$$k_E+1 \leq \inf_F \frac{k_F+1}{\mathrm{ord}_F(\mathfrak{a}^E_\bullet)} = \mathrm{lct}(\mathfrak{a}^E_\bullet).$$

To prove the previous inequality, note that $\mathfrak{a}_m \subseteq \mathfrak{a}^E_{\mathrm{ord}_E(\mathfrak{a}_m)}$. Therefore,

$$\mathrm{ord}_F(\mathfrak{a}_m) \geq \mathrm{ord}_F(\mathfrak{a}^E_{\mathrm{ord}_E(\mathfrak{a}_m)}) \geq \mathrm{ord}_E(\mathfrak{a}_m)\,\mathrm{ord}_F(\mathfrak{a}^E_\bullet).$$

After dividing by m and taking infimums, the desired inequality follows.

Lastly, we show (3) implies (1). Assume $\mathrm{lct}(\mathfrak{a}^E_\bullet) = k_E+1$. By Theorem 1.5, the graded sequence $\mathfrak{a}^E_\bullet$ is finitely generated. Choose n so that $\mathfrak{a}^E_\bullet$ is finitely generated in degree n. By our assumption,

$$k_E+1 = \mathrm{lct}(\mathfrak{a}^E_\bullet) = n\cdot \mathrm{lct}(\mathfrak{a}^E_n).$$

Since $\mathrm{ord}_E(\mathfrak{a}^E_n) = n$, E computes the log canonical threshold of $\mathfrak{a}^E_n$. □

Remark 5.1. When X is a smooth surface, the implication (3) implies (1) does not require Theorem 1.5 (which relies on the MMP). Indeed, Lemma 6.1, which is an easy consequence of [21], implies $\mathfrak{a}^E_\bullet$ is finitely generated.

Question 5.2. Let E be a divisor over X such that E computes a minimal log discrepancy.

(1) Does it follow that $\mathfrak{a}^E_\bullet$ is finitely generated?

(2) Does this imply that $\operatorname{lct}(\mathfrak{a}_\bullet^E) > k_E$ or $= k_E + 1$?

By Proposition 1.3 and Theorem 1.5, the answer is yes to both if we replace "minimal log discrepancy" with "log canonical threshold." By Theorem 1.5, an affirmative answer to the second question implies an affirmative answer to the first.

Let us consider a divisor E such that $\operatorname{lct}(\mathfrak{a}_\bullet^E) < k_E + 1$ and ask whether or not it can compute a minimal log discrepancy. We know that there must exist some divisor F such that $\frac{k_F+1}{\operatorname{ord}_F(\mathfrak{a}_\bullet^E)} < k_E + 1$. Additionally, it might be that $c_X(E) = c_X(F)$ and $k_F \leq k_E$. The proposition below shows that if such an F exists, then E cannot compute a minimal log discrepancy.

Proposition 5.3. *Let E, F be divisors over a klt variety X with*

(1) $c_X(E) = c_X(F)$,
(2) $k_F \leq k_E$, *and*
(3) $\frac{k_F+1}{\operatorname{ord}_F(\mathfrak{a}_\bullet^E)} < k_E + 1$.

Then for any lc pair $(X, \mathfrak{b}^\lambda)$ where $c_X(E) \subset \operatorname{Cosupp}(\mathfrak{b})$ and $\lambda > 0$

$$a\big(F; X, \mathfrak{b}^\lambda\big) < a\big(E; X, \mathfrak{b}^\lambda\big).$$

Before proving the above proposition, we define the log discrepancy of a divisor F over X along $\mathfrak{a}_\bullet^\lambda$, where $\mathfrak{a}_\bullet$ is a graded sequence of ideals on X and $\lambda \in \mathbf{R}_{\geq 0}$, as

$$a\big(F; X, \mathfrak{a}_\bullet^\lambda\big) := k_F + 1 - \lambda \operatorname{ord}_F(\mathfrak{a}_\bullet)$$

and prove the following lemma.

Lemma 5.4. *Let E, F be divisors over X satisfying the conditions in Proposition 5.3. Then,*

$$a\big(F; X, \big(\mathfrak{a}_\bullet^E\big)^\lambda\big) < a\big(E; X, \big(\mathfrak{a}_\bullet^E\big)^\lambda\big)$$

for all $\lambda \in \mathbf{R}_{>0}$ such that $a\big(F; X, \big(\mathfrak{a}_\bullet^E\big)^\lambda\big) \geq 0$.

Proof. Note that $a\big(F; X, \big(\mathfrak{a}_\bullet^E\big)^\lambda\big)$ and $a\big(F; X, \big(\mathfrak{a}_\bullet^E\big)^\lambda\big)$ are real valued linear functions in λ. When $\lambda = 0$, we compare the values of the two functions:

$$a\big(E; X, \big(\mathfrak{a}_\bullet^E\big)^0\big) = k_E + 1 \geq k_F + 1 = a\big(F; X, \big(\mathfrak{a}_\bullet^E\big)^0\big).$$

Set λ_F and λ_E to be the values of λ so that $a\big(F; X, \big(\mathfrak{a}_\bullet^E\big)^\lambda\big) = 0$ and $a\big(F; X, \big(\mathfrak{a}_\bullet^E\big)^\lambda\big) = 0$, respectively. (The existence of λ_F relies on assumption (3), which implies $\operatorname{ord}_F(\mathfrak{a}_\bullet^E) > 0$.) Note that

$$\lambda_F = \frac{k_E + 1}{\operatorname{ord}_F(\mathfrak{a}_\bullet^E)} < k_E + 1 = \frac{k_E + 1}{\operatorname{ord}_E(\mathfrak{a}_\bullet^E)} = \lambda_E,$$

where the inequality comes from (3). By analyzing these linear functions, we see that the desired inequality holds. $\square$

Proof of Proposition 5.3. Let E, F and $(X, \mathfrak{b}^\lambda)$ satisfy the hypotheses of the proposition. We have

$$\begin{aligned} a\big(F; X, \mathfrak{b}^\lambda\big) &\le a\Big(F; X, \Big(\mathfrak{a}^E_{\operatorname{ord}_E(\mathfrak{b})}\Big)^\lambda\Big) \\ &\le a\big(F; X, \big(\mathfrak{a}^E_\bullet\big)^{\lambda \operatorname{ord}_E(\mathfrak{b})}\big) \\ &< a\big(E; X, \big(\mathfrak{a}^E_\bullet\big)^{\lambda \operatorname{ord}_E(\mathfrak{b})}\big) \\ &= a\big(E; X, \mathfrak{b}^\lambda\big), \end{aligned}$$

where the first inequality follows from $\mathfrak{b} \subset \mathfrak{a}^E_{\operatorname{ord}_E(\mathfrak{b})}$, the second from the definition of $\operatorname{ord}_F(\mathfrak{a}^E_\bullet)$ as an infimum, the third from the previous lemma, and the last from Lemma 2.5. □

6. Divisors computing mlds on surfaces

Before proving Theorem 1.2, we prove the following lemma and proposition. The lemma is closely related to the discussion in [12, Section 7.3].

Lemma 6.1. *Let $x \in X$ be a point on a surface with at worst rational singularities and E a divisor over X with $c_X(E) = \{x\}$. If*

$$f : Y \to X$$

is a projective birational morphism and Y is a smooth surface that contains E as a prime divisor, then there is an $m \in \mathbf{Z}_{>0}$ such that $\mathfrak{a}^E_\bullet$ is finitely generated in degree m and $\mathfrak{a}^E_m \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D)$, for some divisor D on Y with $\operatorname{Supp}(D) \subseteq \operatorname{Exc}(f)$.

Proof. We consider the intersection form on the curves in $\operatorname{Exc}(f)$. By [16, Theorem 10.1], the intersection form is negative definite. Thus, we may define a $\mathbf{Q}$-divisor $\check{E}$ with support on $\operatorname{Exc}(f)$ such that

$$\check{E} \cdot C = \begin{cases} 1, & \text{if } C = E, \\ 0, & \text{if } C \neq E \text{ and } C \subset \operatorname{Exc}(f). \end{cases}$$

Since $\check{E}$ intersects non-negatively with all exceptional curves of f, $\check{E}$ is f-nef. By [21, Theorem 12.1], $\check{E}$ is also f-base point free (we are using that X has rational singularities). Thus, $\check{E}$ gives rise to a fiber space h, over X, that contracts all curves in $\operatorname{Exc}(f)$ not equal to E.

$$\begin{array}{ccc} Y & \xrightarrow{\quad h \quad} & Z \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & X & \end{array}$$

The only exceptional divisor of the map g (labeled in the above diagram) is the prime divisor $E_Z := h_*(E)$.

We claim that $-E_Z$ is relatively ample over X. For this, note that X is **Q**-factorial by [21, Proposition 17.1]. Since g is a projective birational morphism, [17, Lemma 2.62] implies that there exists an effective exceptional divisor F on Z such that $-F$ is relatively ample. Since the only prime divisor contracted by g is E_Z, we conclude $-E_Z$ must be relatively ample.

Since $-E_Z$ is relatively ample over X and $g_*\mathcal{O}_Z(-mnE_Z) = \mathfrak{a}^E_{mn}$, we see $\bigoplus_{m\geq 0} \mathfrak{a}^E_m$ is a finitely generated $\mathcal{O}_X$-algebra. Hence, we may choose a positive integer n so that $\mathfrak{a}^E_\bullet$ is finitely generated in degree n. Replacing n with a multiple, we may also assume $\mathcal{O}_Z(-nE_Z)$ is also relatively base point free over X. Therefore, $\mathfrak{a}^E_n \cdot \mathcal{O}_Z = \mathcal{O}_Z(-nE_Z)$. If we set $D := nh^*E_Z$, we see

$$\mathfrak{a}^E_n \cdot \mathcal{O}_Y = \mathcal{O}_Z(-nE_Z)\cdot \mathcal{O}_Y = \mathcal{O}_Y(-D)$$

and $\operatorname{Supp}(D) \subseteq h^{-1}(E_Z) = f^{-1}(x)$. □

Proposition 6.2. *Let X be a surface, $x \in X$ a smooth point or a du Val singularity, and $(X, \mathfrak{b}^\lambda)$ a pair such that $x \in \operatorname{Cosupp}(\mathfrak{b})$ and $\lambda > 0$. If E is a divisor over X that computes $\operatorname{mld}_x(X, \mathfrak{b}^\lambda)$, then $\operatorname{lct}(\mathfrak{a}^E_\bullet) = k_E + 1$.*

Proof. We first consider the following maps

$$Y \xrightarrow{f} X' \xrightarrow{g} X,$$

where $g : X' \to X$ is the minimal resolution of X and $f : Y \to X'$ is the map achieved by repeatedly blowing up the center of E until the center is a prime divisor. If $x \in X$ is a smooth point, then $X' = X$ and g is the identity. Similarly, if E corresponds to a prime divisor on X', then $Y = X'$ and f is the identity.

Let $E_1, \ldots, E_s$ denote the exceptional divisors of $g \circ f$. Note that if E_i is not contracted by f, then $f(E_i)$ is an exceptional divisor of g and $k_{E_i} = 0$ (since X is a du Val singularity).

By the previous lemma, there exists an $n \in \mathbf{Z}_{>0}$ so that $\mathfrak{a}^E_\bullet$ is finitely generated in degree n and $Y \to X$ is a log resolution of $\mathfrak{a}^E_n$. By Proposition 2.6,

$$\operatorname{lct}(\mathfrak{a}^E_\bullet) = \min_i \frac{k_{E_i}+1}{\operatorname{ord}_{E_i}(\mathfrak{a}^E_\bullet)}.$$

To show that $\operatorname{lct}(\mathfrak{a}^E_\bullet) = k_E + 1$, it suffices to show

$$k_E + 1 \leq \frac{k_{E_i}+1}{\operatorname{ord}_{E_i}(\mathfrak{a}^E_\bullet)}$$

for all $1 \leq i \leq s$ by Lemma 2.7.

Claim: $k_{E_i} \leq k_E$ for all $1 \leq i \leq s$.

To prove the claim, we first consider the case when f is the identity. In this case, $x \in X$ is a du Val singularity and E corresponds to a prime divisor on the minimal resolution of X. Thus, $k_{E_i} = 0$ for all $1 \leq i \leq s$ and $k_E = 0$.

Next, we consider the case when f is not the identity. Let E_i for $r \leq i \leq s$ denote the exceptional divisors contracted by f. Since these divisors arose via a sequence of blowups, we may assume that the divisors are labelled in the order in which they arose (in particular, $E_s = E$). If $1 \leq i < r$, then E_i is a an exceptional divisor of g and, as stated before, $k_{E_i} = 0$. If $r \leq j \leq s$, then E_j either arose as the blowup of a point lying on a single exceptional divisor or the intersection of two such exceptional divisors. Thus, we have that either

$$k_{E_j} = k_{E_{j-1}} + 1 \quad \text{or} \quad k_{E_j} = k_{E_{j-1}} + k_{E_q} + 1$$

for some $q < j - 1$. We see that $0 \leq k_{E_{i-1}} \leq k_{E_i}$ for all $1 < i \leq s$. Since $k_{E_s} = k_E$, and the proof of the claim is complete.

By the above claim and the fact that E computes $\mathrm{mld}_x(X, \mathfrak{b}^\lambda)$, Proposition 5.3 implies that $k_E + 1 \geq \frac{k_{E_i}+1}{\mathrm{ord}_{E_i}(\mathfrak{a}_\bullet^E)}$ for all $1 \leq i \leq s$. This completes the proof. □

Proof of Theorem 1.2. Let X be a smooth surface and E a divisor over X computing $\mathrm{mld}_x(X, \mathfrak{b}^\lambda)$ where $(X, \mathfrak{b}^\lambda)$ is a pair. If $x \notin \mathrm{Cosupp}(\mathfrak{b})$ or $\lambda = 0$, then $\lambda\,\mathrm{ord}_F(\mathfrak{b}) = 0$ for all divisors F over X with $\{x\} = c_X(F)$. Thus,

$$\mathrm{mld}_x(X, \mathfrak{b}^\lambda) = \min\{k_E + 1 \,|\, E \text{ is a divisor with } c_X(E) = \{x\}\} = 2$$

and the minimum is achieved solely by the divisor corresponding to the exceptional divisor of the blowup of X at $\{x\}$. Note that this divisor also computes a log canonical threshold, namely $\mathrm{lct}(\mathfrak{m}_x)$, where $\mathfrak{m}_x$ is the ideal of functions vanishing at x.

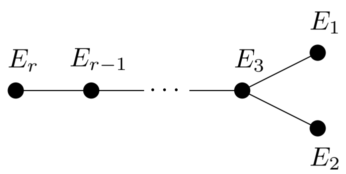
If $x \in \mathrm{Cosupp}(\mathfrak{b})$ and $\lambda > 0$, then we apply the previous proposition to see that $\mathrm{lct}(\mathfrak{a}_\bullet^E) = k_E + 1$. By Theorem 1.3, E must also compute a log canonical threshold. □

7. Examples

Below, we compute $\mathrm{lct}(\mathfrak{a}_\bullet^E)$ for some divisors over smooth varieties.

Example 7.1. For a trivial example consider $\mathbf{A}^n$ when $n > 1$ and let E be the exceptional divisor of the blowup of $\mathbf{A}^n$ at the origin. Then, $\mathfrak{a}_m^E = \mathfrak{m}_0^m$, where $\mathfrak{m}_0 \subset \mathcal{O}_{\mathbf{A}^n}$ is the ideal of functions that vanish at the origin. Thus, $\mathrm{lct}(\mathfrak{a}_\bullet^E) = k_E + 1 = n$ and is computed by E.

Example 7.2. Let X be a smooth surface and $Y \to X$ the composition of $r \geq 3$ point blowups resulting in the following dual tree of exceptional curves



where E_i denotes the strict transform of the exceptional divisor arising from the i-th blowup and set $E := E_r$. To understand $\mathfrak{a}_\bullet^E$, we apply the argument in the proof of Lemma 6.1 to find a positive integer n and a divisor D on Y satisfying: $\mathfrak{a}_\bullet^E$ is finitely generated in degree n and $\mathfrak{a}_n^E \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D)$.

First, note that $E_1^2 = -3$, $E_i^2 = -2$ for $2 \leq i \leq r-1$, and $E_r^2 = -1$. Since $D \cdot E_i = 0$ for all $1 \leq i \leq r-1$ (by the proof of Lemma 6.1) and $\mathrm{coeff}_E(D) = n$, we see

$$D = \frac{n}{r+3}\Big(2E_1 + 3E_2 + \sum_{i=3}^{r}(i+3)E_i\Big).$$

Since $k_{E_1} = 1$, $k_{E_2} = 2$, and $k_{E_i} = i+1$ for $3 \leq i \leq r$ and $\mathrm{ord}_{E_i}(\mathfrak{a}_n^E) = \mathrm{coeff}_{E_i}(D)$, we see

$$\mathrm{lct}(\mathfrak{a}_\bullet{}^E) = n\,\mathrm{lct}(\mathfrak{a}_n{}^E) = \min_{i=1,\dots,r} \frac{k_{E_i}+1}{\mathrm{ord}_{E_i}(\mathfrak{a}_n{}^E)} = \frac{k_{E_3}+1}{\mathrm{ord}_{E_3}(\mathfrak{a}_n{}^E)} = \frac{5(r+3)}{6}.$$

We see two behaviors. When $r = 3$, $\mathrm{lct}(\mathfrak{a}_\bullet^E) = k_E + 1$ and E computes $\mathrm{lct}(\mathfrak{a}_\bullet^E)$. When $r > 3$, $\mathrm{lct}(\mathfrak{a}_\bullet^E) < k_E + 1$ and E does not compute an lct (by Proposition 1.3). We may view E_3 as preventing E from computing a minimal log discrepancy. Indeed, for any log canonical pair $(X, \mathfrak{b}^\lambda)$, we have that

$$a\big(E; X, \mathfrak{b}^\lambda\big) > a\big(E_3; X, \mathfrak{b}^\lambda\big)$$

by Proposition 5.3.

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