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RINGS IN WHICH EVERY ELEMENT IS A SUM OF A NILPOTENT AND THREE TRIPOTENTS

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Abstract. In this article, we completely determine the rings for which every element is a sum of a nilpotent and three tripotents that commute with one another. We discuss this property for some extensions of rings, including group rings.

1. Introduction

Throughout, R is an associative ring with identity. The set of all units, the set of all nilpotents, the Jacobson radical and the center of R are denoted by $U(R)$, $\mathrm{Nil}(R)$, $J(R)$ and $C(R)$, respectively. We write $\mathbb{Z}_n$ for the ring of integers modulo n, $\mathbb{M}_n(R)$ for the $n \times n$ matrix ring and $\mathbb{T}_n(R)$ for the $n \times n$ upper triangular matrix ring over R, respectively.

Rings whose elements are sums of certain special elements have been widely studied in ring theory. In [4], Hirano and Tominaga determined the rings for which every element is a sum of two commuting idempotents. An element a of a ring is called a tripotent if $a^3 = a$. Tripotents are a natural generalization of idempotents. In [8], the authors determined the rings for which every element is a sum of two commuting tripotents. In [3], Diesl defined and discussed (strongly) nil-clean rings: A ring is called (strongly) nil-clean if every element is a sum of a nilpotent and an idempotent (that commute with each other). The structure of strongly nil-clean rings was made available in [5] and [6]. In [1], Chen and Sheibani determined the rings for which every element is a sum of a nilpotent and a tripotent that commute. In [9], the author characterized the rings for which every element is a sum of a nilpotent and two tripotents that commute with one another.

This is a further investigation of this subject. The main objective of this article is to present the structure of rings for which every element is a sum of a nilpotent and three tripotents that commute with one another (see Theorem

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10). We also discuss this property for some extensions of rings, including group rings.

2. The structure theorem

We give the following definition for convenience.

Definition 1. A ring R is said to have property $\mathcal{P}$ if every element of R is a sum of a nilpotent and three tripotents that commute with one another.

One can easily check that the class of rings with property $\mathcal{P}$ is closed under finite direct products and homomorphic images.

Lemma 2. *A ring R has property $\mathcal{P}$ if and only if $R = R_1 \oplus R_2 \oplus R_3 \oplus R_4$ where R_1, R_2, R_3, R_4 have property $\mathcal{P}$ with $2 \in \mathrm{Nil}(R_1)$, $3 \in \mathrm{Nil}(R_2)$, $5 \in \mathrm{Nil}(R_3)$ and $7 \in \mathrm{Nil}(R_4)$.*

Proof. It suffices to show the necessity, so let us assume that R has property $\mathcal{P}$. We first show that $2 \cdot 3 \cdot 5 \cdot 7$ is nilpotent. Write $4 = b + e + f + g$, where $b \in \mathrm{Nil}(R)$, $e^3 = e$, $f^3 = f$, $g^3 = g$ and b, e, f, g commute with one another. Note that $(4-b)^3 - (4-b) = (15 - 8b + b^2)(4-b)$ and $4 - b = e + f + g$. With $t := 15 - 8b + b^2$, we have

$$\begin{aligned} t(e+f+g) &= (e+f+g)^3 - (e+f+g) \\ &= 3e^2f + 3e^2g + 3ef^2 + 3f^2g + 3eg^2 + 3fg^2 + 6efg. \end{aligned} \tag{2.1}$$

Multiplying both sides of (2.1) by $e^2f^2g^2$ gives $6efg = (t-6)efg(ef+eg+fg)$, so

$$\begin{aligned} 12efg &= (t-6)efg(2ef+2eg+2fg) \\ &= (t-6)efg[(4-b)^2 - e^2 - f^2 - g^2] \\ &= (t-6)efg(4-b)^2 - (t-6)efg(e^2+f^2+g^2) \\ &= (t-6)efg(4-b)^2 - 3(t-6)efg, \end{aligned}$$

which implies that $[(t-6)(4-b)^2 - 3(t-6) - 12]efg = 0$. As b is nilpotent, we deduce that $(9 \cdot 4^2 - 27 - 12)efg = 105efg$ is nilpotent.

Now multiplying both sides of (2.1) by e^2, f^2 and g^2 respectively, we obtain:

$$te + (t-3)e^2f + (t-3)e^2g = 3ef^2 + 3eg^2 + 3e^2f^2g + 3e^2fg^2 + 6efg; \tag{2.2}$$

$$tf + (t-3)ef^2 + (t-3)f^2g = 3e^2f + 3g^2f + 3e^2f^2g + 3ef^2g^2 + 6efg; \tag{2.3}$$

$$tg + (t-3)eg^2 + (t-3)fg^2 = 3e^2g + 3f^2g + 3e^2fg^2 + 3ef^2g^2 + 6efg. \tag{2.4}$$

In view of (2.1), (2.2) + (2.3) + (2.4) gives

$$(t-3)(e^2f + e^2g + ef^2 + gf^2 + eg^2 + fg^2) = 6efg(2 + ef + eg + fg).$$

It follows that $35(t-3)(e^2f+e^2g+ef^2+gf^2+eg^2+fg^2) = 105\cdot 2efg(2+ef+eg+fg) \in \mathrm{Nil}(R)$, since $105efg$ is nilpotent. Thus, in view of (2.1), we see that

$$\begin{aligned}
&35(t-3)t(4-b)\\
&= 35(t-3)(3e^2f+3e^2g+3ef^2+3gf^2+3eg^2+3fg^2+6efg)\\
&= 35(t-3)(3e^2f+3e^2g+3ef^2+3gf^2+3eg^2+3fg^2)+105efg\cdot 2(t-3)
\end{aligned}$$

is nilpotent. Since b is nilpotent, it follows that $2^4\cdot 3^2\cdot 5^2\cdot 7 \in \mathrm{Nil}(R)$, and thus, $2\cdot 3\cdot 5\cdot 7 \in \mathrm{Nil}(R)$. Hence there exists an integer $n \geq 1$ such that $2^nR\cap 3^nR\cap 5^nR\cap 7^nR = 0$. By the Chinese Reminder Theorem, $R = R_1\oplus R_2\oplus R_3\oplus R_4$ where $R_1 \cong R/2^nR$, $R_2 \cong R/3^nR$, $R_3 \cong R/5^nR$, $R_4 \cong R/7^nR$, and R_1, R_2, R_3 and R_4 have property $\mathcal{P}$. □

Lemma 3. *If R is a subdirect product of $\mathbb{Z}_7$'s and $x \in R$, then there exist polynomials $\theta(t), \zeta(t), \eta(t) \in \mathbb{Z}[t]$ such that $x = \theta(x)+\zeta(x)+\eta(x)$ and $\theta(x), \zeta(x), \eta(x)$ are tripotents.*

Proof. Let R be a subdirect product of $\{R_\alpha : \alpha \in \Lambda\}$ where $R_\alpha = \mathbb{Z}_7$ for all $\alpha \in \Lambda$. Then R is a subring of ΠR_α. Write $x = (x_\alpha) \in R$ and let Λ be a disjoint union of Λ_0, Λ_1, Λ_2, Λ_3, Λ_4, Λ_5, Λ_6 such that $x_\alpha = i \Leftrightarrow \alpha \in \Lambda_i$ for $i = 0,1,2,3,4,5,6$. Without lose of generality, let $x = (0_{\Lambda_0}, 1_{\Lambda_1}, 2_{\Lambda_2}, 3_{\Lambda_3}, 4_{\Lambda_4}, 5_{\Lambda_5}, 6_{\Lambda_6})$, and set

$$\begin{aligned}
e_1 &= (0_{\Lambda_0}, 1_{\Lambda_1}, 0_{\Lambda_2}, 0_{\Lambda_3}, 0_{\Lambda_4}, 0_{\Lambda_5}, 0_{\Lambda_6}),\\
e_2 &= (0_{\Lambda_0}, 0_{\Lambda_1}, 1_{\Lambda_2}, 0_{\Lambda_3}, 0_{\Lambda_4}, 0_{\Lambda_5}, 0_{\Lambda_6}),\\
e_3 &= (0_{\Lambda_0}, 0_{\Lambda_1}, 0_{\Lambda_2}, 1_{\Lambda_3}, 0_{\Lambda_4}, 0_{\Lambda_5}, 0_{\Lambda_6}),\\
e_4 &= (0_{\Lambda_0}, 0_{\Lambda_1}, 0_{\Lambda_2}, 0_{\Lambda_3}, 1_{\Lambda_4}, 0_{\Lambda_5}, 0_{\Lambda_6}),\\
e_5 &= (0_{\Lambda_0}, 0_{\Lambda_1}, 0_{\Lambda_2}, 0_{\Lambda_3}, 0_{\Lambda_4}, 1_{\Lambda_5}, 0_{\Lambda_6}),\\
e_6 &= (0_{\Lambda_0}, 0_{\Lambda_1}, 0_{\Lambda_2}, 0_{\Lambda_3}, 0_{\Lambda_4}, 0_{\Lambda_5}, 1_{\Lambda_6}).
\end{aligned}$$

One can show that there exist polynomials $\theta_i(t) \in \mathbb{Z}[t]$ such that $e_i = \theta_i(x)$ for $i = 1,2,\ldots,6$. Indeed,

$$\begin{aligned}
e_1 &= x^6 - y^6 \text{ where } y = x - x^6,\\
e_2 &= y^6 - z^6 \text{ where } z = y - y^6,\\
e_3 &= z^6 - u^6 \text{ where } u = z - z^6,\\
e_4 &= u^6 - v^3 \text{ where } v = u - u^6,\\
e_5 &= 2v^3 - v,\\
e_6 &= v - v^3.
\end{aligned}$$

Thus, $e_i \in R$ for $i = 1,2,\ldots,6$. Let $e := e_1+e_2+e_3+6e_4+6e_5+6e_6$, $f := e_2+e_3+6e_4+6e_5$ and $g := e_3+6e_4$. Then, e, f and g are tripotents in

R and $x = e + f + g$. With

$$\begin{aligned}\theta(t) &= \theta_1(t) + \theta_2(t) + \theta_3(t) + 6\theta_4(t) + 6\theta_5(t) + 6\theta_6(t),\\ \zeta(t) &= \theta_2(t) + \theta_3(t) + 6\theta_4(t) + 6\theta_5(t),\\ \eta(t) &= \theta_3(t) + 6\theta_4(t),\end{aligned}$$

we have $e = \theta(x)$, $f = \zeta(x)$ and $g = \eta(x)$. □

Lemma 4. *Let R be a ring with $2 \in \mathrm{Nil}(R)$. The following are equivalent*:

(1) *R has property $\mathcal{P}$.*
(2) *Every element of R is a sum of a nilpotent and an idempotent that commute.*
(3) *$a - a^2$ is nilpotent for all $a \in R$.*

Proof. (2) ⇔ (3) See [9, Proposition 2.5].

(2) ⇒ (1) The implication is clear.

(1) ⇒ (3) Let $a \in R$, and write $a = b + e + f + g$, where $b \in \mathrm{Nil}(R)$, $e^3 = e$, $f^3 = f$, $g^3 = g$ and b, e, f, g commute with one another. If x is a tripotent of R, then $(x - x^2)^2 = 2x(x - 1)$; so $x - x^2 \in \mathrm{Nil}(R)$. As e, f, g are tripotents, $a - a^2 = b(1 - b - 2e - 2f - 2g) - 2(ef + eg + fg) + (e - e^2) + (f - f^2) + (g - g^2) \in \mathrm{Nil}(R)$. □

Lemma 5. *Let R be a ring with $3 \in \mathrm{Nil}(R)$. The following are equivalent*:

(1) *R has property $\mathcal{P}$.*
(2) *Every element of R is a sum of a nilpotent and a tripotent that commute.*
(3) *$a - a^3$ is nilpotent for all $a \in R$.*

Proof. (2) ⇔ (3) See [9, Proposition 2.8].

(2) ⇒ (1) The implication is clear.

(1) ⇒ (3) Let $a \in R$ and write $a = b + e + f + g$, where $b \in \mathrm{Nil}(R)$, $e^3 = e$, $f^3 = f$, $g^3 = g$ and b, e, f, g commute with one another. As $3 \in \mathrm{Nil}(R)$,

$$\begin{aligned}a - a^3 = b(1 &- b^2 - 3be - 3bf - 3bg - 3e^2 - 3f^2 - 3g^2 - 6ef - 6eg - 6fg)\\ &- 3(e^2 f + e^2 g + ef^2 + eg^2 + f^2 g + fg^2 + 2efg)\end{aligned}$$

is nilpotent. □

Lemma 6. *Let R be a ring with $5 \in \mathrm{Nil}(R)$. The following are equivalent*:

(1) *R has property $\mathcal{P}$.*
(2) *$J(R)$ is nil and $R/J(R)$ is a subdirect product of $\mathbb{Z}_5$'s.*
(3) *$a - a^5$ is nilpotent for all $a \in R$.*
(4) *Every element of R is a sum of a nilpotent and two tripotents that commute.*

Proof. In view of [9, Proposition 2.19], (2) ⇔ (3) ⇔ (4) follows.

(4) ⇒ (1) The implication is clear.

(1) $\Rightarrow$ (2) Let $a \in R$ and write $a = b + e + f + g$, where $b \in \mathrm{Nil}(R)$, $e^3 = e$, $f^3 = f$, $g^3 = g$ and b, e, f, g commute with one another. There exist polynomials $\theta(t_1, t_2, t_3, t_4), \eta(t_1, t_2, t_3, t_4)$ in $\mathbb{Z}[t_1, t_2, t_3, t_4]$ such that $a^5 - a = 5 \cdot \theta(b, e, f, g) + b \cdot \eta(b, e, f, g)$; so $a^5 - a \in \mathrm{Nil}(R)$. □

Lemma 7 ([9, Lemma 2.6]). *If $2 \in U(R)$ and $a^3 - a$ is nilpotent, then there exists a polynomial $\theta(t) \in \mathbb{Z}[t]$ such that $\theta(a)^3 = \theta(a)$ and $a - \theta(a)$ is nilpotent.*

Lemma 8 ([7, Theorem 2.1]). *Let R be a ring with $J(R) = 0$ such that every nonzero right ideal contains a nonzero idempotent. If $a^n = 0$ but $a^{n-1} \neq 0$, then there exists an idempotent $e^2 = e \in RaR$ such that $eRe \cong \mathbb{M}_n(T)$ for some non-trivial ring T.*

Recall that a ring R is reduced if it contains no nonzero nilpotents.

Lemma 9. *Let R be a ring with $7 \in \mathrm{Nil}(R)$. The following are equivalent:*

(1) *R has property $\mathcal{P}$.*
(2) *$a - a^7$ is nilpotent for all $a \in R$.*
(3) *$J(R)$ is nil and $R/J(R)$ is a subdirect product of $\mathbb{Z}_7$'s.*

Proof. (1) $\Rightarrow$ (2) Let $a \in R$ and write $a = b + e + f + g$, where $b \in \mathrm{Nil}(R)$, $e^3 = e$, $f^3 = f$, $g^3 = g$ and b, e, f, g commute with one another. There exist polynomials $\theta(t_1, t_2, t_3, t_4), \eta(t_1, t_2, t_3, t_4)$ in $\mathbb{Z}[t_1, t_2, t_3, t_4]$ such that $a^7 - a = 7 \cdot \theta(b, e, f, g) + b \cdot \eta(b, e, f, g)$; so $a^7 - a \in \mathrm{Nil}(R)$.

(2) $\Rightarrow$ (3) For $j \in J(R)$, $j - j^7 = j(1 - j^6)$ is nilpotent. As $1 - j^6$ is a unit in R, j is nilpotent. Hence $J(R)$ is nil. For any $a \in R$, $a - a^7$ is nilpotent; so there exists an integer $n \geq 1$ such that $a^n \in a^{n+1}R \cap Ra^{n+1}$. Thus, R and further $\overline{R} := R/J(R)$ are strongly π-regular rings. Now suppose $\overline{a}^2 = \overline{0}$ for some $\overline{0} \neq \overline{a} \in \overline{R}$. By Lemma 8, there exists $\overline{0} \neq \overline{w}^2 = \overline{w} \in \overline{R}\overline{a}\overline{R}$ such that $\overline{w}\overline{R}\overline{w} \cong \mathbb{M}_2(T)$ where T is a nontrivial ring. Let $x = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right] \in \mathbb{M}_2(T)$. As $7 \in J(R)$, 7=0 in $\overline{R}$, so $x^7 - x = \left[\begin{smallmatrix} -1 & -2 \\ -2 & 1 \end{smallmatrix}\right]$ and $(x^7 - x)^2 = \left[\begin{smallmatrix} 5 & 0 \\ 0 & 5 \end{smallmatrix}\right] \in U(\overline{R})$, a contradiction. This shows that $\overline{R}$ is reduced, so $x^7 = x$ for all $x \in \overline{R}$. Hence $\overline{R}$ is a subdirect product of $\mathbb{Z}_7$'s.

(3) $\Rightarrow$ (1) Let $a \in R$. By Lemma 3, then there exist polynomials $\theta(t)$, $\zeta(t)$, $\eta_3(t) \in \mathbb{Z}[t]$ such that $\overline{a} = \theta(\overline{a}) + \zeta(\overline{a}) + \eta(\overline{a})$ and $\theta(\overline{a}), \zeta(\overline{a}), \eta(\overline{a})$ are tripotents in $\overline{R} := R/J(R)$. Since $\theta(\overline{a}) = \overline{\theta(a)}, \zeta(\overline{a}) = \overline{\zeta(a)}$ and $\eta(\overline{a}) = \overline{\eta(a)}$, we see that $\theta(a)^3 - \theta(a), \zeta(a)^3 - \zeta(a)$ and $\eta(a)^3 - \eta(a)$ are in $J(R)$, so are nilpotent. As $2 \in U(R)$, by Lemma 7, there exist tripotents e, f and g in $\mathbb{Z}[a]$ such that $\theta(a) - e, \zeta(a) - f$ and $\eta(a) - g$ are nilpotent. Then $b := a - e - f - g = (a - \theta(a) - \zeta(a) - \eta(a)) + (\theta(a) - e) + (\zeta(a) - f) + (\eta(a) - g)$ is nilpotent, b, e, f and g commute with one another, and $a = b + e + f + g$. □

Here is the structure of rings with property $\mathcal{P}$.

Theorem 10. *The following are equivalent for a ring R:*

(1) *R has property $\mathcal{P}$.*

(2) *$R = R_1 \oplus R_2 \oplus R_3 \oplus R_4$, where R_1 is zero or $R_1/J(R_1)$ is a Boolean ring with $J(R_1)$ nil, R_2 is zero or $R_2/J(R_2)$ is a subdirect product of $\mathbb{Z}_3$'s with $J(R_2)$ nil, R_3 is zero or $R_3/J(R_3)$ is a subdirect product of $\mathbb{Z}_5$'s with $J(R_3)$ nil, R_4 is zero or $R_4/J(R_4)$ is a subdirect product of $\mathbb{Z}_7$'s with $J(R_4)$ nil.*

(3) *$R = A \oplus B$, where $a - a^5 \in \mathrm{Nil}(A)$ for all $a \in A$ and $b - b^7 \in \mathrm{Nil}(B)$ for all $b \in B$ with $7 \in \mathrm{Nil}(B)$.*

Proof. (1) $\Leftrightarrow$ (2) The equivalence follows from Lemma 2 and Lemmas 4, 5, 6 and 9.

(2) $\Rightarrow$ (3) Suppose that (2) holds. Let $A = R_1 \oplus R_2 \oplus R_3$ and $B = R_4$. Then, $a - a^5 = (1 + a + a^2 + a^3)(a - a^2) = (1 + a^2)(a - a^3) \in \mathrm{Nil}(A)$ for all $a \in A$. By Lemma 9, $b - b^7 \in \mathrm{Nil}(B)$ for all $b \in B$ with $7 \in \mathrm{Nil}(B)$.

(3) $\Rightarrow$ (1) In view of [9, Theorem 2.11], every element of A is a sum of a nilpotent and two tripotents that commute with one another, so A has property $\mathcal{P}$. By Lemma 9, B has property $\mathcal{P}$. Hence R has property $\mathcal{P}$. □

The following result is useful to verify property $\mathcal{P}$ of a ring.

Theorem 11. *A ring R has $\mathcal{P}$ property if and only if the following conditions are satisfied:*

(1) *$13 \in U(R)$ and $a - a^{13}$ is nilpotent for all $a \in R$;*

(2) *$1 + a + a^2 \in U(R)$ for all $a \in R$ whenever $2 \in \mathrm{Nil}(R)$.*

Proof. ($\Rightarrow$) By the proof of Lemma 2, $2 \cdot 3 \cdot 5 \cdot 7 = 210 \in \mathrm{Nil}(R)$, so $13 \cdot 97 = 1 + 210 \cdot 6 \in U(R)$, and hence $13 \in U(R)$. Moreover, R has the decomposition $R = A \oplus B$ as stated in Theorem 10(3). For $a \in A$, $a - a^{13} = (a - a^5)(1 + a^4 + a^8)$ is nilpotent; for $a \in B$, $a - a^{13} = (a - a^7)(1 + a^6)$ is nilpotent. Thus, $a - a^{13}$ is nilpotent for all $a \in R$. Now assume $2 \in \mathrm{Nil}(R)$. By Lemma 4, $a - a^2$ is nilpotent for all $a \in R$. So $1 + a + a^2 = 1 + [(a - a^2) + 2a^2] \in U(R)$, and hence (2) holds.

($\Leftarrow$) It is clear that $J(R)$ is nil. Since $2^{13} - 2 = 2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \in \mathrm{Nil}(R)$ and $13 \in U(R)$, $2 \cdot 3 \cdot 5 \cdot 7$ is nilpotent. So $2^n \cdot 3^n \cdot 5^n \cdot 7^n = 0$ for some $n > 0$. By the Chinese Reminder Theorem, $R = R_1 \oplus R_2 \oplus R_3 \oplus R_4$, where $R_1 \cong R/2^nR$, $R_2 \cong R/3^nR$, $R_3 \cong R/5^nR$, $R_4 \cong R/7^nR$.

For $a \in R_1$, as $2 \in \mathrm{Nil}(R_1) \subseteq \mathrm{Nil}(R)$, there exists a polynomial $\theta(t) \in \mathbb{Z}[t]$ such that $(a - a^4)^4 = a^3(a - a^{13}) + 2\theta(a) \in \mathrm{Nil}(R_1)$, so $a - a^4 = (a - a^2)(1 + a + a^2)$ is nilpotent. It follows that $a - a^2$ is nilpotent since $1 + a + a^2 \in U(R)$. By Lemma 4, R_1 has property $\mathcal{P}$.

For $a \in R_2$, $(a - a^5)^3 = a^2(a - a^{13}) - 3(a^7 - a^{11}) \in \mathrm{Nil}(R_2)$. So, by [9, Theorem 2.11], every element of R_2 is a sum of a nilpotent and two tripotents that commute with one another. Hence, R_2 has property $\mathcal{P}$.

As $\overline{R_3}$ is semiprimitive, $\overline{R_3}$ is a subdirect product of a family of right primitive rings $\{R_\alpha : \alpha \in \Lambda\}$. Assume that R_α is not simple Artinian. Then there exists a subring $S_\alpha \subseteq R_\alpha$ such that $\mathbb{M}_2(D)$ is a factor ring of S_α, where D is a division ring. It follows that $y - y^{13} \in \mathrm{Nil}(\mathbb{M}_2(D))$ for any $y \in \mathbb{M}_2(D)$. Take

$y = \left[\begin{smallmatrix} 2 & 3 \\ 3 & 1 \end{smallmatrix}\right]$. Noticing that $5 = 0$ in R_α, we have $y - y^{13} = \left[\begin{smallmatrix} -1 & 1 \\ 1 & 2 \end{smallmatrix}\right] \in U(\mathbb{M}_2(D))$, so $y - y^{13}$ is not a nilpotent, a contradiction. This shows that R_α is simple Artinian for each $\alpha \in \Lambda$, so $R_\alpha \cong \mathbb{M}_n(D)$ for some $n \geq 1$. As argued above, $n = 1$, i.e., $R_\alpha \cong D$ is a division ring. Therefore $a^{13} = a$ for all $a \in R_\alpha$. Further, let $b \in R_\alpha$. Then $(b - b^5)^5 = b^5 - b^{25} = b^5 - b^{13} = b^5 - b \in R_\alpha$. If $b - b^5 \neq 0$, then $(b - b^5)^4 = -1$, and $(b - b^5)^{12} = -1$, so $(b - b^5)^{13} = -(b - b^5)$. But $(b - b^5)^{13} = b - b^5$. So $2(b - b^5) = 0$, and hence $b - b^5 = 0$, a contradiction. Thus $b = b^5$ for all $b \in R_\alpha$. It follows that $\overline{a} - \overline{a}^5 = 0$ for all $a \in R_3$. So, $a - a^5 \in J(R_3)$ is a nilpotent. By Lemma 6, R_3 has property $\mathcal{P}$.

Similarly, $\overline{R_4} := R_4/J(R_4)$ is a subdirect product of right primitive rings $\{R_\beta : \beta \in \Gamma\}$. Assume that R_β is not simple Artinian. Then there exists a subring $S_\beta \subseteq R_\beta$ such that $\mathbb{M}_2(D)$ is a factor ring of S_β, where D is a division ring. It follows that $z - z^{13} \in \mathrm{Nil}(\mathbb{M}_2(D))$ for any $z \in \mathbb{M}_2(D)$. Take $z = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Then $z - z^{13} = \left[\begin{smallmatrix} 2 & -1 \\ -1 & 3 \end{smallmatrix}\right] \in U(\mathbb{M}_2(D))$. This contradiction shows that R_β is simple Artinian. So $R_\beta \cong \mathbb{M}_n(D)$ for some $n \geq 1$. As argued above, $n = 1$, i.e., $R_\beta \cong D$ is a division ring. Thus, for each $a \in R_\beta$, $a^{13} = a$, so $(a - a^7)(1 + a^6) = 0$. Assume $a - a^7 \neq 0$. Then $1 + a^6 = 0$, so $a^7 = -a$ and $(1 + a)^7 = 1 + a^7 = 1 - a$. Now $(1 + a) - (1 + a)^7 = a - a^7 \neq 0$ and $[(1+a) - (1+a)^7][1 + (1+a)^6] = 0$ (indeed, $(x - x^7)(1 + x^6) = 0$ for all $x \in R_\beta$). Thus, $1 + (1 + a)^6 = 0$ and so $(1 + a)^7 = -(1 + a)$. Hence, $1 - a = -1 - a$, i.e., $2 = 0$ in R_β, a contradiction. Therefore, $a^7 = a$ for all R_β. It follows that, for each $a \in R_4$, $a - a^7 \in J(R_4)$ is nilpotent. By Lemma 9, R_4 has property $\mathcal{P}$. Hence R has property $\mathcal{P}$. □

Corollary 12. *If a ring R has property $\mathcal{P}$, then so does its center $C(R)$.*

For $R = \mathbb{Z}_{13}$, $2 \in U(R)$ and $a - a^{13} = 0$ for all $a \in R$, but $13 \notin U(R)$; so R does not have $\mathcal{P}$ property. For $R = \mathbb{M}_2(\mathbb{Z}_2)$, $13 \in U(R)$ and $a - a^{13} \in \mathrm{Nil}(R)$ for all $a \in R$. But $2 \in \mathrm{Nil}(R))$ and $1 + a + a^2 = 0$ for $a = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$; so R does not have property $\mathcal{P}$.

Next we consider property $\mathcal{P}$ for some extensions of rings. By Theorem 10, if R has property $\mathcal{P}$, then $R/J(R)$ is commutative. Thus any matrix ring of size greater than 1 does not have property $\mathcal{P}$.

Proposition 13. *If a ring R has property $\mathcal{P}$, then so does eRe for any $e^2 = e \in R$.*

Proof. By Theorem 10, $R = A \oplus B$, where $a - a^5 \in \mathrm{Nil}(A)$ for all $a \in A$ and $b - b^7 \in \mathrm{Nil}(B)$ for all $b \in B$ with $7 \in \mathrm{Nil}(B)$. Write $e = (e_1, e_2)$, where ${e_1}^2 = e_1 \in A$ and ${e_2}^2 = e_2 \in B$. So, $eRe = e_1Ae_1 \oplus e_2Be_2$. We have $x - x^5 \in \mathrm{Nil}(e_1Ae_1)$ for all $x \in e_1Ae_1$, $y - y^7 \in \mathrm{Nil}(e_2Be_2)$ for all $y \in e_2Be_2$ and $7e_2 \in \mathrm{Nil}(e_2Be_2)$. Hence, by Theorem 10, eRe has property $\mathcal{P}$. □

Proposition 14. *For a nil ideal I of a ring R, R has property $\mathcal{P}$ if and only if R/I has property $\mathcal{P}$.*

Proof. The necessity is obvious; the sufficiency is a quick consequence of Theorem 11. □

Corollary 15. *A ring R has property $\mathcal{P}$ if and only if $\mathbb{T}_n(R)$ has property $\mathcal{P}$.*

Corollary 16. *A ring R has property $\mathcal{P}$ if and only if $R/J(R)$ has property $\mathcal{P}$ and $J(R)$ is nil.*

Proof. If R has property $\mathcal{P}$, then $R/J(R)$ has property $\mathcal{P}$ and $J(R)$ is nil (by Theorem 10); the sufficiency is by Proposition 14. □

3. Group rings

In this section, we determine when a group ring of an abelian group has property $\mathcal{P}$. The center of a group G is denoted by $\mathcal{Z}(G)$. A group G is called locally finite if every finitely generated subgroup of G is finite. Let p be a prime number. A group G is called a p-group if the order of each element of G is a power of p. The cyclic group of order n is denoted by C_n.

If R is a ring and G is a group, RG denotes the group ring of the group G over R. The ring homomorphism $\omega : RG \to R$, $\Sigma r_g g \mapsto \Sigma r_g$ is called the augmentation map, and $\ker(\omega)$ is called the augmentation ideal of the group ring RG and is denoted by $\triangle(RG)$. Note that if the group ring RG has property $\mathcal{P}$, so does R.

Lemma 17. *Let R be a ring and G be a group. If RG is has property $\mathcal{P}$, then $C(R)\mathcal{Z}(G)$ has property $\mathcal{P}$.*

Proof. This is by Corollary 12, because $C(RG) = C(R)\mathcal{Z}(G)$. □

Lemma 18. *Let R be a ring and G be a group. Suppose that RG has property $\mathcal{P}$.*

(1) *If $2 \in J(R)$, then $\mathcal{Z}(G)$ is a 2-group.*
(2) *If $3 \in J(R)$, then $\mathcal{Z}(G)$ is a direct product of a group of exponent 2 and a 3-group.*
(3) *If $5 \in J(R)$, then $\mathcal{Z}(G) = H \times K$, where $h^4 = 1$ for all $h \in H$ and K is a 5-group.*
(4) *If $7 \in J(R)$, then $\mathcal{Z}(G)$ is a direct product of a group of exponent 2, a group of exponent 3 and a 7-group.*

Proof. Let $p \in \{2, 3, 5, 7\}$. If $p \in J(R)$, then $(R/J(R))G$ has property $\mathcal{P}$ and $p = 0$ in $R/J(R)$. So, without loss of generality, we can assume $J(R) = 0$. Then $x - x^p$ is nilpotent for all $x \in RG$ by Lemmas 4, 5, 6 and 9. For $g \in \mathcal{Z}(G)$, $g - g^p$ is nilpotent, so $1 - g^{p-1}$ is nilpotent. Thus, there exists $n > 0$ such that $(1 - g^{p-1})^{p^n} = 0$, i.e., $g^{(p-1)\cdot p^n} = 1$ as $p = 0$ in R.

If $p = 2$, then for each $g \in \mathcal{Z}(G)$, $g^{2^n} = 1$ for some $n \geq 1$; so $\mathcal{Z}(G)$ is a 2-group.

If $p = 3$, then for each $g \in \mathcal{Z}(G)$, $g^{2\cdot 3^n} = 1$ for some $n \geq 1$; so $\mathcal{Z}(G)$ is a direct product of its 2-component and 3-component. Write $\mathcal{Z}(G) = H \times K$

where H is a 2-group and K is a 3-group. We next verify that H is of exponent 2. By Lemma 17, $C(R)\mathcal{Z}(G)$ has property $\mathcal{P}$, so $C(R)H$ has property $\mathcal{P}$. Let F be a field that is an image of $C(R)$. Then $3 = 0$ in F and FH has property $\mathcal{P}$. If $x := a_0 + a_1h_1 + \cdots + a_kh_k \in FH$ is nilpotent, then, for some $n > 0$,

$$0 = x^{3^n} = (a_0)^{3^n} + (a_1)^{3^n}(h_1)^{3^n} + \cdots + (a_k)^{3^n}(h_k)^{3^n}.$$

Note that, for $i \neq j$, $(h_i)^{3^n} \neq (h_j)^{3^n}$. To see this, note that H is a 2-group, so $(h_i)^{2^m} = (h_j)^{2^m} = 1$ for some $m > 0$. Write $1 = s2^m + t3^n$ where $s, t \in \mathbb{Z}$. Then $h_i = (h_i)^{s2^m+t3^n} = ((h_i)^{2^m})^s((h_i)^{3^n})^t = ((h_i)^{3^n})^t$, and $h_j = (h_j)^{s2^m+t3^n} = ((h_j)^{2^m})^s((h_j)^{3^n})^t = ((h_j)^{3^n})^t$. So $h_i \neq h_j$ implies $(h_i)^{3^n} \neq (h_j)^{3^n}$. Therefore, it follows that $(a_i)^{3^n} = 0$ for $i = 0, 1, \ldots, k$. This is, $a_i = 0$ for $i = 0, 1, \ldots, k$. This shows that FH has no nonzero nilpotents, so every element of FH is a tripotent by Lemma 5. In particular, for any $h \in H$, $h = h^3$, i.e., $h^2 = 1$. Hence H is a group of exponent 2.

If $p = 5$, then for each $g \in \mathcal{Z}(G)$, $g^{4\cdot 5^n} = 1$ for some $n \geq 1$; so $\mathcal{Z}(G)$ is a direct product of its 2-component and 3-component. Write $\mathcal{Z}(G) = H \times K$ where H is a 2-group and K is a 3-group. As argued above, $h^4 = 1$ for all $h \in H$.

If $p = 7$, then for each $g \in \mathcal{Z}(G)$, $g^{2\cdot 3\cdot 7^n} = 1$ for some $n \geq 1$; so $\mathcal{Z}(G)$ is a direct product of its 2-component, 3-component and 7-component. Write $\mathcal{Z}(G) = H \times K \times J$ where H is a 2-group, K is a 3-group and J is a 7-group. As argued above, for $h \in H$ and $k \in K$, $h^6 = 1$ and $k^6 = 1$. As H is a 2-group and K is a 3-group, it follows that $h^2 = 1$ and $k^3 = 1$. So H is a group of exponent 2 and K is a group of component 3. □

Lemma 19. *Let $p \in \{2, 3, 5, 7\}$. If R has property $\mathcal{P}$ with $p \in J(R)$ and G is a locally finite p-group, then RG has property $\mathcal{P}$.*

Proof. As G is locally finite, we can assume that G is a finite p-group. By Theorem 10, $J(R)$ is nil, so $p \in J(R)$ is nilpotent. By [2, Theorem 9], $\triangle(RG)$ is nilpotent. Since $(RG)/\triangle(RG) \cong R$, it follows from Proposition 14 that RG has property $\mathcal{P}$. □

Theorem 20. *Let R be a ring and G be an abelian group. Then RG has property $\mathcal{P}$ if and only if one of the following cases holds*:

(1) *$R \cong A$ and G is a 2-group,*
(2) *$R \cong B$ and G is a direct product of a group of exponent 2 and a 3-group.*
(3) *$R \cong C$ and $G = H \times K$, where $h^4 = 1$ for all $h \in H$ and K is a 5-group.*
(4) *$R \cong D$ and $G = H \times K \times J$, where H is a group of exponent 2, K is a group of exponent 3 and J is a 7-group.*
(5) *$R \cong A \oplus C$, and $g^4 = 1$ for all $g \in G$.*
(6) *$R \in \{A \oplus B, A \oplus D, B \oplus C, B \oplus D, C \oplus D, A \oplus B \oplus C, A \oplus B \oplus D, B \oplus C \oplus D, A \oplus B \oplus C \oplus D\}$, and G is a group of exponent 2,*

where $A/J(A)$ is Boolean with $J(A)$ nil, $B/J(B)$ is a subdirect product of $\mathbb{Z}_3$'s with $J(B)$ nil, $C/J(C)$ is a subdirect product of $\mathbb{Z}_5$'s with $J(C)$ nil and $D/J(D)$ is a subdirect product of $\mathbb{Z}_7$'s with $J(D)$ nil.

Proof. ($\Rightarrow$) Suppose that RG has property $\mathcal{P}$. Then R has property $\mathcal{P}$, so, by Theorem 10, $R = A \oplus B \oplus C \oplus D$ where A is zero or $A/J(A)$ is Boolean with $J(A)$ nil, B is zero or $B/J(B)$ is a subdirect product of $\mathbb{Z}_3$'s with $J(B)$ nil, C is zero or $C/J(C)$ is a subdirect product of $\mathbb{Z}_5$'s with $J(C)$ nil, and D is zero or $D/J(D)$ is a subdirect product of $\mathbb{Z}_7$'s. Then one of the following cases occurs, in view of Lemma 18.

Case 1: $R = A$. Then G is 2-group.

Case 2: $R = B$. Then G is a direct product of a group of exponent 2 and a 3-group.

Case 3: $R = C$. Then $G = H \times K$, where $h^4 = 1$ and K is a 5-group.

Case 4: $R = D$. Then G is a direct product of a group of exponent 2, a group of exponent 3 and a 7-group.

Case 5: $A \neq 0$, and $B \neq 0$ or $D \neq 0$. Then G satisfies the conditions in Cases 1,2 and 4, so G is a group of exponent 2.

Case 6: $R = A \oplus C$. Then G satisfies the conditions in Cases 1 and 3, so $g^4 = 1$ for all $g \in G$.

($\Leftarrow$) Firstly, by Lemma 19, (1) implies that RG has property $\mathcal{P}$.

We next show that, if G is a group of exponent 2, then $(A \oplus B \oplus C \oplus D)G$ has property $\mathcal{P}$. Indeed, $(A \oplus B \oplus C \oplus D)G \cong AG \oplus (B \oplus C \oplus D)G$. As AG has property $\mathcal{P}$ by (1), we only need to show that $(B \oplus C \oplus D)G$ has property $\mathcal{P}$, and we can assume that G is a finite group. So G is a direct product of finite copies of C_2, and hence, as 2 is a unit in $B \oplus C \oplus D$, $(B \oplus C \oplus D)G$ is a direct sum of finite copies of $B \oplus C \oplus D$. Hence, $(B \oplus C \oplus D)G$ has property $\mathcal{P}$. Thus, (6) implies RG has property $\mathcal{P}$.

Suppose (2) holds. Write $G = H \times K$ where H is a group of exponent 2 and K is a 3-group. Then $RH \cong BH$ has property $\mathcal{P}$ by (6), so $RG \cong (BH)K$ has property $\mathcal{P}$ by Lemma 19.

Suppose that (3) holds. Write $G = H \times K$, where $h^4 = 1$ for all $h \in H$ and K is a 5-group. Then $RG \cong (CH)K$, thus to show RG has property $\mathcal{P}$ it suffices to show that CH has property $\mathcal{P}$ by Lemma 19. We can assume that H is a finite group. Thus, H is a direct product of finite copies of C_2 and finite copies of C_4. Therefore, we only need to show that CC_2 and CC_4 have property $\mathcal{P}$. Note that CC_2 has property $\mathcal{P}$ by (6). Since $J(C)$ is nil, $J(C)C_4$ is nil. As $(CC_4)/(J(C)C_4) \cong (C/J(C))C_4$, to show CC_4 has property $\mathcal{P}$ it suffices to show that $(C/J(C))C_4$ has property $\mathcal{P}$ by Proposition 14. As $C/J(C)$ is commutative with $x^5 = x$ for all $x \in C/J(C)$ and $g^5 = g$ for all $g \in C_4$, one quickly sees that $y^5 = y$ for all $y \in (C/J(C))C_4$, so $(C/J(C))C_4$ has property $\mathcal{P}$ by Lemma 6. Hence (3) implies RG has property $\mathcal{P}$.

Suppose (4) holds. Write $G = H \times K \times J$, where H is a group of exponent 2, K is a group of exponent 3 and J is a 7-group. Then $RG \cong DG \cong ((DH)K)J$.

Thus to show RG has property $\mathcal{P}$ it suffices to show that $(DH)K$ has property $\mathcal{P}$ by Lemma 19. By (6), DH has property $\mathcal{P}$. Since $J(DH)$ is nil, $J(DH)K$ is nil. As $(DH)K/J(DH)K \cong ((DH)/J(DH))K$, to show that $(DH)K$ has property $\mathcal{P}$ it suffices to show that $((DH)/J(DH))K$ has property $\mathcal{P}$ by Proposition 14. As $(DH)/J(DH)$ is commutative with $x^7 = x$ for all $x \in (DH)/J(DH)$ and $g^7 = g$ for all $g \in K$, one quickly sees that $y^7 = y$ for all $y \in ((DH)/J(DH))K$, so $((DH)/J(DH))K$ has property $\mathcal{P}$ by Lemma 9. Hence (4) implies RG has property $\mathcal{P}$.

Finally suppose (5) holds. Then $RG \cong AG \oplus CG$. By (1), AG has property $\mathcal{P}$. By (3), CG has property $\mathcal{P}$. So RG has property $\mathcal{P}$. □

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