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SHARP CONDITIONS FOR THE EXISTENCE OF AN EVEN $[a, b]$-FACTOR IN A GRAPH

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Abstract. Let a and b be positive integers, and let $V(G)$, $\delta(G)$, and $\sigma_2(G)$ be the vertex set of a graph G, the minimum degree of G, and the minimum degree sum of two non-adjacent vertices in $V(G)$, respectively. An even $[a, b]$-factor of a graph G is a spanning subgraph H such that for every vertex $v \in V(G)$, $d_H(v)$ is even and $a \leq d_H(v) \leq b$, where $d_H(v)$ is the degree of v in H. Matsuda conjectured that if G is an n-vertex 2-edge-connected graph such that $n \geq 2a + b + \frac{a^2-3a}{b} - 2$, $\delta(G) \geq a$, and $\sigma_2(G) \geq \frac{2an}{a+b}$, then G has an even $[a, b]$-factor. In this paper, we provide counterexamples, which are highly connected. Furthermore, we give sharp sufficient conditions for a graph to have an even $[a, b]$-factor. For even an, we conjecture a lower bound for the largest eigenvalue in an n-vertex graph to have an $[a, b]$-factor.

1. Introduction

Throughout all sections, a graph G is finite, simple, and undirected. We denote by $V(G)$ the set of vertices of G and by $E(G)$ the set of edges of G. For $S \subseteq V(G)$, we denote by $G - S$ the subgraph of G obtained from G by deleting the vertices in S together with the edges incident to vertices in S. Similarly, for $A \subseteq E(G)$, we denote by $G - A$ the subgraph of G obtained from G by deleting the edges of A. For $S, T \subseteq V(G)$, we denote by $[S, T]$ the set of edges joining S and T. The *degree* of a vertex v in G, written $d_G(v)$ (or $d(v)$ if G is clear from the context), is the number of edges in $E(G)$ incident to the vertex v. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A graph is *even* (or *eulerian*) if every vertex has even degree.

A subgraph H of G is a *spanning subgraph* of G if $V(H) = V(G)$. An $[a, b]$-*factor* of G is a spanning subgraph H such that $a \leq d_H(v) \leq b$ for all

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$v \in V(G)$. A subgraph H of G is an *even* $[a,b]$*-factor* of G if H is an $[a,b]$-factor of G and $d_H(v)$ is even for all $v \in V(H)$. If $a = b$, then we call it an *a-factor*. A graph G is *k-edge-connected* if for $A \subseteq E(G)$ with $|A| < k$, $G - A$ is connected. The *edge-connectivity* of G, denoted $\kappa'(G)$, is the maximum k such that G is k-edge-connected. A graph G is *k-vertex-connected* if $|V(G)| \geq k+1$ and for $S \subseteq V(G)$ with $|S| < k$, $G - S$ is connected. The *vertex-connectivity* of G, denoted $\kappa(G)$, is the maximum k such that G is k-vertex-connected.

Veblen [19] showed that a graph has a cycle decomposition if and only if it is an even graph. An Euler's Theorem says that a connected graph has an Euler cycle if and only if it is an even graph. Those results says that even graphs are very interesting objects that have lots of useful properties. Thus many researchers investigated sufficient conditions for a graph to have an even $[a,b]$-factor [7–9, 14, 16].

Kouider and Vestaargard [8, 9] explored minimum degree conditions for a graph to have an even $[a,b]$-factor.

Theorem 1.1 ([8,9])**.** *Let a and b be even integers such that $2 \leq a \leq b$, and let G be an n-vertex 2-edge-connected graph.*

(i) *If $a \geq 4$, $n \geq \max\{\frac{(a+b)^2}{b}, \frac{3(a+b)}{2}\}$, and $\delta(G) \geq \frac{an}{a+b}$, then G has an even $[a,b]$-factor.*
(ii) *If $a \geq 4$, $n \geq \frac{(a+b)^2}{b}$ and $\delta(G) \geq \frac{an}{a+b} + \frac{a}{2}$, then G has an even $[a,b]$-factor.*
(iii) *If $a \geq 4$, $n \geq \frac{(a+b)^2}{b}$, $\kappa'(G) \geq (a + \min\{\sqrt{a}, \frac{b}{a}\})$, and $\delta(G) \geq \frac{an}{a+b}$, then G has an even $[a,b]$-factor.*
(iv) *If $a = 2, n \geq 3$, and $\delta(G) \geq \max\{3, \frac{2n}{b+2}\}$, then G has an even $[2,b]$-factor.*

Note that for $a = b$, the lower bounds for n in (i), (ii), and (iii) of Theorem 1.1 are greater than $4a-5$. In the papers [2,5], the conditions $n \geq 4a-5$, $\delta(G) \geq \frac{n}{2}$ and G being just connected are enough to guarantee the existence of an even $[a,b]$-factor. The lower bound for $\delta(G)$ in (ii) of Theorem 1.1 is greater than $\frac{an}{a+b}$. For $a = 2$, the condition $\delta(G) \geq \frac{an}{a+b}$ guarantees that G has an even $[a,b]$-factor. Thus it may be possible to improve the conditions on n, $\delta(G)$, and $\kappa'(G)$ in (i), (ii), and (iii) of Theorem 1.1.

The conditions in (iv) of Theorem 1.1 are sharp as shown in Remark 1 of [14]. In 2005, Matsuda [14] improved the result (iv) of Theorem 1.1 by considering $\sigma_2(G) \geq \frac{2an}{a+b}$ instead of $\delta(G) \geq \frac{an}{a+b}$ for $a = 2$, where $\sigma_2(G) = \min_{uv \notin E(G)} (d(u) + d(v))$ and proposed a conjecture for the existence of an even $[a,b]$-factor in a graph as follows:

Conjecture 1.2 ([14])**.** *Let $2 \leq a \leq b$ be even integers. If G is a graph with n vertices such that* (i) *$\kappa'(G) \geq 2$,* (ii) *$n \geq 2a + b + \frac{a^2-3a}{b} - 2$,* (iii) *$\delta(G) \geq a$, and* (iv) *$\sigma_2(G) \geq \frac{2an}{a+b}$, then G contains an even $[a,b]$-factor.*

However, Conjecture 1.2 is not true even when $a = 2$. Remark 3 in [14] says that if $n = b + 2$, then Conjecture 1.2 does not hold. Theorem 8 in [14] says that if we replace $n \geq b+2$ by $n \geq b+3$, then G contains an even $[2, b]$-factor. A result of Iida and Nishimura [4] implies that Conjecture 1.2 is true when $a = b$.

For $a \geq 4$, all other conditions in the conjecture are sharp, except $\kappa'(G) \geq 2$ as shown in the examples in Section 6 of [14]. In Section 2 of this paper, we provide counterexamples to Conjecture 1.2, which are $(a-1)$-edge-connected. Note that Tsuchiya and Yashima [16] also provided some counterexamples, which are 3-edge-connected. Furthermore, we show that there are $(a-1)$-vertex-connected graphs satisfying all conditions in Conjecture 1.2, which do not contain an even $[a, b]$-factor. Thus to guarantee the existence of an even $[a, b]$-factor in a graph, we may need high vertex-(or edge-)connectivity. By reinforcing the condition $\sigma_2(G) \geq \frac{2an}{a+b}$ to $\delta(G) \geq \frac{an}{a+b}$, we give sharp sufficient conditions for a graph to have an even $[a, b]$-factor in Theorem 1.3. Example 2.3 of this paper and examples in Section 6 of [14] show that the conditions in Theorem 1.3 are sharp.

Theorem 1.3 (Main Theorem)**.** *Let* $4 \leq a \leq b$ *be even integers. If* G *is a graph with* n *vertices such that* (i) $\kappa(G) \geq a$, (ii) $n \geq 2a + b + \frac{a^2-3a}{b} - 2$, *and* (iii) $\delta(G) \geq \frac{an}{a+b}$, *then* G *contains an even* $[a, b]$*-factor.*

Katerinis [5], and Egawa and Enomoto [2] independently showed that Theorem 1.3 is true when $a = b$. In this paper, we extend their results by proving for all $4 \leq a \leq b$ including the case $a = b$. In the papers [2, 5], to have an $[a, a]$-factor (or a-factor), one of the sufficient conditions is just "connected" instead of $\kappa(G) \geq a$. However, if there is an enough gap between a and b, then to have an even $[a, b]$-factor, a graph must be highly connected (See Section 2).

Note that Conditions (ii) and (iii) in Theorem 1.3 imply that $\delta(G) \geq a + 1$. If $\delta(G) \leq a$, then Condition (iii) says $\frac{an}{a+b} \leq a$. Thus we have $n \leq a + b$, which contradicts Condition (ii).

In Section 3, we prove Theorem 1.3 by using Corollary 1.5 of Lovasz's (g, f)-factor theory.

Theorem 1.4 (Lovasz's parity (g, f)-factor theory [10])**.** *Let* G *be a graph and let* g, f *be two integer valued functions defined on* $V(G)$ *such that* $0 \leq g(v) \leq f(v) \leq d_G(v)$ *and* $g(v) \equiv f(v) \pmod 2$ *for all* $v \in V(G)$*. Then* G *has a* (g, f)*-factor* F *such that* $d_F(v) \equiv f(v) \pmod 2$ *for all* $v \in V(G)$ *if and only if*

$$\sum_{v\in T} (d_G(v) - g(v)) + \sum_{u\in S} f(u) - |[S, T]| - q(S, T) \geq 0$$

for all disjoint subsets S *and* T *of* $V(G)$*, where* $q(S, T)$ *is the number of components* Q *of* $G - (S \cup T)$ *such that*

$$|[Q, T]| + \sum_{v\in V(Q)} f(v) \equiv 1 \pmod 2.$$

Corollary 1.5. *Let a and b be even integers with $2 \le a \le b$. A graph G has an even $[a,b]$-factor if*

$$q(S,T) - b|S| + a|T| - \sum_{v \in T} d_{G-S}(v) \le 0$$

for all disjoint choices $S, T \subseteq V(G)$, where $q(S,T)$ is the number of components Q of $G - (S \cup T)$ such that $|[Q,T]|$ is odd.

By applying Theorem 1.4 when $g(x) = a$ and $f(x) = b$, we obtain Corollary 1.5.

We point out that Tutte [18] proved that the Lovasz's (g,f)-factor theory [10] can be demonstrated by using Tutte's f-factor theory [17].

The Parity Lemma is also used in the proof of our main result.

Lemma 1.6 (Parity Lemma). *Let a and b be positive integers with the same parity. Then $q(S,T) - b|S| + a|T| - \sum_{v \in T} d_{G-S}(v)$ has the same parity as a and b for any disjoint sets $S, T \subseteq V(G)$.*

2. Sharp examples

In this section, by providing Example 2.1 and Example 2.3, we show why high edge-(or vertex-)connectivity in Theorem 1.3 requires. Note that Matsuda [14] showed in the last section that Conditions (ii) and (iii) in Theorem 1.3 are sharp.

Example 2.1 shows that if a graph satisfying Conditions (ii), (iii), and (iv) in Conjecture 1.2 is not a-edge-connected, then we cannot guarantee the existence of an even $[a,b]$-factor in the graph. Thus the graph in Example 2.1 is a counterexample to Conjecture 1.2, which has edge-connectivity equal to $a-1$.

Example 2.1. Let a and b be even integers such that $12 \le 3a \le b$, and let t be an integer such that $t \ge \frac{(a+b)^2 - 3a - 4b}{2b} (= \frac{2a+b-4}{2} + \frac{a^2-3a}{2b} > a)$. For $i \in \{1,2\}$, let H_i be a copy of the complete graph on t vertices, and let $V(H_i) = \{x_{i1}, \dots, x_{it}\}$. Let H_3 be a copy of the complete graph on 2 vertices and let $V(H_3) = \{y, z\}$. Suppose that H is the graph obtained from H_1, H_2, and H_3 by adding edges between y and $x_{11}, \dots, x_{1(\frac{a}{2}-1)}, x_{2\frac{a}{2}}, \dots, x_{2(a-1)}$, and between z and $x_{21}, \dots, x_{2(\frac{a}{2}-1)}, x_{1\frac{a}{2}}, \dots, x_{1(a-1)}$ (see Figure 1).

Proposition 2.2. *The graph in Example 2.1 has edge-connectivity equal to $a-1$ and satisfies all conditions in Conjecture 1.2. Furthermore, it does not contain an even $[a,b]$-factor.*

Proof. Since there are $\frac{a}{2} - 1$ edges between y and H_1 and $\frac{a}{2}$ edges between y and H_2, and H_1 and H_2 are both complete graphs, there are exactly $a-1$ edge-disjoint $y-z$ paths including the yz edge. Also, since there are exactly $a-1$ edges between H_i to H_3, we have $\kappa'(H) = a-1$.

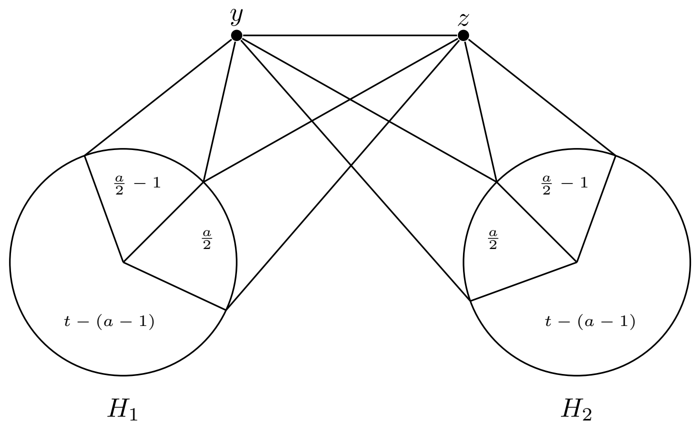


FIGURE 1. The graph H in Example 2.1

The order of H is

$$|V(H)| = 2t + 2 \geq \frac{(a+b)^2 - 3a - 4b}{b} + 2 = 2a + b + \frac{a^2 - 3a}{b} - 2.$$

Since every vertex in $V(H_1) \cup V(H_2)$ has degree at least a and $d_H(y) = d_H(z) = a$, we have $\delta(H) = a$.

Since $\sigma_2(H) = a + (t-1)$ and $t = \frac{|V(H)|-2}{2}$, we have

$$\sigma_2(H) = a + \frac{|V(H)|}{2} - 2 \geq \frac{|V(H)|}{2} = \frac{2a|V(H)|}{4a} \geq \frac{2a|V(H)|}{a+b}.$$

Thus H satisfies all conditions in Conjecture 1.2.

Now, we prove that H does not contain an even $[a, b]$-factor. Assume to the contrary that H has an even $[a, b]$-factor F. Since $d_H(y) = a$, all edges incident to y must be in F. Since $H_1 \cap F$ is also a graph, $\sum_{v \in V(H_1 \cap F)} d_{H_1 \cap F}(v)$ must be even by the degree-sum formula. Note that the $a-1$ edges incident to both H_1 and H_3 in F are not in $H_1 \cap F$. Thus we have

$$\sum_{v \in V(H_1 \cap F)} d_{H_1 \cap F}(v) = \sum_{v \in V(H_1 \cap F)} d_F(v) - (a-1).$$

However, the degree sum is odd since $a-1$ is odd and every vertex in F has even degree. Thus we have the desired result. $\square$

Example 2.3 shows that if a graph satisfying Conditions (ii), (iii), and (iv) in Conjecture 1.2 is not a-vertex-connected, then we cannot guarantee the existence of even $[a, b]$-factor in the graph. Thus the graph in Example 2.3 is also a counterexample to Conjecture 1.2, which is $(a-1)$-vertex-connected. Thus the condition $\kappa(G) \geq a$ in Theorem 1.3 is sharp. Note that in Example 2.1, we require $b \geq 3a$ while in Example 2.3, we require $b \geq \frac{a^2-3a+a\sqrt{(a-3)(a+1)}}{2}$.

Example 2.3. Let a and b be even integers at least 4 with

$$b \geq \frac{a^2 - 3a + a\sqrt{(a-3)(a+1)}}{2}.$$

Let L_0 be the trivial graph on $a-1$ vertices, and let $V(L_0) = \{y_1, \dots, y_{(a-1)}\}$. For $1 \leq i \leq a$, let L_i be a copy of the complete graph on $a+2$ vertices and let $V(L_i) = \{x_{i1}, \dots, x_{i(a+2)}\}$. Let t be a positive integer such that $(a+2 \leq)$ $-a^2 - a + b + \frac{a^2-3a}{b} - 1 \leq t \leq -a^2 - 2a + b + \frac{b}{a} + 2$. Let L_{a+1} be a copy of the complete graph on t vertices and let $V(L_{a+1}) = \{x_{(a+1)1}, \dots, x_{(a+1)t}\}$. Suppose that L is the graph obtained from $L_0, \dots, L_{a+1}$ by adding edges between y_j and x_{ij} for all $i \in \{1, \dots, a+1\}$ and for all $j \in \{1, \dots, a-1\}$ (see Figure 2).

Proposition 2.4. *The graph in Example 2.3 has vertex-connectivity equal to $a-1$ and satisfies all conditions in Conjecture 1.2. Furthermore, it does not contain an even $[a,b]$-factor.*

Proof. For each $i \in [a+1]$, there are $a-1$ vertex-disjoint paths between any vertex in L_i and L_0 by using the vertex $x_{i1}, x_{i2}, \dots, x_{i(a-1)}$. Also, for $i \neq j$, there are $a+1$ vertex-disjoint paths between y_i and y_j by using the path $y_i x_{ki} x_{kj} y_j$ for $k \in [a+1]$. Thus we have $\kappa(L) = (a-1)$.

Since $t \geq -a^2 - a + b + \frac{a^2-3a}{b} - 1$, the order of L is

$$|V(L)| = a - 1 + (a+2)a + t \geq 2a + b + \frac{a^2 - 3a}{b} - 2.$$

Since for $i \in \{0, 1, \dots, a+1\}$, every vertex in $V(L_i)$ has degree at least $a+1$ and every vertex in L_0 has degree $a+1$, we have $\delta(L) = a+1$.

Since $\sigma_2(L) = 2(a+1)$ and $t \leq -a^2 - 2a + b + \frac{b}{a} + 2$, we have

$$\frac{2a|V(L)|}{a+b} = \frac{2a(a^2 + 3a - 1 + t)}{a+b} \leq \frac{2a(a + b + \frac{b}{a} + 1)}{a+b} = 2(a+1) = \sigma_2(L).$$

Thus F satisfies all conditions in Conjecture 1.2.

Now, we prove that L does not contain an even $[a,b]$-factor. Assume to the contrary that L has an even $[a,b]$-factor F. Then we have $d_F(v) = a$ for every vertex in L_0 since a is even. Since $\sum_{v \in V(L_i \cap F)} d_{L_i \cap F}(v)$ must be even by the degree-sum formula, there are at most $a-2$ edges coming out from $V(L_i)$ in F. Thus we have

$$a(a-1) \leq (a-2)(a+1),$$

which is a contradiction. □

Proposition 2.4 shows that Condition (i) in Theorem 1.3 is tight.

3. Proof of Theorem 1.3

As we observed in Section 1, Conjecture 1.2 holds for $a = 2$ if we replace Condition (ii) by $n \geq b+3$ and does not hold when $n = b+2$ (by Theorem 8 and Remark 3 of [14]). The same thing is true for Theorem 1.3 when $a = 2$ by

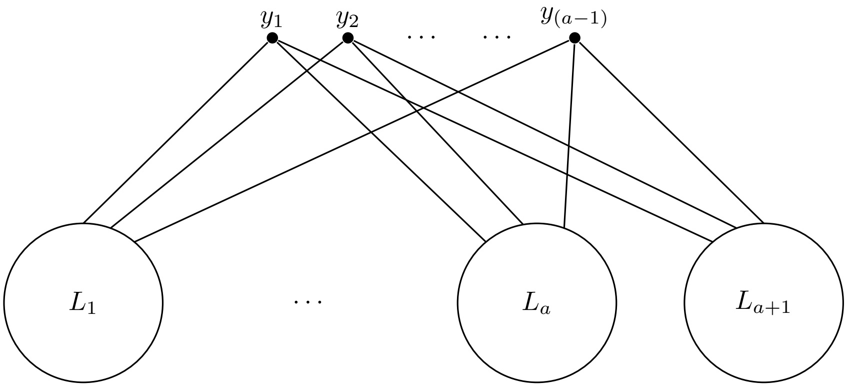


FIGURE 2. The graph L in Example 2.3

the same theorem and example. Therefore, from now, we assume that $a \geq 4$. The examples in Section 2 and the last section in [14] say that we require the conditions in Theorem 1.3 for a graph to have an even $[a, b]$-factor.

In this section, we prove Theorem 1.3. Note that for $a = b$, Theorem 1.3 is true by Katerinis [5], Egawa and Enomoto [2], and Iida and Nishimura [4]. In this paper, we prove for all $4 \leq a \leq b$ including the case $a = b$.

To prove Case 3 and Case 4-1 in the proof of Theorem 1.3, we use Proposition 3.1.

Proposition 3.1. *Let $a, b, n,$ and p be integers such that $4 \leq a \leq b$ and $p > 0$, and let $f(x) = n + (a - 1 - \frac{an}{a+b})x + (x - 1 - b)\frac{ax-p}{b}$.*

(i) *If $n \geq 2a + b + \frac{a^2-3a}{b} - 2$, then $f(b + 1) < 0$ and $f(a + b - 3) < 0$.*
(ii) *If $n \geq 2a + b + \frac{a^2-3a}{b} + 1$, then $f(a + b - 1) < 0$ and $f(a + b - 2) < 0$.*

Proof. (i) Assume that $n \geq 2a + b + \frac{a^2-3a}{b} - 2$. Then we have

$$\begin{aligned} f(b+1) &= n + (a - 1 - \frac{an}{a+b})(b+1) \\ &= (1-a)(\frac{bn}{a+b} - b - 1) \\ &\leq (1-a)\left[\frac{b}{a+b}(2a + b + \frac{a^2-3a}{b} - 2) - b - 1\right] \\ &= (1-a)\left[\frac{b(a-3) + a(a-4)}{a+b}\right] < 0 \end{aligned}$$

and

$$f(a+b-3) = n + (a - 1 - \frac{an}{a+b})(a+b-3)$$

$$
\begin{aligned}
&+ [(a+b-3)-1-b]\frac{a(a+b-3)-p}{b} \\
&= (4a+b-a^2-ab)(\frac{n}{a+b} - \frac{a+b-3}{b}) - \frac{(a-4)p}{b} \\
&\leq (4a+b-a^2-ab)(\frac{2a+b+\frac{a^2-3a}{b}-2}{a+b} - \frac{a+b-3}{b}) - \frac{(a-4)p}{b} \\
&= \frac{a(4-a)+b(1-a)}{a+b} - \frac{(a-4)p}{b} < 0.
\end{aligned}
$$

(ii) Assume that $n \geq 2a+b+\frac{a^2-3a}{b}+1$. Then we have

$$
\begin{aligned}
f(a+b-1) &= n + (a-1-\frac{an}{a+b})(a+b-1) \\
&\quad + [(a+b-1)-1-b]\frac{a(a+b-1)-p}{b} \\
&= (2a+b-a^2-ab)(\frac{n}{a+b} - \frac{a+b-1}{b}) - \frac{(a-2)p}{b} \\
&\leq (2a+b-a^2-ab)(\frac{2a+b+\frac{a^2-3a}{b}+1}{a+b} - \frac{a+b-1}{b}) - \frac{(a-2)p}{b} \\
&= \frac{(a(2-a)+b(1-a))(-2a+2b)}{b(a+b)} - \frac{(a-2)p}{b} < 0.
\end{aligned}
$$

Since $f(x)$ is a quadratic function which has a positive leading coefficient and $f(b+1) < 0$ by (i), we have $f(x) < 0$ for all $x \in [b+1, a+b-1]$ so that $f(a+b-2) < 0$. $\square$

Now, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Assume to the contrary that G has no even $[a, b]$-factor. Then there exist disjoint subsets S and T in $V(G)$ such that

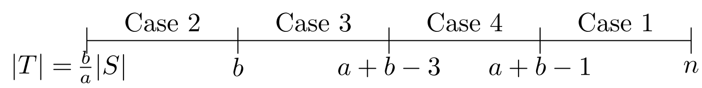
$$0 < q(S,T) - b|S| + a|T| - \sum_{v \in T} d_{G-S}(v)$$

by Corollary 1.5. Let $p = -b|S| + a|T|$ so that

$$0 < q(S,T) + p - \sum_{v \in T} d_{G-S}(v).$$

Note that $p > 0$ since $q(S,T) - \sum_{v \in T} d_{G-S}(v) \leq 0$.

We consider four cases depending on $|T|$.



In Case 4, we consider two subcases Case 4-1 and Case 4-2 depending on n. To prove Case 1, Case 3, and Case 4-1, we use the same argument as in [9]. For Case 2 and Case 4-2, we prove by using a new technique.

Case 1: $|T| \geq a+b$. Since $n \geq |S|+|T|+q(S,T)$, we have

$$|S| = \frac{a|T|-p}{b} \leq \frac{a(n-|S|-q(S,T))-p}{b} \iff |S| \leq \frac{a(n-q(S,T))-p}{a+b}$$

$$\Longrightarrow |[S,T]| \leq |S||T| \leq \frac{a(n-q(S,T))-p}{a+b}|T|. \tag{1}$$

With the inequality (1), we have

$$\begin{aligned}
0 &< q(S,T) - b|S| + a|T| - \sum_{v\in T} d_{G-S}(v) \\
&= q(S,T) + p - \sum_{v\in T} d_G(v) + |[S,T]| \\
&\leq q(S,T) + p - \delta(G)|T| + \frac{a(n-q(S,T))-p}{a+b}|T| \\
&\leq q(S,T) + p - \frac{an}{a+b}|T| + \frac{a(n-q(S,T))-p}{a+b}|T| \\
&= q(S,T) + p - \frac{aq(S,T)+p}{a+b}|T| \\
&\leq (1-a)q(S,T) \leq 0,
\end{aligned}$$

which is a contradiction.

Case 2: $|T| \leq b$. Since $\delta(G) \geq a+1$, we have

$$\begin{aligned}
0 &< q(S,T) - b|S| + a|T| - \sum_{v\in T} d_{G-S}(v) \\
&= q(S,T) - b|S| + a|T| - \sum_{v\in T} d_G(v) + |[S,T]| \\
&\leq q(S,T) - b|S| + a|T| - \delta(G)|T| + |S||T| \\
&\leq q(S,T) - b|S| + [a-(a+1)+|S|]\,|T| \\
&\leq q(S,T) - b|S| + b|S| - |T| = q(S,T) - |T|,
\end{aligned}$$

which implies $q(S,T) > |T| \geq 0$. Let l be the minimum of $|[Q,T]|$ over all components Q of $G-(S\cup T)$ such that $|[Q,T]|$ is odd. Then we have $l \geq 1$. Also, we have $\sum_{v\in T} d_{G-S}(v) \geq lq(S,T) \iff \frac{1}{l}\sum_{v\in T} d_{G-S}(v) \geq q(S,T)$. Thus we have

$$\begin{aligned}
0 &< q(S,T) - b|S| + a|T| - \sum_{v\in T} d_{G-S}(v) \\
&\leq \frac{1-l}{l}\sum_{v\in T} d_{G-S}(v) - b|S| + a|T| \\
&= \frac{1-l}{l}\sum_{v\in T} d_G(v) - \frac{1-l}{l}|[S,T]| - b|S| + a|T| \\
&\leq \frac{1-l}{l}\delta(G)|T| - \frac{1-l}{l}|S||T| - b|S| + a|T|
\end{aligned}$$

$$\leq \frac{(1-l)(a+1)}{l}|T| + |S|(\frac{l-1}{l}|T| - b) + a|T|$$

$$\leq \frac{a+1-l}{l}|T| - |S|\frac{b}{l}. \tag{2}$$

If $a+1-l \leq 0$ in the inequality (2), then it is a contradiction. Thus $a+1-l > 0$, and since $|T| \leq b$, we have

$$0 < \frac{a+1-l}{l}|T| - |S|\frac{b}{l} \leq \frac{a+1-l-|S|}{l}b.$$

This gives $a+1-l-|S| > 0$ so that $|S| \leq a-l$.

Claim 1: $|S| = a-l$. Assume to the contrary that $|S| < a-l$. Then each component Q in $G-(S\cup T)$ with $|[Q,T]| = l$ can have at most $a-1$ neighbors in $S\cup T$. Since G is a-vertex-connected, there is only one component in $G-(S\cup T)$, namely Q, and every vertex of T must have a neighbor in Q. Thus $|T| \leq l$. For $v \in V(T)$, we have

$$d(v) = |[\{v\},S]| + \big(d_{G[T]}(v) + |[\{v\},Q]|\big) \leq (a-l-1)+l = a-1 < a+1 \leq \delta(G),$$

which is a contradiction. Thus $|S| = a-l$, which implies $S \neq \emptyset$ since a is even and l is odd. Since $|S| \geq 1$ and $|T| \leq b$, we have

$$\begin{aligned}
0 &< q(S,T) - b|S| + a|T| - \sum_{v\in T} d_{G-S}(v)\\
&= q(S,T) - b|S| + a|T| - \sum_{v\in T} d_G(v) + |[S,T]|\\
&\leq q(S,T) - b|S| + a|T| - \delta(G)|T| + |S||T|\\
&\leq q(S,T) - b|S| + [a-(a+1)+|S|]|T|\\
&\leq q(S,T) - b|S| + (|S|-1)b = q(S,T) - b. \qquad (3)
\end{aligned}$$

By the inequality (3), we have $q(S,T) \geq b+1$. Let $q(S,T) = b+\alpha$ for some $\alpha \geq 1$.

Let q_l be the number of components Q of $G-(S\cup T)$ such that $|[Q,T]| = l$. Since $|S| = a-l$, $|T| \leq b$, and $q(S,T) = b+\alpha$, we have

$$\begin{aligned}
0 &< q(S,T) - b|S| + a|T| - \sum_{v\in T} d_{G-S}(v)\\
&\leq q(S,T) - b(a-l) + a|T| - [lq_l + (l+2)(q(S,T)-q_l)]\\
&\leq (-l-1)q(S,T) + bl + 2q_l\\
&= (-l-1)(b+\alpha) + bl + 2q_l\\
&= -b - (l+1)\alpha + 2q_l. \qquad (4)
\end{aligned}$$

By the inequality (4), we have $q_l > \frac{b+(1+l)\alpha}{2}$. Note that b and $1+l$ are even integers so that $\frac{b+(1+l)\alpha}{2}$ is an integer. Thus $q_l \geq \frac{b+(1+l)\alpha}{2} + 1$.

Let m be the minimum of $|V(Q)|$ over all components Q in $G-(S\cup T)$ such that $|[Q,T]| = l$. Let O be a component of $G-(S\cup T)$ such that $|[O,T]| = l$

and $|V(O)| = m$. Then there exists a vertex v in $V(O)$ such that $|[\{v\}, T]| \le \frac{l}{m}$ by the pigeonhole principle. Thus we have

$$\begin{aligned}
\delta(G) &\le d(v) = d_O(v) + |[\{v\}, T]| + |[\{v\}, S]| \le (m-1) + (\frac{l}{m}) + (a-l) \\
\Longrightarrow m &\le \frac{\delta(G)+l+1-a-\sqrt{(\delta(G)+l+1-a)^2-4l}}{2} \\
\text{or } m &\ge \frac{\delta(G)+l+1-a+\sqrt{(\delta(G)+l+1-a)^2-4l}}{2}.
\end{aligned}$$

Note that we have

$$\frac{\delta(G)+l+1-a-\sqrt{(\delta(G)+l+1-a)^2-4l}}{2} < 1 \iff \delta(G) \ge a+1. \tag{5}$$

Since $m \ge 1$, we have $m \ge \frac{\delta(G)+l+1-a+\sqrt{(\delta(G)+l+1-a)^2-4l}}{2}$ by the inequality (5). Note that we have

$$\begin{aligned}
&\frac{\delta(G)+l+1-a+\sqrt{(\delta(G)+l+1-a)^2-4l}}{2} \ge \delta(G) - |S| \\
&\iff (\delta(G)+l+1-a)^2 - 4l \ge (\delta(G) - 2|S| - 1 + a - l)^2 \\
&\iff (\delta(G) - |S|)(1 - a + l + |S|) \ge l \\
&\iff \delta(G) - a + l \ge l \iff \delta(G) \ge a. \qquad (6)
\end{aligned}$$

By the inequality (6), we have $m \ge \delta(G) - |S|$, implying $n \ge |S| + |T| + q_l(\delta(G) - |S|)$. Since $q_l \ge \frac{b+(1+l)\alpha}{2} + 1$, $\delta(G) \ge \frac{an}{a+b}$, $2b - a(b + (1+l)\alpha) = (2-a)b - (1+l)\alpha < 0$, and $a|T| - b|S| > 0$, we have

$$\begin{aligned}
&\frac{a+b}{a}\delta(G) \ge n \ge |S| + |T| + [\frac{b+(1+l)\alpha}{2} + 1](\delta(G) - |S|) \\
&\iff \delta(G) \le \frac{2a|T| - a[b+(1+l)\alpha]|S|}{2b - a[b+(1+l)\alpha]} = \frac{2(a|T| - b|S|)}{2b - a[b+(1+l)\alpha]} + |S| < |S|,
\end{aligned}$$

which is a contradiction.

Case 3: $b+1 \le |T| \le a+b-3$. Since $q(S,T) \le n - |S| - |T|$ and $\delta(G) \ge \frac{an}{a+b}$, we have

$$\begin{aligned}
0 &< q(S,T) - b|S| + a|T| - \sum_{v \in T} d_{G-S}(v) \\
&\le (n - |S| - |T|) - b|S| + a|T| - \frac{an}{a+b}|T| + |S||T| \\
&= n + (a - 1 - \frac{an}{a+b})|T| + (|T| - 1 - b)|S| \\
&= n + (a - 1 - \frac{an}{a+b})|T| + (|T| - 1 - b)\frac{a|T| - p}{b}. \qquad (7)
\end{aligned}$$

Let $f(|T|) = n + (a - 1 - \frac{an}{a+b})|T| + (|T| - 1 - b)\frac{a|T|-p}{b}$. Since f is a quadratic function which has a positive leading coefficient, the maximum value of f occurs when $|T| = b+1$ or $|T| = a+b-3$. By Proposition 3.1, both $f(b+1)$ and $f(a+b-3)$ are negative, which contradicts the inequality (7).

Case 4: $|T| = a+b-2$ or $a+b-1$.

Case 4-1: $n \geq 2a+b+\frac{a^2-3a}{b}+1$. By using the same argument with Case 3 and Proposition 3.1, we have the desired result.

Case 4-2: $2a+b+\frac{a^2-3a}{b}-2 \leq n < 2a+b+\frac{a^2-3a}{b}+1$. Let $|T| = a+b-k$ and $2a+b+\frac{a^2-3a}{b}-j \leq n < 2a+b+\frac{a^2-3a}{b}-j+1$ where $k \in \{1,2\}$ and $j \in \{0,1,2\}$. Let $n = 2a+b+\frac{a^2-3a}{b}-j+\epsilon$, where $0 \leq \epsilon < 1$.

Claim 2: *If $\delta(G) \geq j-k+i+n-|T| = i+a+\frac{a^2-3a}{b}+\epsilon$, then $a(3-k)-\epsilon b \geq (a-k)(j-k)+i(a+b-k)+(a-k-1)q(S,T)+2$, where i is an integer.*

By Lemma 1.6, we have

$$
\begin{aligned}
2 &\leq q(S,T) - b|S| + a|T| - \sum_{v\in T} d_G(v) + |[S,T]| \\
&\leq q(S,T) - b|S| + a|T| - \delta(G)|T| + |S||T| \\
&\leq q(S,T) - b|S| + a|T| - (i + a + \frac{a^2-3a}{b} + \epsilon)|T| + |S||T| \\
&\leq q(S,T) + (a-k)(n-|T|-q(S,T)) - (i + \frac{a^2-3a}{b} + \epsilon)(a+b-k) \\
&= (k+1-a)q(S,T) + (a-k)(a-i-j+k) - b(i + \frac{a^2-3a}{b} + \epsilon) \\
&= (k+1-a)q(S,T) + (a-k)(k-j) - i(a+b-k) + a(3-k) - \epsilon b.
\end{aligned}
$$

Thus we have the desired result.

Since $\delta(G) \geq \frac{an}{a+b}$, we have

$$
\begin{aligned}
\delta(G) - n + |T| &\geq \frac{an}{a+b} - n + |T| \\
&= -\frac{bn}{a+b} + (a+b-k) \\
&= \frac{-b(2a+b+\frac{a^2-3a}{b}-j+\epsilon) + (a+b-k)(a+b)}{a+b} \\
&= \frac{(3-k)a + (j-k)b - \epsilon b}{a+b} \qquad (8) \\
&> j-k-1,
\end{aligned}
$$

which is true for $j \in \{0,1,2\}$ and $\epsilon \in [0,1)$. Since $\delta(G)-n+|T|$ is an integer, we obtain $\delta(G)-n+|T| \geq j-k$, which satisfies the condition on $\delta(G)$ when $i=0$ in Claim 2. Thus we have $a(3-k)-\epsilon b \geq (a-k)(j-k)+(a-k-1)q(S,T)+2$. By the inequality (8), we have

$$
\delta(G) - n + |T| \geq \frac{(3-k)a + (j-k)b - \epsilon b}{a+b}
$$

$$\begin{aligned}&\geq \frac{(j-k)b+(a-k)(j-k)+(a-k-1)q(S,T)+2}{a+b}\\&= j-k+\frac{-k(j-k)+(a-k-1)q(S,T)+2}{a+b} > j-k,\end{aligned}$$

which is true for $k \in \{1,2\}$ and $j \in \{0,1,2\}$ and $a \geq 4$. Since $\delta(G)-n+|T|$ is an integer, we obtain $\delta(G)-n+|T| \geq j-k+1$, which satisfies the condition on $\delta(G)$ when $i=1$ in Claim 2. Thus we have

$$a(3-k)-\epsilon b \geq (a-k)(j-k)+(a+b-k)+(a-k-1)q(S,T)+2. \tag{9}$$

When $k=j=1$, the inequality (9) becomes $a-(\epsilon+1)b \geq (a-2)q(S,T)+1$ which is a contradiction since $a-(\epsilon+1)b \leq 0$ and $(a-2)q(S,T)+1 > 0$. Similarly, we have a contradiction when $(k,j) \in \{(1,2),(2,1),(2,2)\}$ by using the inequality (9). The remaining case is when $j=0$. By the inequality (8) and (9), we improve $\delta(G)$ as follows:

$$\begin{aligned}&\delta(G)-n+|T|\\&\geq \frac{(3-k)a+(j-k)b-\epsilon b}{a+b}\\&\geq \frac{(j-k)b+(a-k)(j-k)+(a+b-k)+(a-k-1)q(S,T)+2}{a+b}\\&= j-k+1+\frac{-k(j-k+1)+(a-k-1)q(S,T)+2}{a+b} > j-k+1,\end{aligned}$$

which is true for $(k,j) \in \{(1,0),(2,0)\}$ and $a \geq 4$. Since $\delta(G)-n+|T|$ is an integer, we obtain $\delta(G)-n+|T| \geq j-k+2$, which satisfies the condition on $\delta(G)$ when $i=2$ in Claim 2. Thus we have

$$a(3-k)-\epsilon b \geq (a-k)(j-k)+2(a+b-k)+(a-k-1)q(S,T)+2. \tag{10}$$

When $k=1$ and $j=0$, the inequality (10) becomes $a-(\epsilon+2)b \geq (a-2)q(S,T)+2$ which is a contradiction since $a-(\epsilon+2)b < 0$ and $(a-2)q(S,T)+2 > 0$. Similarly, we get a contradiction when $k=2$ and $j=0$, which completes the proof. □

4. Concluding remarks

In this section, we provide some questions and a conjecture. By finding counterexamples to Conjecture 1.2, we investigated alternative sharp conditions replacing the conditions in the conjecture and proved Theorem 1.3 in this paper. As a consequence, we could replace $\kappa'(G) \geq 2$ by $\kappa(G) \geq a$ and $\sigma_2(G) \geq \frac{2an}{a+b}$ by $\delta(G) \geq \frac{an}{a+b}$ and proved that these conditions are sharp in a sense that we cannot replace $\kappa(G) \geq a$ by $\kappa(G) \geq a-1$ or $\delta(G) \geq \frac{an}{a+b}$ by $\delta(G) \geq \frac{an}{a+b}-1$. However, we do not know whether we can replace the condition $\kappa(G) \geq a$ by $\kappa'(G) \geq a$ or $\delta(G) \geq \frac{an}{a+b}$ by $\sigma_2(G) \geq \frac{2an}{a+b}$, which for both cases, improves Theorem 1.3. Therefore, it is natural to ask the following questions.

Question 4.1. If we replace "$\kappa(G)$" in Theorem 1.3 by "$\kappa'(G)$", then do we have the same conclusion?

Question 4.2. If we replace "$\delta(G) \geq \frac{an}{a+b}$" in Theorem 1.3 by "$\sigma_2(G) \geq \frac{2an}{a+b}$", then do we have the same conclusion?

We might be also interested in some sufficient conditions for a certain eigenvalue in a certain graph G to have an (even or odd) $[a, b]$-factor, where a graph is *odd* if every vertex has odd degree. If G has an $[a, b]$-factor, then we have $\lambda_1(G) \geq a$ since $\lambda_1(G) \geq \delta(G)$, where $\lambda_1(G)$ is the largest eigenvalue of G. Is there a sufficient condition for $\lambda_1(G)$ in a graph G to have an $[a, b]$-factor? If we restrict our attention to a complete bipartite graph, which looks the simplest case, then it is easy to get a sufficient condition for the largest eigenvalue.

Observation 4.3. *Let G be the complete bipartite graph $K_{x,n-x}$ such that $n \geq 2x > 0$. Then G has an $[a, b]$-factor if and only if*

$$\lambda_1(G) \geq \begin{cases} \sqrt{a(n-a)} & \text{if } n < a+b, \\ \frac{\sqrt{ab}}{a+b}n & \text{if } n \geq a+b. \end{cases}$$

Proof. G has an $[a, b]$-factor F if and only if

$$x \geq a \text{ and } (n-x-b)x \leq (n-x)(x-a)(\Leftrightarrow x \geq \frac{an}{a+b})$$

since $\delta(F) \geq a$ and $\Delta(F) \leq b$.

Thus we have the desired result with $\lambda_1(G) = \sqrt{x(n-x)}$. □

Among n-vertex graphs G without an $[a, b]$-factor, we guess that the n-vertex graph $H_{n,a}$ obtained from one vertex and a copy of K_{n-1} by adding $a-1$ edges between them has the largest eigenvalue. Note that there are $n-a$ vertices with degree $n-2$, $a-1$ vertices with degree $n-1$, and 1 vertex with degree $a-1$ in the graph $H_{n,a}$. Thus $H_{n,a}$ cannot have an $[a, b]$-factor.

Conjecture 4.4. *Let an be an even integer at least 2, where $n \geq a+1$, and let $\rho(n, a)$ be the largest eigenvalue of $H_{n,a}$. If G is an n-vertex graph with $\lambda_1(G) > \rho(n, a)$, then G has an $[a, b]$-factor.*

We mention that $\lambda_1(H_{n,a})$ equals the largest root of $x^3 - (n-3)x^2 - (a + n - 3)x - a^2 + (a-1)n + 1 = 0$ without giving a reason in detail.

Recently, the third author with Kim, Park, and Ree [6] proved a sharp lower bound for the third largest eigenvalue in an n-vertex r-regular graph G to guarantee the existence of an odd $[1, b]$-factor improving the bound in the paper [13]. Also, the third author [15] found a sharp lower bound for the third largest eigenvalue in an n-vertex r-regular graph G to have an even or odd $[a, b]$-factor. When $a = b$, his result [15] implies the result of Gu [3] extending the result of Bollobás, Saito, and Wormald [1] and the ones of Lu [11, 12].

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