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SYMMETRY AND UNIQUENESS OF EMBEDDED MINIMAL HYPERSURFACES IN $\mathbb{R}^{n+1}$

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Abstract. In this paper, we prove some rigidity results about embedded minimal hypersurface $M \subset \mathbb{R}^{n+1}$ with compact ∂M that has one end which is regular at infinity. We first show that if $M \subset \mathbb{R}^{n+1}$ meets a hyperplane in a constant angle $\geq \pi/2$, then M is part of an n-dimensional catenoid. We show that if M meets a sphere in a constant angle and ∂M lies in a hemisphere determined by the hyperplane through the center of the sphere and perpendicular to the limit normal vector n_M of the end, then M is part of either a hyperplane or an n-dimensional catenoid.

We also show that if M is tangent to a C^2 convex hypersurface S, which is symmetric about a hyperplane P and n_M is parallel to P, then M is also symmetric about P. In special, if S is rotationally symmetric about the x_{n+1}-axis and $n_M = e_{n+1}$, then M is also rotationally symmetric about the x_{n+1}-axis.

1. Introduction

In [7], Schoen defined the notion of an end $\mathcal{E}$ of a minimal hypersurface $M \subset \mathbb{R}^{n+1}$ being *regular at infinity*, and showed that a complete minimal immersion $M \subset \mathbb{R}^{n+1}$ with two ends, which are regular at infinity, is either an n-dimensional catenoid or a pair of hyperplanes [7]. In $\mathbb{R}^3$, Osserman showed that an end of a complete minimal surface is regular at infinity if and only if the end has finite total curvature and is embedded [5].

In [1], Choe used the Weierstrass representation formula for minimal surfaces in $\mathbb{R}^3$ and the fact that the Gauss map of a minimal surface in $\mathbb{R}^3$ is meromorphic to show that a minimal surface meeting a plane in a constant angle can be reflected across the plane. In special, Choe showed that if a complete minimal surface has finite total curvature and one end of the surface meets a plane in a constant angle, then the minimal surface is a catenoid. In [6], the authors showed that a minimal hypersurface in $\mathbb{R}^{n+1}$ meeting a sphere in a constant

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angle and staying in a half space, determined by a hyperplane passing through the center of the sphere, is part of an n-dimensional catenoid or a hyperplane.

We generalize the above results using a variation of the Alexandrov's reflection argument based on the spherical reflection developed in [6]. Actually, we prove some rigidity and symmetry results about embedded minimal hypersurface $M \subset \mathbb{R}^{n+1}$ with one end, which is regular at infinity, meeting a hyperplane or a sphere in a constant angle. Throughout the paper, we assume that ∂M is compact. First we show that if $M \subset \mathbb{R}^{n+1}$ meets a hyperplane Π in a constant angle $\gamma \geq \pi/2$, then M is part of an n-dimensional catenoid. (See §2 for the choice of γ.) Next we show that if M meets a sphere in a constant angle and ∂M lies in a hemisphere determined by the hyperplane perpendicular to n_M (the limit normal vector of the end), then M is part of either a hyperplane or an n-dimensional catenoid.

We also show that if $M \subset \mathbb{R}^{n+1}$ is tangent to a C^2 convex hypersurface S, which is symmetric about a hyperplane P and n_M is parallel to P, then M is also symmetric about P. If S is rotationally symmetric about the x_{n+1}-axis, then M is symmetric about each hyperplane containing x_{n+1}-axis and parallel to n_M. Moreover, if $n_M = e_{n+1}$, then M is rotationally symmetric.

2. Complete minimal hypersurfaces meeting a hyperplane or a sphere in constant angle

An end $\mathcal{E}$ of a minimal hypersurface in $\mathbb{R}^{n+1}$ is *regular at infinity* if i) after a suitable rotation, $\mathcal{E}$ is the graph of a function u having bounded slope on the exterior of some bounded domain in $\Pi = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$, and ii) for the coordinates $x = (x_1, \ldots, x_n)$ on Π, u satisfies the following asymptotic behavior for $|x|$ large: if $n = 2$

$$u(x) = a \log |x| + b + \frac{c_1 x_1 + c_2 x_2}{|x^2|} + O\left(|x|^{-2}\right), \tag{1}$$

if $n \geq 3$

$$u(x) = b + a|x|^{2-n} + \sum_{j=1}^{n} c_j x_j |x|^{-n} + O\left(|x|^{-n}\right) \tag{2}$$

for constants a, b, c_j [7]. If $n = 2$ and $a \neq 0$, then $\mathcal{E}$ is asymptotic to an end of a catenoid, and is called *catenoidal*. If $n = 2$ and $a = 0$ or $n \geq 3$, then $\mathcal{E}$ is asymptotic to a (hyper)plane. If $n = 2$, then $\mathcal{E}$ is called *planar*. The limit unit normal vector n_M of the end is the limit of the unit normal vector of the end as $|x| \to \infty$.

Let

$$\Pi_t = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = t\},$$

$$\Pi_{[a,b]} = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : a \leq x_{n+1} \leq b\}.$$

Let $\Pi_b^+ = \Pi_{[b,\infty)}$. Hence $\Pi = \Pi_0$, $\Pi^+ = \Pi_0^+$ and $\Pi^- = \Pi_0^-$. For a nonzero vector u, let

$$\Pi_u = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : (x_1, \ldots, x_{n+1}) \cdot u = 0\}.$$

We use similar notations for Π_u. Let $e_{n+1} = (0, \ldots, 0, 1)$.

In the hyperplane case, we assume that M meets Π_1. In the sphere case, we assume that M meets the unit sphere $S^n(O, 1)$. We note that M divides the half space Π_1^+ (in the hyperplane case) or $\mathbb{R}^{n+1} \setminus B^{n+1}(O, 1)$ (in the sphere case) into two parts. Let U_M be the component of $\Pi_1^+ \setminus M$ or $\mathbb{R}^{n+1} \setminus B^{n+1}(O, 1)$ which stays above M for $x \in \Pi$ with large $|x|$. The unit normal vector ν on M is chosen to point into U_M, and n_M is the limit of ν. Note that $n_M \cdot e_{n+1} \geq 0$. The contact angle γ between M and Π_1 or $S^n(O, 1)$ is measured between the outward conormals η along ∂M and η_{U_M} of $\Pi_1 \cap \overline{U_M}$ or $S^n(O, 1) \cap \overline{U_M}$ along the boundary.

We note that, outside some compact set, M is a graph over Π_{n_M}. If M is not flat and has an asymptotic hyperplane $\Pi^l_{n_M}$ and meets a sphere or a convex hypersurface in a constant contact angle, then ∂M lies on one side of $\Pi^l_{n_M}$. Otherwise, one may use the maximum principle for the 2nd order elliptic pde [3] to see that $\eta \cdot n_M > 0$ and $\eta \cdot n_M < 0$ at the points p_1 and p_2 where $x \cdot n_M$ attains maximum and minimum on ∂M respectively. Hence, $\gamma - \pi/2$ must have different signs at p_1 and p_2. In the following, we assume that either $a > 0$ in (1) or (2), or ∂M lies below $\Pi^l_{n_M}$.

For completeness, we prove the following.

Lemma 2.1. *Let M be an embedded minimal hypersurface in $\mathbb{R}^{n+1}$ $(n \geq 2)$ having one regular at infinity end and meeting Π_1 in a constant angle with compact ∂M. Then the limit unit normal n_M of M is perpendicular to Π_1.*

Proof. The flux of M along ∂M is

$$\mathrm{Flux}(\partial M) = \int_{\partial M} \eta.$$

If $n = 2$, then $\mathrm{Flux}(\partial M) = c\, n_M$. Moreover, $c \neq 0$ if the end is catenoidal, and $c = 0$ if the end is planar [2]. Similarly, when $n \geq 3$, one may use (2) to see that $\mathrm{Flux}(\partial M) = c\, n_M$ with $c \neq 0$ if $a \neq 0$ and $c = 0$ if $a = 0$. Clearly, n_M is perpendicular to the asymptotic hyperplane of the end.

Along ∂M, we have $\eta = -(\sin\gamma) e_{n+1} + (\cos\gamma)\eta_{U_M}$. Since γ is constant, we use the divergence theorem to get

$$\int_{\partial M} \eta = \int_{\partial M} (-(\sin\gamma) e_{n+1} + (\cos\gamma)\eta_{U_M}) = -(\sin\gamma)\mathrm{Vol}(\partial M) e_{n+1}.$$

If $n = 2$ and the end is planar or $n > 2$ and $a = 0$ in (2), then we have $\gamma = 0$. Hence M is a hyperplane and consequently $M = \Pi_1$, which contradicts that ∂M is compact. Otherwise, we have n_M and e_{n+1} are parallel. Hence, $n_M \perp \Pi_1$. □

The spherical reflection

$$SR_1 : \mathbb{R}^{n+1} \setminus \{O\} \to \mathbb{R}^{n+1}$$

about the unit sphere $S^n(O,1)$ is defined by

$$SR_1(x) = \frac{x}{|x|^2}.$$

It is easy to see that for an end $\mathcal{E}$ of (1) or (2), $SR_1(\mathcal{E}) \cup \{O\}$ is C^1 at O. From [4], we recall the following.

Lemma 2.2. *For a smooth hypersurface $M \subset \mathbb{R}^{n+1}$ and $x \in M$, the mean curvature $\tilde{H}$ of $SR_1(M)$ at $SR_1(x)$ is given by*

$$\tilde{H}(SR_1(x)) = H(x)|x|^2 + 2x \cdot \nu, \tag{3}$$

where $H(x)$ is the mean curvature of M at x with respect to the unit normal vector ν of M.

For a unit vector $v \in \Pi_{n_M}$, let V_θ be the hyperplane passing through O and containing the $(n-1)$-dimensional plane perpendicular to v in Π_{n_M} and making angle θ with v. (θ is chosen in such a way that V_θ is above v for small θ.) Let R_θ be the reflection about V_θ. For a subset $D \subset \mathbb{R}^{n+1}$ and $0 \leq \theta < \pi/2$, let D^-_θ be the subset of D on or below V_θ, and let D^+_θ be the subset of D on or above V_θ.

Theorem 2.3. *Let M be an embedded minimal hypersurface in $\mathbb{R}^{n+1}$ having one end, which is regular at infinity. Suppose that $M \subset \Pi^+_1$ meets the hyperplane Π_1 in a constant angle $\gamma > \pi/2$ along ∂M, which is compact. Then M is part of an n-dimensional catenoid.*

Proof. From Lemma 2.1, we may assume that $M \subset \Pi^+_{[1,\infty)}$. We first show that M is a graph over Π_1. Fix $t_1 > 0$ so that $M_{[t_1/2,\infty)} = M \cap \Pi_{[t_1/2,\infty)}$ is a graph over Π in (1) or (2), and the projection $p : M \to \Pi$ is one-to-one on $\{(x, x_{n+1}) \in M : x_{n+1} \geq t_1/2\}$.

Suppose that $M_{[1,t_1]}$ is not a graph over Π. Decreasing t from t_1 to 1, there exists t_0 such that the reflection $M^{R|t_0}_{[t_0,t_1]}$ of $M_{[t_0,t_1]}$ about Π_{t_0} meets $M_{[1,t_0]}$ for the first time i) tangentially at an interior point of $M_{[1,t_0]}$, ii) tangentially at a boundary point of $M_{[1,t_0]}$ or iii) transversally at a point in ∂M. If i) or ii) holds, then $M^{R|t_0}_{[t_0,t_1]}$ and $M_{[1,t_0]}$ coincide by the comparison principles of the 2nd order elliptic pdes. Hence Π_{t_0} is a symmetry plane of M. Since $M^{R|t_0}_{[t_0,t_1]}$ is a graph over Π, we should have $\gamma < \pi/2$, which is a contradiction. If iii) holds, then x_{n+1} of $M^{R|t_0}_{[t_0,t_1]}$ has an interior local minimum, which is impossible.

Using a parallel translation, we may assume that $N = (0, \ldots, 0, 1) \notin \partial M$. We note that $\tilde{M} = SR_1(M) \cup \{O\}$ is C^1 at O, $SR_1(M) \subset B^{n+1}(N', 1/2)$. Using a parallel translation and a proper homothety, we may assume that $SR_1(\partial M)$ lies in the upper hemisphere $\tilde{S}^+$ of $S^n(N', 1/2)$. Let

$$\hat{M} = SR_1(\partial U_M) \cup \{O\}.$$

For each unit vector $v \in \Pi$, we show that $V_{\pi/2}$ is symmetry hyperplane of M. Let $W_t \subset \Pi_1$ be the $(n-1)$-plane perpendicular to v where $t = v \cdot y$ for $y \in W_t$. For big $t > 0$, $W_t \cap \partial M = \emptyset$. Decreasing t, there exists t_0 such that the reflection of $(\partial M)^+_{t_0} = \partial M \cap \left(\bigcup_{t \geq t_0} W_t\right)$ about W_{t_0} is tangent to $(\partial M)^-_{t_0} = \partial M \cap \left(\bigcup_{t \leq t_0} W_t\right)$ for the first time at some point, say x_0, in $(\partial M)^-_{t_0}$. Using a parallel translation and a homothety centered at N, we may assume that $t_0 = 0$ and $\partial M \subset B^n(N, 1) \subset \Pi_1$. We have $V_0 \cap \hat{M} = \{O\}$, $W_0 \subset V_\theta$ and

$$(SR_1)^{-1} \circ R_\theta \circ SR_1 = R_\theta. \tag{4}$$

Since M is a graph over Π_1, we have $R_\theta((\partial M)^-_\theta) \cap M^+_\theta = \emptyset$ for $\theta < \pi/2$, and $R_\theta((\partial \tilde{M})^-_\theta) \cap \tilde{M}^+_\theta = \{O\}$ for $\theta < \pi/2$. Increasing θ from 0 to $\pi/2$, there exists θ_v such that $R_{\theta_v}(\hat{M}^-_{\theta_v})$ and $\hat{M}^+_{\theta_v}$ meets for the first time i) tangentially at $x \in \mathrm{int}(\hat{M}^+_{\theta_v})$ or ii) at $x \in \partial\left(\hat{M}^+_{\theta_v}\right)$ tangentially. Let $x_v \in \hat{M}^-_{\theta_v}$ for which $R_{\theta_v}(x_v) = x$.

We show that $\theta_v = \pi/2$. Otherwise, $x \in \tilde{M}^+_{\theta_v}$ and $x \neq O$. Suppose that i) holds. Since $R_\theta((\partial \tilde{M})^-_\theta) \cap \tilde{M}^+_\theta = \emptyset$ for $\theta < \pi/2$, we have $x_v \in \mathrm{int}(\tilde{M}^-_{\theta_v})$, and $R_{\theta_v}(\tilde{M}^-_{\theta_v})$ and $\tilde{M}^+_{\theta_v}$ are tangent at x. Applying $(SR_1)^{-1}$, we see that $R_{\theta_v}(M^-_{\theta_v})$ and $M^+_{\theta_v}$ are tangent at $SR_1^{-1}(x)$ and $R_{\theta_v}(M^-_{\theta_v})$ lies on one side of $M^+_{\theta_v}$ near $SR_1^{-1}(x)$. Since $R_{\theta_v}(M^-_{\theta_v})$ and $M^+_{\theta_v}$ are both minimal, $R_{\theta_v}(M^-_{\theta_v})$ and $M^+_{\theta_v}$ coincide by the comparison principles for the 2nd order elliptic pdes. Hence V_{θ_v} is a symmetry hyperplane of M, which contradicts $n_M = e_{n+1}$. If ii) holds, then $x \in \mathrm{int}(\tilde{M}) \cap V_\theta$. It is easy to see that V_{θ_v} is also a symmetry hyperplane of M as above, which is a contradiction.

Hence $\theta_v = \pi/2$. By the choice of x_0 and the fact that γ is constant, $R_{\pi/2}\left(\tilde{M}^-_{\pi/2}\right)$ and $\tilde{M}^+_{\pi/2}$ are tangent at $SR_1(x_0) \in \tilde{M}^+_{\pi/2} \setminus \{O\}$. Note that $SR_1(x_0)$ might be a corner point on $\tilde{M}^+_{\pi/2} \cap V_{\pi/2}$. One may apply the comparison principles for the 2nd order elliptic pdes at a boundary point or at a corner point [4] to see that $R_{\pi/2}\left(M^-_{\pi/2}\right)$ and $M^+_{\pi/2}$ coincide as above. Hence $V_{\pi/2}$ is a symmetry hyperplane of M. Since $v \in \Pi$ is arbitrary, M is rotationally symmetric. □

In [6], the authors showed that an embedded minimal hypersurface $M \subset \mathbb{R}^{n+1}$ with one regular at infinity end that meets $S^n(O, 1)$ in a constant angle is a hyperplane or an n-dimensional catenoid if M stays in a half space determined by a hyperplane passing through O.

The following lemma is a generalization of the result in [6]. In the following lemma, the end of M is a graph of a function u on the exterior of some compact set in Π_{n_M}. We assume that ∂M lies in the upper hemisphere and either $n = 2$ and $a > 0$ in (1) or ∂M lies below the asymptotic hyperplane $\Pi^l_{n_M}$ of the end. Hence M may intersect Π^- while ∂M stays in Π^+.

Lemma 2.4. *Let M an embedded minimal hypersurface in $\mathbb{R}^{n+1}$ with one regular at infinity end, which is a graph of a function u as in* (1) *on the exterior of some compact set in Π_{n_M} and $n_M \cdot e_{n+1} > 0$. Suppose that M meets $S^n(O,1)$ in a constant contact angle and lies outside of $S^n(O,1)$. If ∂M lies in the upper hemisphere and either $n = 2$ and $a > 0$ in* (1) *or ∂M lies below the asymptotic hyperplane $\Pi^l_{n_M}$ of the end, then M is either a hyperplane or an n-dimensional catenoid.*

Proof. We may assume that M is not flat. There is $\delta > 0$ such that $\partial M \subset \Pi^+_\delta$. Clearly, $M^-_\delta = M \cap \Pi^-_\delta$ is a graph over Π. Note that $\tilde{M} = SR_1(M) \cup \{O\}$ is C^1 with $T_O\tilde{M} = \Pi_{n_M}$. Clearly, $\tilde{M}^+$ lies above the reflection $R(\tilde{M}^-)$ of $\tilde{M}^-$ about Π, and is transversal to $R(\tilde{M}^-)$ along the boundary.

We fix a unit vector $v \in \Pi$. Let U_θ be the hyperplane containing the $(n-1)$-plane perpendicular to v in Π and making angle θ with v. There exists θ^v such that $U_{\theta^v} \perp \Pi_{n_M}$. Increasing θ from 0 to θ^v, there is θ_v for which $R_{\theta_v}(\tilde{M}^-_{\theta_v})$ and $\tilde{M}^+_{\theta_v}$ meet tangentially for the first time either at an interior point $\tilde{x}$ of $\tilde{M}^+_{\theta_v}$ or at a point on $\partial\tilde{M}^+_{\theta_v}$. If $\theta_v = \theta^v$ and $\tilde{x} = O$, then we repeat the above process with $-v$ instead of v. Then we get a new θ_{-v} and $\tilde{x}$ such that either $\theta_{-v} \neq \theta^{-v}$ or $\theta_{-v} = \theta^{-v}$ and $\tilde{x} \neq O$. Since $R_{\theta_v}(\tilde{M}^-_{\theta_v})$ and $\tilde{M}^+_{\theta_v}$ are tangent at $\tilde{x} \neq O$, $SR_1^{-1}(R_{\theta_v}(\tilde{M}^-_{\theta_v}))$ and $SR_1^{-1}(\tilde{M}^+_{\theta_v})$ are tangent at $SR_1^{-1}(\tilde{x})$. We see that $SR_1^{-1}(R_{\theta_v}(\tilde{M}^-_{\theta_v}))$ and $SR_1^{-1}(\tilde{M}^+_{\theta_v})$ coincide and U_{θ_v} is a symmetry hyperplane of M. Moreover, we should have $\theta_v = \theta^v$. Since v is arbitrary, M is rotationally symmetric about the line parallel to n_M. □

In the following theorem, ∂M lies in the lower hemisphere, while M lies in the upper hemisphere for $|x|$ large. The reflection $R(\tilde{M}^-)$ of $\tilde{M}^-$ about Π may not be a graph over Π.

Theorem 2.5. *Let $M \subset \mathbb{R}^{n+1}$ be an embedded minimal hypersurface having one end, which is regular at infinity with $n_M = e_{n+1}$. Suppose that M lies outside of $S^n(O,1)$ and meets $S^n(O,1)$ in a constant angle γ along ∂M, which is compact and lies in the lower hemisphere $S^n(O,1) \cap \Pi^-$. Then M is part of an n-dimensional catenoid or part of a hyperplane.*

Proof. Since M is minimal and M lies in Π^+_{-1} for $|x|$ large, $M \cap \Pi_{-1} = \emptyset$ unless $M = \Pi_{-1}$. We assume that M is not flat. There exists $0 < \delta < 1$ such that $M \subset \Pi^+_{-1+\delta}$ and $\partial M \cap \Pi_{-1+\delta} \neq \emptyset$. It follows that $\gamma < \pi/2$. Fix $t_1 > 0$ so that $M^+_{t_1/2}$ is a graph over Π, and the projection $p : M \to \Pi$ is one-to-one on $\{(x, x_{n+1}) \in M : x_{n+1} \geq t_1/2\}$. We may assume that M_{t_1} is connected.

Step I) We show that M is a graph over Π. Suppose that $M_{(-1,t_1]}$ is not a graph over Π. Decreasing t from t_1 to -1, there exists t_0 such that the reflection

$M^{R|t_0}_{[t_0,t_1]}$ of $M_{[t_0,t_1]}$ about Π_{t_0} meets $M_{(-1,t_0]}$ for the first time i) tangentially at an interior point of $M_{(-1,t_0]}$, ii) tangentially at a boundary point of $M_{(-1,t_0]}$ or iii) transversally at a point $x_f \in \partial M$.

If i) or ii) holds, then $M^{R|t_0}_{[t_0,t_1]}$ and $M_{(-1,t_0]}$ coincide by the comparison principles for the 2nd order elliptic pdes. Since $\partial M \subset \Pi^-$ and M lies outside of $S^n(O,1)$, both $M_{(-1,t_0]}$ and $M^{R|t_0}_{[t_0,t_1]}$ cannot be a graph over Π. This is a contradiction.

Now suppose that iii) holds. Then $M^{R|t_0}_{[t_0,t_1]} \cap B^{n+1}(O,1) \neq \emptyset$. Clearly, $M_{[t_0,\infty]}$ is a graph over Π and $M_{[t_0,t_1]}$ is connected. We first show that the projection $p : M_{[0,t_1]} \to B^n(O,1) \subset \Pi$ is onto. Otherwise, there is $q \in B^n(O,1) \backslash p(M_{[0,t_1]})$. Let $M^q_{t_0}$ be the component of $M \cap \Pi_{t_0}$ containing (q,t_0) inside and no other component of $M \cap \Pi_{t_0}$. Let $M^q_{(-1,t_0]}$ be the component of $M_{(-1,t_0]}$ having $M^q_{t_0}$ as boundary. Since $M_{[t_0,t_1]}$ is a graph over Π and $q \in B^n(O,1) \setminus p(M_{[0,t_1]})$ and $M^{R|t_0}_{[t_0,t_1]} \cap B^{n+1}(O,1) \neq \emptyset$, we have $p(M^q_{t_0}) \cap B^n(O,1) \neq \emptyset$. Moreover, $M^q_{(-1,t_0]}$ lies between $M^{R|t_0}_{[t_0,t_1]}$ and $S^n(O,1)$. Since ∂M lies in the lower hemisphere and $M^q_{t_0}$ surrounds (q,t_0) in Π_{t_0} and $p^{-1}(q,0) \cap M = \emptyset$, it follows that $M^q_{(-1,t_0]}$ cannot exist. Hence $p : M_{[0,t_1]} \to B^n(O,1)$ is onto.

Using the Sard's theorem, we assume that $M_0 = M \cap \Pi$ is regular. (One may use Π_ϵ instead of Π for small $\epsilon > 0$.) If M_0 is connected and encloses $B^n(O,1) \subset \Pi$, then, for some point $a \in B^n(O,1)$, $p^{-1}(a) \cap (M \cap \Pi^+)$ should contain at least 2 points. In this case, iii) cannot happen. Hence M_0 either consists of at least 2 components or is connected and encloses a region disjoint from $B^n(O,1) \subset \Pi$. It follows that $T = \{x \in M \cap \Pi : \overline{Ox} \cap M \text{ contains at least 2 points}\}$ is not empty.

Let $\mathrm{dist}(O,x)$, for $x \in T$, attains maximum at P and let $w = \overrightarrow{OP}/|\overrightarrow{OP}|$. For small $\epsilon > 0$, let $w_\epsilon = w - \epsilon e_{n+1}$. We apply the Alexandrov's reflection argument to M using the hyperplanes $\Pi^s_{w_\epsilon}$. Let R_s be the reflection about $\Pi^s_{w_\epsilon}$. For large s, $\Pi^s_{w_\epsilon}$ is disjoint from $M_{(-1,t_1]}$, and $R_s(M_{w_\epsilon,s}^{\ +})$, where $M_{w_\epsilon,s}^{\ +} = \{x \in M : x \cdot w_\epsilon \geq s\}$, stays above $M_{w_\epsilon,s}^{\ -} = \{x \in M : x \cdot w_\epsilon \leq s\}$. Decreasing s, there exists $s_0 > 1$ such that $R_s\left(M_{w_\epsilon,s_0}^{\ +}\right)$ meets $M_{w_\epsilon,s_0}^{\ -}$ for the first time either i) tangentially at a point of $M_{w_\epsilon,s_0}^{\ -}$ or ii) transversally at $x_{w_\epsilon} \in \partial M$. It is easy to see that i) is impossible for sufficiently small $\epsilon > 0$.

We show that ii) is impossible for sufficiently small $\epsilon > 0$. Since $P \in T$, $\overline{OP} \cap M$ contains a point $P' \in \Pi^{1+\delta}_w$ for some small $\delta > 0$. It follows that, for sufficiently small $\epsilon > 0$, the line $P + tw_\epsilon$, $t \in \mathbb{R}$, intersects M at a point $\tilde{P} \in \Pi^{1+\tilde{\delta}}_{w_\epsilon}$ close to P', for some $\tilde{\delta} > 0$. Let $B = \frac{P+\tilde{P}}{2}$. Then

$$\mathrm{dist}\left(B, \Pi^0_{w_\epsilon}\right) < s_0 \text{ and } \mathrm{dist}(P, \Pi^{s_0}_{w_\epsilon}) \leq \mathrm{dist}(P,B).$$

On the other hand, for sufficiently small $\epsilon > 0$

$$\mathrm{dist}(P,B) \leq \mathrm{dist}(\tilde{P}, \Pi^{s_0}_{w_\epsilon}) \leq \mathrm{dist}(x_{w_\epsilon}, \Pi^{s_0}_{w_\epsilon}).$$

Hence

$$s_0 - R_{s_0}(P)\cdot w_\epsilon/|w_\epsilon| \le s_0 - x_{w_\epsilon}\cdot w_\epsilon/|w_\epsilon|. \tag{5}$$

Let $w^\perp$ be a vector with $w^\perp \perp w_\epsilon$, $w^\perp \perp (\Pi\cap\Pi^{s_0}_{w_\epsilon})$ and $w^\perp\cdot e_{n+1} > 0$. Suppose that $\Pi^t_{w^\perp}$ passes through P. From the choice of P, we see that the function $s_0 - x\cdot w_\epsilon/|w_\epsilon|$ on $R_{s_0}\left(M^+_{w_\epsilon,s_0}\right)\cap\Pi^t_{w^\perp}$ attains maximum at P. From (5), it follows that $s_0 - x\cdot w_\epsilon/|w_\epsilon|$ on $R_{s_0}\left(M^+_{w_\epsilon,s_0}\right)$ attains an interior local maximum, which is a contradiction. Hence iii) cannot happen, which completes the proof of Step I).

Step II) We show that M is rotationally symmetric. The proof is similar to the proof of Lemma 2.4. $\tilde{M} = SR_1(M)\cup\{O\}$ is C^1 with $T_O\tilde{M} = \Pi$, and meets $S^n(O,1)$ in constant angle γ. Since M is a graph over Π, $R_0(\tilde{M}^+_0)$ lies above $\tilde{M}^-_0$, and is transversal to $\tilde{M}^-_0$ along $R_0(\tilde{M}^+_0)\cap\tilde{M}^-_0$. Otherwise, there is either i) a point $r\in(R_0(\tilde{M}^+_0)\cap\tilde{M}^-_0)\setminus\Pi$ or ii) $r\in\tilde{M}^-_0)\cap\Pi$ where $R_0(\tilde{M}^+_0)$ and $\tilde{M}^-_0$ are tangent. In both cases, M cannot be a graph over Π.

Fix a unit vector $v\in\Pi$. Increasing θ from 0 to $\pi/2$, there is θ_v for which $R_{\theta_v}(\tilde{M}^+_{\theta_v})$ and $\tilde{M}^-_{\theta_v}$ meet tangentially for the first time either at an interior point $\tilde{x}$ of $\tilde{M}^-_{\theta_v}$ or at a point on $\partial\tilde{M}^-_{\theta_v}$. If $\theta_v = \pi/2$ and $\tilde{x} = O$, then we repeat the above process with $-v$ instead of v. Then we get a new θ_{-v} and $\tilde{x}$ such that either $\theta_{-v}\ne -\pi/2$ or $\theta_{-v} = \pi/2$ and $\tilde{x}\ne O$. As in the proof of Lemma 2.4, we see that V_{θ_v} is a symmetry hyperplane of M, and $\theta_v = \pi/2$. Since v is arbitrary, M is rotationally symmetric about the x_{n+1}-axis. □

If ∂M intersects both Π^+ and Π^-, then the projection p might not be onto $B^n(O,1)\subset\Pi$. It would be interesting to prove Theorem 2.5 without conditions on ∂M.

3. Minimal hypersurfaces tangent to a convex rotational hypersurface

We assume that either $a > 0$ in (1) or the asymptotic hyperplane $\Pi^l_{n_M}$ stays above ∂M as in §2, and $n_M\cdot e_{n+1}\ge 0$.

Theorem 3.1. *Let $S\subset\mathbb{R}^{n+1}$ be a convex C^2 hypersurface symmetric about a hyperplane P. Let $M\subset\mathbb{R}^{n+1}$ be an embedded minimal hypersurface with one end which is regular at infinity. Assume ∂M is compact and M is tangent to S. If n_M is parallel to P, then M is symmetric about P.*

Since we assume that ∂M is compact, S can be assumed to be compact. We fix the unit normal vector $\nu_S = \vec{H}_S/|\vec{H}_S|$ on S, where $\vec{H}_S$ is the mean curvature vector of S. Let B be the convex body bounded by S. Then M divides $\mathbb{R}^{n+1}\setminus B$ into two parts. As in §2, let U_M be the component of $\mathbb{R}^{n+1}\setminus B$ that stays above M.

Proof of Theorem 3.1. There is t_0 such that $M\subset\Pi^+_{n_M,t_0}$ and $M\cap\Pi_{n_M,t_0}\subset\partial M$. It follows that that $\nu = \nu_S$ along ∂M.

By translating in the direction of n_M, we may assume that $S \cup M$ lies in $\Pi_{n_M,1}^+$ and outside of $B^{n+1}(O,1)$. Let $\hat{M} = SR_1(M \cup (S \setminus \partial U_M)) \cup \{O\}$, which is C^1 and complete. Since M and S are embedded respectively, $\hat{M}$ is also embedded except for the points where $SR_1(M)$ and $SR_1(S)$ are tangent. We note that $\hat{M}$ is Alexandrov embedded, that is, bounds an embedded region in $\mathbb{R}^{n+1}$.

For each unit vector $v \in \Pi_{n_M}$ perpendicular to P, we show that $V_{\pi/2}$ is a symmetry hyperplane of M. Note that $V_0 = \Pi_{n_M}$ and $V_0 \cap \hat{M} = \{O\}$. Increasing θ from 0 to $\pi/2$, there exists $\theta_v \leq \pi/2$ for which $R_{\theta_v}(\hat{M}_{\theta_v}^-)$ meets $\hat{M}_{\theta_v}^+$ tangentially for the first time at $x_f \in \hat{M}_{\theta_v}^+$.

First we show that $\theta_v = \pi/2$. Suppose that $\theta_v < \pi/2$. Since $SR_1(S)$ is symmetric about $V_{\pi/2}$, the point $x_v \in \hat{M}_{\theta_v}^-$ for which $R_{\theta_v}(x_v) = x_f$ is an interior point of $SR_1(M)$. The mean curvatures of $R_{\theta_v}(\hat{M}_{\theta_v}^-)$ and $\hat{M}_{\theta_v}^+$ satisfy

$$H_{\hat{M}_{\theta_v}^+}(x_f) \leq H_{R_{\theta_v}(\hat{M}_{\theta_v}^-)}(x_f).$$

We note that $\tilde{H}(x_f) = H_{\hat{M}_{\theta_v}^+}(x_f)$ and $\tilde{H}(x_v) = H_{R_{\theta_v}(\hat{M}_{\theta_v}^-)}(x_f)$.

Let $\check{x} = (SR_1)^{-1}(x_f)$ and $\check{x_v} = (SR_1)^{-1}(x_v)$. By (3),

$$\tilde{H}(x_f) = H(\check{x})|\check{x}|^2 + 2\check{x} \cdot \nu(\check{x}),$$

and

$$\tilde{H}(x_v) = 2\check{x_v} \cdot \nu'(\check{x_v}),$$

where $\nu' = \nu$ if $x_v \in SR_1(M)$ and $\nu' = \nu_S$ if $x_v \in SR_1(S)$. Since SR_1 is conformal and $R_{\theta_v}(\hat{M}_{\theta_v}^-)$ and $\hat{M}_{\theta_v}^+$ are tangent at x_f, we have $\check{x} \cdot \nu_{\hat{M}}(\check{x}) = \check{x_v} \cdot \nu'(\check{x_v})$. If $x_f \notin \tilde{S} \setminus \tilde{M}$, then $H(\check{x}) > 0$ by the choice of ν_S and

$$\tilde{H}(x_v) < \tilde{H}(x_f).$$

This is a contradiction. Hence $x_f \in SR_1(M)$. Note that $R_{\theta_v}(M_{\theta_v}^-)$ is tangent to $M_{\theta_v}^+$ at $\check{x_v}$ and lies on one side of $M_{\theta_v}^+$. Since $R_{\theta_v}(M_{\theta_v}^-)$ and $M_{\theta_v}^+$ are both minimal, they coincide. Hence V_{θ_v} is a symmetry hyperplane of M and $n_M \in V_{\theta_v} \neq V_{\pi/2}$, which is a contradiction.

Now we show that $V_{\frac{\pi}{2}}$ is a symmetry hyperplane of M. If $V_{\frac{\pi}{2}}$ is not a symmetry hyperplane of M, then we repeat the above argument with $-v$. We must have $\theta_{-v} < \pi/2$, which is a contradiction. Therefore $V_{\pi/2}$ is a symmetry hyperplane of M. □

Suppose that M satisfies the conditions of the above Theorem.

Corollary 3.2. *If S is rotationally symmetric about the x_{n+1}-axis, then M is symmetric about each hyperplane containing x_{n+1}-axis and parallel to n_M. If $n_M = e_{n+1}$, then M is either a hyperplane or an n-dimensional catenoid.*

Proof. If S is rotationally symmetric about the x_{n+1}-axis, then S is symmetric about each hyperplane containing the x_{n+1}-axis. Therefore M is symmetric about each hyperplane containing the x_{n+1}-axis and parallel to n_M.

If $n_M = e_{n+1}$, then M is symmetric about each hyperplane containing x_{n+1}-axis. Hence M is rotationally symmetric about the x_{n+1}-axis. □

References

[1] J. Choe, *On the analytic reflection of a minimal surface*, Pacific J. Math. **157** (1993), no. 1, 29–36. http://projecteuclid.org/euclid.pjm/1102634862

[2] Y. Fang, *Lectures on minimal surfaces in* $\mathbb{R}^3$, Proceedings of the Centre for Mathematics and its Applications, Australian National University, **35**, Australian National University, Centre for Mathematics and its Applications, Canberra, 1996.

[3] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[4] J. McCuan, *Symmetry via spherical reflection and spanning drops in a wedge*, Pacific J. Math. **180** (1997), no. 2, 291–323. https://doi.org/10.2140/pjm.1997.180.291

[5] R. Osserman, *A Survey of Minimal Surfaces*, second edition, Dover Publications, Inc., New York, 1986.

[6] S.-H. Park and J. Pyo, *Free boundary minimal hypersurfaces with spherical boundary*, Math. Nachr. **290** (2017), no. 5-6, 885–889. https://doi.org/10.1002/mana.201500399

[7] R. M. Schoen, *Uniqueness, symmetry, and embeddedness of minimal surfaces*, J. Differential Geom. **18** (1983), no. 4, 791–809 (1984). http://projecteuclid.org/euclid.jdg/1214438183

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