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ZERO MEAN CURVATURE SURFACES IN ISOTROPIC THREE-SPACE

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Abstract. We examine the theory of surfaces in the isotropic three-space, with emphases on the surfaces related to the zero mean curvature.

1. Introduction

We became interested in the geometry of the simply isotropic space $\mathbb{I}^3$ while trying to find a way to relate the geometry of the Lorentzian three space $\mathbb{L}^3$ and the geometry of the Euclidean three space $\mathbb{E}^3$. The Wick rotation is one such tool but we are not satisfied with it; we want to know if it is possible to transform the Euclidean geometry to the Lorentzian geometry in a continuous manner in the realm of the real differential geometry. In particular, we are interested in whether or not minimal surfaces in $\mathbb{E}^3$ and maximal surfaces in $\mathbb{L}^3$ can be transformed continuously to each other.

One approach is to consider the three dimensional subspaces of $\mathbb{L}^4$; by rotating and rescaling properly, one can transform both $\mathbb{E}^3$ and $\mathbb{L}^3$ to the lightlike subspace. Another approach is to consider $\mathbb{R}^3 \ni (x, y, z)$ equipped with the family of metrics $ds_\epsilon^2 := dx^2 + dy^2 + \epsilon dz^2$ for $\epsilon \in \mathbb{R}$. If $\epsilon = 1$ or -1, then we obtain $\mathbb{E}^3$ or $\mathbb{L}^3$, where the metric determines the geometry more or less completely.

When considering $\epsilon = 0$, it turns out that the isotropic three-space $\mathbb{I}^3$ is the proper geometry to look at. The study of $\mathbb{I}^3$ has been initiated by Strubecker in the 1930's, and gained some momentum recently. See for example the article by da Silva [5] and the references therein for a brief account of history and applications of $\mathbb{I}^3$. We just add a comment that essentially the Weierstrass

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representation formula for zero mean curvature (ZMC) surfaces in $\mathbb{I}^3$ has been derived by many people, for example Alías and B. Palmer [2] and, Ma, Wang and Wang [14], Pember [15], Sato [18], in different contexts.

Our goal in this article is to examine how much of the theory of ZMC surfaces in $\mathbb{E}^3$ and in $\mathbb{L}^3$ can be applied to the theory of the ZMC surfaces in $\mathbb{I}^3$. We present some known facts and some results of our own about totally umbilic surfaces, ZMC surfaces, constant mean curvature surfaces, and a special class of linear Weingarten surfaces, etc, in $\mathbb{I}^3$. We can see that much of the surface theory in $\mathbb{E}^3$ and $\mathbb{L}^3$ can be applied to the surfaces in $\mathbb{I}^3$ and that there is a great possibility for future research.

This article is organized as follows: In Section 2, we collect some known facts. In Section 3, we provide an elementary account of the relation between $\mathbb{I}^3$ and the lightlike subspaces of $\mathbb{L}^4$. In Section 4, we classify the totally umbilic surfaces in $\mathbb{I}^3$. In Section 5, we provide a geometric interpretation of the map g in the Weierstrass representation formula. In Section 6, we inspect some examples of ZMC surfaces. In Section 7, we take a look at constant mean curvature surfaces. In Section 8, we present the Björling representation formula for ZMC surfaces in $\mathbb{I}^3$. In Section 9, we study a special class of linear Weingarten surfaces in $\mathbb{I}^3$. In Section 10, we study the holomorphic representation formula for ZMC surfaces in the three-dimensional lightcone.

2. Preliminaries

2.1. Guiding principle

There is one simple principle which guided us in our study.

> If you see that $+1$ in an expression for an object in $\mathbb{E}^3$ changes to -1 in the expression for the corresponding object in $\mathbb{L}^3$, then change the $+1$ (or equally -1) to 0. That will be the expression for the corresponding object in $\mathbb{I}^3$.

This has been inspired by the Wick rotation which relates the geometry of $\mathbb{L}^3$ to the geometry of $\mathbb{E}^3$. For example, the mean curvature for the graph of f in $\mathbb{E}^3$ or in $\mathbb{L}^3$ is as follows:

$$H = \frac{(1 \pm f_y^2) f_{xx} \mp 2 f_x f_y f_{xy} + (1 \pm f_x^2) f_{yy}}{2\sqrt{(1 \pm (f_x^2 + f_y^2))}^3}.$$

If we apply the guiding principle to this, then we expect that

$$H = \frac{f_{xx} + f_{yy}}{2},$$

which is indeed the case [16]. Another example is the expression for the null holomorphic one-forms for minimal surfaces in $\mathbb{E}^3$ and the maximal surfaces in $\mathbb{L}^3$:

$$\phi = \big(1 \mp g^2, i(1 \pm g^2), 2g\big)\, \omega.$$

So we expect to get

$$\phi = (1, i, 2g)\,\omega$$

for the zero mean curvature surfaces in $\mathbb{I}^3$.

2.2. Curvatures of curves and surfaces in $\mathbb{I}^3$

For the basic theory of curvature, we adopt the definitions and propositions from Pottman and Liu [16].

For the graph of $y = f(x)$ viewed as a curve in $\mathbb{E}^2$ or in $\mathbb{L}^2$, respectively, its curvature is

$$\kappa 2[f](x) = \frac{f''}{(1+f'^2)^{3/2}} \quad \text{or} \quad \kappa 2[f](x) = \frac{f''}{(1-f'^2)^{3/2}}.$$

So, following the Guiding principle, it seems plausible to define its curvature as a curve in $\mathbb{I}^2$ as

$$\kappa 2[f](x) = f''(x),$$

which is indeed the case. Note that with this definition the graph of $f(x) = \frac{1}{2}x^2$ has constant curvature 1.

In [16] the standard isotropic unit circle or sphere is defined as the set of (x, ℓ) or (x, y, ℓ) which satisfies $\ell = \frac{1}{2}x^2$ or $\ell = \frac{1}{2}(x^2 + y^2)$, respectively, but we slightly modify them as follows, whose reason is explained in Remark 11.

Definition 1. We call the following set the standard isotropic unit circle.

$$\mathbb{P}^1 := \{(x, \ell) + (0, 1) \in \mathbb{I}^2 : \ell = \frac{1}{2}x^2\}.$$

Note that we do not define its center. Now we define the Gauss map σ of curves in $\mathbb{I}^2$ as follows:

Definition 2 ([16])**.** Given a point P on a curve γ, $\sigma(P)$ is defined to be the point in the standard isotropic unit circle such that the tangent line to the curve at P is parallel to the tangent line to the standard isotropic unit circle at $\sigma(P)$.

Note that σ is not perpendicular to the curve.

Lemma 3. *We have*

$$\sigma(x, f(x)) = \left(f'(x), \frac{1}{2}f'(x)^2\right) + (0, 1), \quad \sigma(x(t), \ell(t)) = \left(\frac{\ell'}{x'}, \frac{1}{2}\left(\frac{\ell'}{x'}\right)^2\right) + (0, 1).$$

Given a regular curve $\gamma(t) = (x(t), \ell(t))$ which satisfies $\ell(t) = f(x(t))$, we see $\ell'(t) = f_x(x(t))x'(t)$, $\ell''(t) = f_{xx}(x(t))(x'(t))^2 + f_x(x(t))x''(t)$, hence $f_{xx}(x(t))$ is equal to $\frac{\ell''(t)x'(t) - \ell'(t)x''(t)}{x'(t)^3}$, which is equivalent to

$$\frac{\langle \sigma'(t), \gamma'(t)\rangle}{\langle \gamma'(t), \gamma'(t)\rangle}, \tag{1}$$

where $\langle\ ,\ \rangle$ is the isotropic inner product, that is

$$\langle (x_1,\ell_1),(x_2,\ell_2)\rangle = x_1x_2, \quad \langle (x_1,y_1,\ell_1),(x_2,y_2,\ell_2)\rangle = x_1x_2+y_1y_2.$$

Definition 4. Given a regular curve $\gamma(t) := (x(t),\ell(t))$ of arbitrary speed, we define $\kappa 2[\gamma](t)$ as (1).

The standard unit sphere and the Gauss map of surfaces in $\mathbb{I}^3$ are as follows:

Definition 5. We call the following set the standard isotropic unit sphere.

$$\mathbb{P}^2 := \{(x,y,\ell)+(0,0,1)\in \mathbb{I}^3 : \ell = \frac{1}{2}(x^2+y^2)\}.$$

Definition 6 ([16])**.** For any surface S in $\mathbb{I}^3$, we define $\sigma : S \to \mathbb{P}^2$ in such a way that $\sigma(P)$ is the point in $\mathbb{P}^2$ which has the parallel tangent plane to S at P. σ is called the Gauss map of the surface.

Lemma 7 ([16])**.** *Given a function $\ell = f(x,y)$, we have*

$$\sigma(a,b,f(a,b)) = \left(f_x(a,b), f_y(a,b), \frac{f_x^2(a,b)+f_y^2(a,b)}{2}\right) + (0,0,1).$$

It is immediate that σ for a parametric surface is as follows:

Lemma 8. *Given a surface $X(u,v) := (x(u,v),y(u,v),\ell(u,v))$, we see that*

$$\sigma\circ X(u,v) = \left(-\frac{N_1}{N_3}, -\frac{N_2}{N_3}, \frac{1}{2}\left(\frac{N_1^2+N_2^2}{N_3^2}\right)\right) + (0,0,1),$$

where

$$N_1 := \begin{vmatrix} y_u & \ell_u \\ y_v & \ell_v \end{vmatrix}, \qquad N_2 := \begin{vmatrix} \ell_u & x_u \\ \ell_v & x_v \end{vmatrix}, \qquad N_3 := \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}.$$

It should be remarked that this is essentially the same as but different in appearance from [5, (2.9)].

Proof of Lemma 8. (N_1,N_2,N_3) is Euclidean-normal to the image of X at $X(u,v)$, and $(x_0,y_0,-1)$ is Euclidean-normal to $\mathbb{P}^2$ at (x_0,y_0,z_0). Therefore, $\sigma\circ X(u,v) = (x_0,y_0,z_0)$ if and only if (N_1,N_2,N_3) and $(x_0,y_0,-1)$ are parallel if and only if $-N_1/N_3 = x_0, -N_2/N_3 = y_0$, from which the conclusion follows. □

By abusing notation we denote $\sigma\circ X$ simply by σ whenever there is no danger of confusion.

Lemma 9. *Suppose $X(u,v) = (u,v,f(u,v))$. Then,*

$$\begin{pmatrix}\sigma_u\\ \sigma_v\end{pmatrix} = \begin{pmatrix} f_{uu} & f_{uv} \\ f_{vu} & f_{vv}\end{pmatrix}\begin{pmatrix}X_u\\ X_v\end{pmatrix}.$$

Proof. It is obvious. □

Definition 10 (Shape operator)**.** Given a surface $X(u,v) := (x(u,v), y(u,v), \ell(u,v))$, define

$$S(aX_u + bX_v) := a\sigma_u + b\sigma_v,$$

which we call the shape operator for X.

It can be easily checked that the above definition does not depend upon the parametrization X.

Motivated by (1), we define, for a tangent vector v_p to the surface at p,

$$\kappa_n(v_p) := \frac{\langle S(v_p), v_p\rangle}{\langle v_p, v_p\rangle}$$

and call it the normal curvature of the surface at p in the direction of v. It is easy to see that S is a linear symmetric transformation of any tangent space to X hence that it has a pair of orthogonal eigenvectors, say e_1 and e_2 with corresponding eigenvalues $\kappa_1 \geq \kappa_2$, respectively. It turns out that $\kappa_1 = \kappa_{\max}$ and $\kappa_2 = \kappa_{\min}$, where $\kappa_{\max}$ and $\kappa_{\min}$ are the maximum and the minimum values of κ_n on the tangent plane. We will use these concepts in Section 4.

Remark 11. For surfaces in $\mathbb{E}^3$ or in $\mathbb{L}^3$, $\kappa_n(v_p)$ can be interpreted as the curvature of the planar curve in the plane which contains both σ_p and v_p. For this, σ_p and v_p need to be linearly independent. In particular, σ_p should not be $\vec{0}$. This is why we slightly alter the definition of the standard unit sphere of [16]. Similar arguments arise when we consider the linear independence of X_u, X_v and σ.

Definition 12 ([16])**.** Given a surface $X(u,v) := (x(u,v), y(u,v), \ell(u,v))$, we define

$$g_{ij} := \langle X_i, X_j\rangle, \quad A_{ij} := \langle X_i, \sigma_j\rangle, \quad K := \det g^{-1}A, \quad H := \frac{1}{2}\operatorname{tr} g^{-1}A.$$

Here, K and H are called the Gauss curvature and the mean curvature, respectively.

Remark 13. Note that $\langle X_i, \sigma_j\rangle \neq -\langle X_{ij}, \sigma\rangle$ in general for surfaces in $\mathbb{I}^3$ since $\langle X_i, \sigma\rangle \neq 0$. So, it is not clear if $A_{12} = A_{21}$.

But we still have:

Lemma 14. *$g_{ij} = g_{ji}$ and $A_{ij} = A_{ji}$.*

Proof. They follow from direct calculations. □

Lemma 15 ([16])**.** *If $X(u,v) = (u, v, f(u,v))$, then $g_{ij} = \delta_{ij}$, $A_{ij} = f_{ij}$ and*

$$K = f_{11}f_{22} - f_{12}f_{21}, \qquad H = (f_{11} + f_{22})/2.$$

Remark 16. If X has a point where the tangent plane is parallel to the ℓ axis, then the above theory breaks down. The situation is pretty similar to the one for surfaces in $\mathbb{L}^3$ with lightlike tangent planes. We need further investigations for these cases.

2.3. About isometries of $\mathbb{I}^3$

The isometries of $\mathbb{I}^3$ are mainly understood from a projective geometric point of view [5]. Unlike $dx^2+dy^2 \pm dz^2$, the degenerate metric dx^2+dy^2 in $\mathbb{R}^3$ alone cannot determine the isometries completely. For the convenience of the reader, we present a derivation of the isometries of $\mathbb{I}^3$ from a metric point of view. Another derivation of the isometries is presented in the next section.

Lemma 17. *For an $f : \mathbb{I}^3 \to \mathbb{I}^3$, suppose that*

(1) *f is affine,*
(2) *f preserves the metric,*
(3) *f preserves the standard isotropic sphere.*

Then f is of the following form:

$$(2) \quad f\begin{pmatrix} x \\ y \\ \ell \end{pmatrix} = \begin{pmatrix} \multicolumn{2}{c}{R} & 0 \\ & & 0 \\ g & h & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ \ell \end{pmatrix} + \begin{pmatrix} R\begin{pmatrix} g \\ h \end{pmatrix} \\ (g^2+h^2)/2 \end{pmatrix}, \quad R \in O(2),\ g, h \in \mathbb{R}.$$

The converse is also true.

Proof. Assumption (1) implies that

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{\ell} \end{pmatrix} = f\begin{pmatrix} x \\ y \\ \ell \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} x \\ y \\ \ell \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \\ \ell_0 \end{pmatrix}$$

for some constant $a, \ldots, i, x_0, \ldots, \ell_0$. Assumption (2) immediately implies that $\left(\begin{smallmatrix} a & b \\ d & e \end{smallmatrix}\right) \in O(2)$ and $c = f = 0$. Then

$$\tilde{x} = ax + by + x_0, \quad \tilde{y} = dx + ey + y_0, \quad \tilde{\ell} = gx + hy + i\ell + \ell_0.$$

Therefore $\tilde{\ell} = \frac{1}{2}(\tilde{x}^2 + \tilde{y}^2)$ leads us to

$$gx + hy + i\ell + \ell_0 = \frac{1}{2}\left((ax + by + x_0)^2 + (dx + ey + y_0)^2\right).$$

Now using that $\ell = \frac{1}{2}(x^2 + y^2)$ and comparing the coefficients of both sides we see that

$$i = 1, \quad g = ax_0 + dy_0, \quad h = bx_0 + ey_0, \quad \ell_0 = (x_0^2 + y_0^2)/2,$$

from which the first conclusion follows. It is immediate that the converse is also true. □

Definition 18 ([5])**.** A matrix of the form $\begin{pmatrix} \multicolumn{2}{c}{R} & 0 \\ & & 0 \\ g & h & \pm 1 \end{pmatrix}$, where $R \in O(2)$ and $g, h \in \mathbb{R}$, is called simply isotropic orthogonal. The set of all the simply isotropic orthogonal matrices is denoted by $O\mathbb{I}(3)$.

Lemma 19 ([5])**.** *The group $O\mathbb{I}(3)$ with translations gives us the group of isotropic isometries, which we denote by $ISO(\mathbb{I}^3)$.*

2.4. Invariance of the curvatures under the isometries

For any $F \in ISO(\mathbb{I}^3)$, it is trivial to see that $\langle V_P, W_P\rangle_{\mathbb{I}^3} = \langle F(V_P), F(W_P)\rangle_{\mathbb{I}^3}$ for any $V_P, W_P \in T_P\mathbb{I}^3$, and that the vertical lines remain to be vertical, and non-vertical lines remain to be non-vertical. We also have the followings.

Lemma 20. *For any surface $X : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{I}^3$, both g and A are invariant under any F in $ISO(\mathbb{I}^3)$.*

Proof. We may assume without loss of generality that $F(\vec{0}) = \vec{0}$ and that $R = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. For g, the proof is obvious. For A, we proceed as follows: First, we see

$$\begin{aligned}\tilde{N}_1 := \begin{vmatrix} \tilde{y}_u & \tilde{\ell}_u \\ \tilde{y}_v & \tilde{\ell}_v \end{vmatrix} &= \begin{vmatrix} cx_u + dy_u & \ell_u + gx_u + hy_u \\ cx_v + dy_v & \ell_v + gx_v + hy_v \end{vmatrix} \\ &= c\begin{vmatrix} x_u & \ell_u \\ x_v & \ell_v \end{vmatrix} + ch\begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} + d\begin{vmatrix} y_u & \ell_u \\ y_v & \ell_v \end{vmatrix} + dg\begin{vmatrix} y_u & x_u \\ y_v & x_v \end{vmatrix} \\ &= dN_1 - cN_2 + (ch - dg)N_3.\end{aligned}$$

Similarly, we see

$$\tilde{N}_2 := \begin{vmatrix} \tilde{\ell}_u & \tilde{x}_u \\ \tilde{\ell}_v & \tilde{x}_v \end{vmatrix} = -bN_1 + aN_2 + (bg - ah)N_3, \quad \tilde{N}_3 := \begin{vmatrix} \tilde{x}_u & \tilde{y}_u \\ \tilde{x}_v & \tilde{y}_v \end{vmatrix} = (ad - bc)N_3.$$

Therefore, if $\varepsilon := ad - bc$ (which is in fact ± 1), then we have

$$\begin{aligned}\begin{pmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \end{pmatrix} = \begin{pmatrix} -\frac{\tilde{N}_1}{\tilde{N}_3} \\ -\frac{\tilde{N}_2}{\tilde{N}_3} \end{pmatrix} &= \begin{pmatrix} -\varepsilon(ch - dg) - \varepsilon d\frac{N_1}{N_3} + c\,\varepsilon\frac{N_2}{N_3} \\ -\varepsilon(bg - ah) + \varepsilon b\frac{N_1}{N_3} - \varepsilon a\frac{N_2}{N_3} \end{pmatrix} \\ &= \begin{pmatrix} -\varepsilon(ch - dg) \\ -\varepsilon(bg - ah) \end{pmatrix} - \begin{pmatrix} -\varepsilon d & c\,\varepsilon \\ +\varepsilon b & -\varepsilon a \end{pmatrix}\begin{pmatrix} -\frac{N_1}{N_3} \\ -\frac{N_2}{N_3} \end{pmatrix}.\end{aligned}$$

Since $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in O(2)$, we have $-\left(\begin{smallmatrix} -\varepsilon d & c\,\varepsilon \\ +\varepsilon b & -\varepsilon a \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then,

$$\partial_i\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\partial_i\begin{pmatrix} x \\ y \end{pmatrix}, \quad \partial_j\begin{pmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\partial_j\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \tag{3}$$

from which $\tilde{A}_{ij} = A_{ij}$ follows. □

Note that (3) shows how the coordinate vectors and the Gauss map get transformed under F.

Corollary 21. *Curvatures κ_n, K and H are invariant under any F in $ISO(\mathbb{I}^3)$.*

For ZMC surfaces we have the following.

Lemma 22. *Both $f_1(x, y, \ell) := (c_1x, c_1y, c_1\ell)$ and $f_2(x, y, \ell) := (x, y, c_2\ell)$ transform a ZMC surface to a ZMC surface.*

We call f_2 a vertical dilation.

2.5. Weierstrass representation formula for ZMC surfaces in $\mathbb{I}^3$

This has been known for quite a while. See for example [2, Section 4], [14, Remark 2.1]. [15] and [18] state that

$$X := \mathrm{Re}\int_\gamma (1, i, 2g)\omega, \qquad X := \mathrm{Re}\int_w (F, -\sqrt{-1}F, G)dw$$

are of ZMC in $\mathbb{I}^3$, respectively, where g, ω, F, G should be interpreted appropriately. For a reason to be seen in Lemma 25, we fix once and for all the following form of the above formulae.

Lemma 23. *Given a meromorphic function g and a holomorphic one-form ω on a Riemann surface M, the following*

$$X := \mathrm{Re}\int_\gamma (1, i, g)\omega \tag{4}$$

defines a generalized surface of ZMC in $\mathbb{I}^3$ on a cover of M, which is regular at p if $|\omega(p)|^2 \neq 0$. Its induced metric is $ds^2 := |\omega|^2$, and $2X_z dx = (1, i, g)\omega$.

3. Rotations and spheres of the isotropic three-space as a lightlike subspace of $\mathbb{L}^4$

It can be easily guessed that the isotropic three-space is deeply related to the lightlike subspaces of $\mathbb{L}^4$. Indeed, [15] says that $\mathbb{I}^3$ is in fact the geometry of a lightlike subspace of $\mathbb{L}^4$.

In this section we present an elementary and explicit account of how the some of the isometries of $\mathbb{I}^3$ arise as the restriction of some rotations of $\mathbb{L}^4$. In so doing, we clarify what the rotations in $\mathbb{I}^3$ are.

We consider the Hermitian model of $\mathbb{L}^4$:

$$\mathbb{L}^4 \ni (x_0, x_1, x_2, x_3) \leftrightarrow \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} \in \mathcal{H}erm(2),$$

where $\mathcal{H}erm(2) := \{v \in M(2, \mathbb{C}) : v^* = v\}$. Note that $SL(2, \mathbb{C})$ acts on $\mathbb{L}^4$ as isometries as follows:

$$SL(2, \mathbb{C}) \times \mathcal{H}erm(2) \ni (g, v) \mapsto gvg^* \in \mathcal{H}erm(2).$$

Now fix an arbitrary constant $k \in \mathbb{R}$ and let

$$\Pi_k := \{(x_0, x_1, x_2, x_3) \in \mathbb{L}^4 : x_1 - x_0 = k\},$$

which is our model of $\mathbb{I}^3$, which we identify with $\mathbb{I}^3$ by $(x_2, x_3, x_1 + x_0) \sim (x, y, \ell)$.

Now consider the following matrices in $SL(2, \mathbb{C})$:

$$g_3(\varphi) := \begin{pmatrix} 1 - i\frac{\varphi}{4} & i\frac{\varphi}{4} \\ -i\frac{\varphi}{4} & 1 + i\frac{\varphi}{4} \end{pmatrix}, \qquad g_{10}(\theta) := \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix},$$

$$g_2(\psi) := g_{10}(\pi/2)^{-1} g_3(-\psi) g_{10}(\pi/2), \qquad g_R := g_3(\varphi) g_2(\psi) g_{10}(\theta).$$

Fix an arbitrary $A \in \mathcal{H}erm(2)$ and consider the map

$$X \in \mathcal{H}erm(2) \mapsto Y = g_R(X - A)g_R^* + A \in \mathcal{H}erm(2). \tag{5}$$

Letting $A = \begin{pmatrix} a_0+a_3 & a_1+ia_2 \\ a_1-ia_2 & a_0-a_3 \end{pmatrix}$ and

$$X = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix}, \quad Y = \begin{pmatrix} y_0 + y_3 & y_1 + iy_2 \\ y_1 - iy_2 & y_0 - y_3 \end{pmatrix},$$

direct calculations show that $y_1 - y_0 = x_1 - x_0$, which implies that the map (5) preserves Π_k, and

$$\begin{pmatrix} y_2 \\ y_3 \\ y_1 + y_0 \end{pmatrix} = B \begin{pmatrix} x_2 - a_2 \\ x_3 - a_3 \\ x_1 + x_0 - (a_1 + a_0) \end{pmatrix} + C + \begin{pmatrix} a_2 \\ a_3 \\ a_1 + a_0 \end{pmatrix}, \tag{6}$$

where, if we let $R := \frac{k-(a_1-a_0)}{2}$,
(7)

$$B := \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ \varphi\cos\theta + \psi\sin\theta & -\varphi\sin\theta + \psi\cos\theta & 1 \end{pmatrix}, \quad C := R \begin{pmatrix} \varphi \\ \psi \\ (\varphi^2 + \psi^2)/2 \end{pmatrix},$$

which can be rewritten as, if we let $g := \varphi\cos\theta + \psi\sin\theta$, $h := -\varphi\sin\theta + \psi\cos\theta$,

$$B = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ g & h & 1 \end{pmatrix}, \qquad C = R \begin{pmatrix} g\cos\theta - h\sin\theta \\ g\sin\theta + h\cos\theta \\ (g^2 + h^2)/2 \end{pmatrix}. \tag{8}$$

Note that the equations (6) and (8) are the equation (2) we obtained previously.

Now, while fixing an arbitrary $X \in \Pi_k$ and varying θ, φ, ψ, we want to observe what the set of $F(X)$'s looks like. First of all, one can easily see that the trajectory of a point by (6) with $\varphi = \psi = 0$ is a circle to Euclidean eyes.

Next, (6) with $\theta = \psi = 0$ yields $y_3 = x_3$ and

$$\begin{pmatrix} y_2 \\ y_1 + y_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \varphi & 1 \end{pmatrix} \begin{pmatrix} x_2 - a_2 \\ x_1 + x_0 - (a_1 + a_0) \end{pmatrix} + R \begin{pmatrix} \varphi \\ \varphi^2/2 \end{pmatrix} + \begin{pmatrix} a_2 \\ a_1 + a_0 \end{pmatrix}. \tag{9}$$

The trajectory of $(x_2, x_3, x_1 + x_0)$ is a line if $R = 0$ or a parabola if $R \neq 0$ to Euclidean eyes. The case where $\theta = \varphi = 0$ is similar to this case.

The map (9) can be interpreted as the rotation in the isotropic plane $\mathbb{I}^2$ equipped with the coordinates $x_2, x_1 + x_0$. See [19] or [20] for elementary accounts of the rotations in $\mathbb{I}^2$ from the same point of view as this section. There is no fixed point of (9) as a rotation in $\mathbb{I}^2$ if $R \neq 0$ and an entire line is the set of fixed points if $R = 0$.

Now let's observe the orbit of X by (6) when all the θ, φ, ψ vary. For convenience of notation, let

$$\tilde{x}_i = x_i - a_i, \qquad \tilde{y}_j = y_j - a_j.$$

If $A \in \Pi_k$, then $R = 0$, and we get

$$\tilde{y}_2^2 + \tilde{y}_3^2 = \tilde{x}_2^2 + \tilde{x}_3^2. \tag{10}$$

So, in the $x_2, x_3, x_1 + x_0$ coordinates system, the set of Y's looks like a circular cylinder to Euclidean eyes.

If $A \notin \Pi_k$, then $R \neq 0$, and we get

$$\tilde{y}_1 + \tilde{y}_0 = g\tilde{x}_2 + h\tilde{x}_3 + \tilde{x}_1 + \tilde{x}_0 + \frac{R}{2}(g^2 + h^2) \tag{11}$$

from the third row of (6), and

$$g = R^{-1}(\tilde{y}_2 \cos\theta + \tilde{y}_3 \sin\theta - \tilde{x}_2), \qquad h = R^{-1}(-\tilde{y}_2 \sin\theta + \tilde{y}_3 \cos\theta - \tilde{x}_3)$$

from the first and the second rows of (6). Plugging these into (11) we obtain

$$\tilde{y}_1 + \tilde{y}_0 - \frac{1}{2R}\left(\tilde{y}_2^2 + \tilde{y}_3^2\right) = \tilde{x}_1 + \tilde{x}_0 - \frac{1}{2R}\left(\tilde{x}_2^2 + \tilde{x}_3^2\right). \tag{12}$$

So, in the $x_2, x_3, x_1 + x_0$ coordinates system, the set of Y's looks like a circular paraboloid to Euclidean eyes.

Out of the above computations we may say that (6) and (8) represent a rotation in $\mathbb{I}^3$ and that (10) and (12) are the equations of the spheres in $\mathbb{I}^3$. The number R can be interpreted as the size of the sphere, an account of which can be found in [23] for example.

4. Totally umbilic surfaces in $\mathbb{I}^3$

The following elementary result does not seem to appear elsewhere.

Lemma 24. *Any totally umbilic surface in $\mathbb{I}^3$ is a part of either a plane or $\mathbb{P}^2$ up to translation, homothety and vertical dilation.*

Proof. Let $\kappa_{\max}(u, v) = \kappa_{\min}(u, v) = \kappa(u, v)$. By assumption,

$$\sigma_u = S(X_u) = \kappa X_u, \quad \sigma_v = S(X_v) = \kappa X_v.$$

By differentiating the first with respect to v and the second with respect to u we obtain

$$\sigma_{uv} = \kappa_v X_u + \kappa X_{uv}, \quad \sigma_{vu} = \kappa_u X_v + \kappa X_{vu}.$$

From this we can conclude that κ is in fact constant. If $\kappa = 0$, then $\sigma_u = \sigma_v = 0$, hence X is a plane. If $\kappa \neq 0$, then

$$\sigma = \kappa X + \vec{c}$$

for some constant vector $\vec{c}$. We may assume that $X(u, v) = (u, v, f(u, v))$ for some f. Then the above equation implies that

$$(f_x, f_y, \frac{1}{2}(f_x^2 + f_y^2)) = (\kappa x + c_1, \kappa y + c_2, \kappa f(x, y) + c_3).$$

From this we can conclude that

$$f(x, y) = \frac{1}{\kappa}\left(\frac{1}{2}((\kappa x + c_1)^2 + (\kappa y + c_2)^2) - c_3\right).$$

So the graph of f is a part of $\mathbb{P}^2$ up to translation, homothety and vertical dilation. □

5. Interpretation of g in the Weierstrass representation formula

Let $\Pi(x, y, \ell) := (x, y, 0)$. In [16] it is called the stereographic projection and [5] called it the top view, respectively.

Lemma 25. *For a ZMC surface given by* (4)*, we have*

$$g = \Pi \circ \sigma \circ X.$$

Proof. Let $z := u + iv$ be a local variable for the underlying Riemann surface and let $\omega = f(z)dz$ in this coordinate. Note that $\int \omega = x - iy$, which implies that $f = x_u - iy_u$, hence

$$x_u = -y_v = (f + \bar{f})/2, \qquad y_u = x_v = -(f - \bar{f})/2i.$$

On the other hand, $\operatorname{Re} \int g\omega = \ell$ implies that $gf = \ell_u - i\ell_v$, hence

$$\ell_u = \frac{gf + \bar{g}\bar{f}}{2}, \qquad \ell_v = -\frac{gf - \bar{g}\bar{f}}{2i}.$$

Therefore

$$N_1 = \begin{vmatrix} y_u & \ell_u \\ y_v & \ell_v \end{vmatrix} = \frac{1}{2}(g + \bar{g})f\bar{f}, \qquad N_2 = \begin{vmatrix} \ell_u & x_u \\ \ell_v & x_v \end{vmatrix} = \frac{1}{2i}(g - \bar{g})f\bar{f},$$

$$N_3 = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} = -(x_u^2 + x_v^2) = -f\bar{f}.$$

Hence $(-N_1/N_3, -N_2/N_3) = (\operatorname{Re} g, \operatorname{Im} g)$, from which the conclusion follows. □

Remark 26. It would be interesting to have an interpretation of g in terms of the cross product of $\mathbb{I}^3$.

6. Examples of zero mean curvature surfaces in $\mathbb{I}^3$

Most of the examples presented in this section appear already in various articles. We present them again to present our point of view about them.

6.1. ZMC surfaces and harmonic functions

If $\omega = dz$, $z = u + iv$, then

$$X(u, v) = (u, -v, f(u, v))$$

for some harmonic function f.

6.2. Isotropic catenoid

If $z = re^{i\theta}$, $g = 1/z$ and $\omega = dz$, we obtain (cf. [18] for example.)

$$X(re^{i\theta}) = (r\cos\theta, -r\sin\theta, \log r). \tag{13}$$

It is obtained by revolving the graph of the log around the ℓ axis. Due to the abundancy of rotations in $\mathbb{I}^3$, as we observed in Section 3, there is one more nontrivial ZMC surface of revolution.

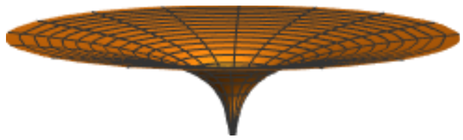

FIGURE 1. Isotropic catenoid

Lemma 27 (ZMC surfaces of revolution in $\mathbb{I}^3$). *A ZMC surface of revolution in $\mathbb{I}^3$ is a non-vertical plane, an isotropic catenoid* (13)*, or*

$$X(u,v) = (u, v, u^2 - v^2) \tag{14}$$

up to congruence, homothety, and vertical dilation.

Proof. Any line in $\mathbb{I}^3$ is congruent to either the ℓ-axis or the y-axis. So it is enough to consider the ZMC surfaces around either of these two axes.

Let's consider a ZMC surface of revolution around the y-axis. Given a function $\ell = g(y)$, consider the curve $(0, y, g(y))$ in the $y\ell$-plane. A surface which is obtained by rotating this curve around the y-axis may be expressed, using (6) and (8) with $\psi = \theta = 0$, as

$$(y,\varphi) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \varphi & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ y \\ g(y) \end{pmatrix} + R \begin{pmatrix} \varphi \\ 0 \\ \varphi^2/2 \end{pmatrix} = \begin{pmatrix} R\varphi \\ y \\ g(y) + R\varphi^2/2 \end{pmatrix}.$$

Setting $u = R\varphi$ and $v = y$, the map can be rewritten as

$$X(u,v) = \left(u, v, u^2/2R + g(v)\right).$$

This surface has ZMC if and only if $\frac{u^2}{2R} + g(v)$ is harmonic if and only if $g(v) = -\frac{v^2}{2R} + \alpha v + \beta$ for some constants α and β. Up to reparametrization, translation, homothery, and vertical dilation, it is equivalent to (14).

In a similar way, one can easily see that the ZMC surface of revolution around the ℓ-axis is a horizontal plane or an isotropic catenoid. □

Remark 28. For $\epsilon = 1, -1, 0$, rotating the graph in the xz-plane of

$$x = \frac{e^z + \epsilon e^{-z}}{2}$$

around the z-axis produces the catenoid in $\mathbb{E}^3, \mathbb{L}^3, \mathbb{I}^3$, respectively.

6.3. Isotropic helicoid

If $z = re^{i\theta}$, $g = 1/z$ and $\omega = idz$, we obtain (cf. [18] for example)

$$X(re^{i\theta}) = (-r\sin\theta, -r\cos\theta, -\theta) \qquad \text{for } r > 0 \quad \text{and } \theta \in \mathbb{R}.$$

Note that this is exactly one half of the standard helicoid (in $\mathbb{E}^3$). It is interesting that the standard helicoid exists in all three kinds of surfaces $\mathbb{E}^3$, $\mathbb{L}^3$ and $\mathbb{I}^3$. In fact, we have the following.

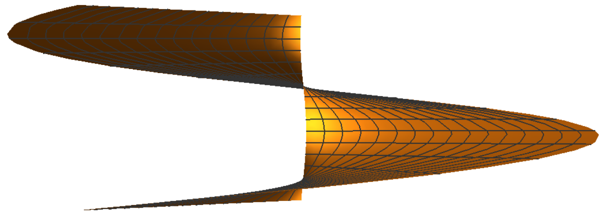

FIGURE 2. Isotropic helicoid

Lemma 29 (O. Kobayashi 1983)**.** *Consider a surface in* $\mathbb{R}^3$. *If it is minimal when* $\mathbb{R}^3$ *is regarded as* $\mathbb{E}^3$ *and also minimal when* $\mathbb{R}^3$ *is regarded as* $\mathbb{I}^3$, *then it is* (*a part of*) *the standard helicoid.*

Proof. It follows right away from [10]. □

Remark 30. This example is interesting in the following senses.

1) It is not a graph.

2) We can definitely analytically extend this surface through the vertical (lightlike) line. Is this a general phenomenon? Is there any other example which contains a vertical (lightlike) line? We refer the readers to the work by Umehara and Yamada [21] for example for related questions and answers for ZMC surfaces in Lorentz-Minkowski three-space.

6.4. ZMC surfaces with multiple helicoidal ends

In $\mathbb{E}^3$, there are minimal surfaces which can be thought of several helicoids glued together (cf. [4], [22]). In $\mathbb{I}^3$, such surfaces also exist and have explicit formulae which can be easily written down.

In general, the graph of

$$f(z) := \mathrm{Re}[(-i)\ln[(z-a_i)\cdots(z-a_n)]], \quad \text{where } a_1,\dots,a_n \in \mathbb{C}$$

is a ZMC surface with helicoidal ends at $a_1,\dots,a_n$, which are the conjugate of the Green's function with multiple poles.

6.5. Translational ZMC surface

If the graph of $\ell = f(x)+g(y)$ in $\mathbb{I}^3$ is of ZMC, then

$$\ell = f(x)+g(y) = x^2 - y^2 \tag{15}$$

up to translation, homothety and vertical dilation (cf. [18] for example).

$$i(x,y) = 2xy$$

is its conjugate. (To find i from ℓ, we simply use the relation that $i_1 = -\ell_2 = 2y, i_2 = \ell_1 = 2x$.) Note that they are in fact congruent to each other in $\mathbb{I}^3$.

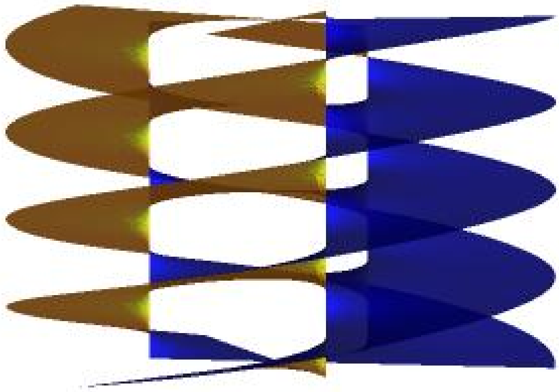

FIGURE 3. The graph of $f(z) := \mathrm{Re}[-i \ln (z^3 - 1)]$ which has three helicoidal ends.

6.6. ZMC surface with W-data of Enneper surface

If $z = u + iv$, $g = z$ and $\omega = dz$, then we obtain (cf. [18] for example)

$$X(u, v) = (u, -v, u^2/2 - v^2/2). \tag{16}$$

The ZMC surfaces in (15) and (16) are related by a vertical dilation. See Lemma 22.

6.7. Cousin of the Scherk first surface

$g = z$ and $\omega = \frac{4dz}{1-z^4}$ are the Weierstrass data for the Scherk first surface. The corresponding ZMC surface of $\mathbb{I}^3$ is drawn in Figure 4.

6.8. ZMC surfaces with more than one catenoidal ends

Green's functions with more than one poles are examples of the ZMC surfaces with more than one catenoidal ends.

6.9. ZMC surfaces with immersed catenoidal ends

For minimal surfaces in $\mathbb{E}^3$, there exists immersed catenoidal ends. For ZMC surfaces in $\mathbb{I}^3$, there also exist immersed catenoidal ends. For example, consider

$$g = \frac{1}{z^2}, \quad \omega = (2z + 3z^2)dz.$$

Then $\Phi = \int \phi = (z^2 + z^3, -i(z^2 + z^3), 2 \ln z + 3z)$ and the image of a punctured neighborhood of $z = 0$ by $X := \mathrm{Re}\,\Phi$ is an immersed catenoidal end. This is a kind of immersed pole for the Green's function.

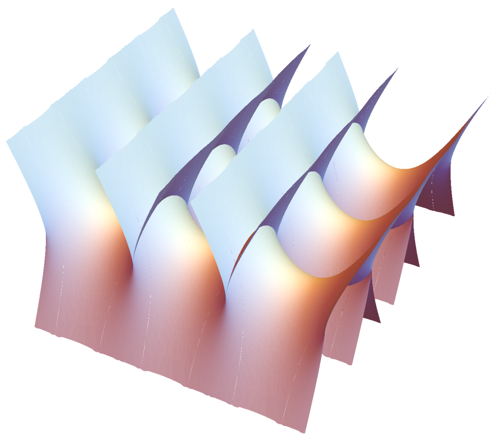

FIGURE 4. Cousin of the Scherk first surface

6.10. Enneper-catenoidal ZMC surface

Consider

$$f(x,y) = \ln\sqrt{x^2+y^2} + x^2 - y^2 = \ln r + r^2\cos 2\theta.$$

It has one catenoidal end and an Enneper end.

7. CMC surfaces in $\mathbb{I}^3$

Suppose that the graph of $\ell = f(x,y)$ is of CMC-c. Let $g(x,y) := \frac{1}{2}(x^2+y^2)$. Then $h(x,y) := f(x,y) - cg(x,y)$ is of ZMC. That is, $f = h + cg$ for some ZMC surface h. This gives a local holomorphic representation formula for CMC surfaces in $\mathbb{I}^3$.

Note that [5, Example 6.3] characterizes all the minimal and CMC helicoids in $\mathbb{I}^3$.

8. Björling representation formula

In the Björling problem for minimal surfaces in $\mathbb{E}^3$ or for maximal surfaces in $\mathbb{L}^3$, a curve γ and a unit vector N normal to the curve are prescribed as data. Then we obtain the desired surface by considering $\dot\gamma - iN\times\dot\gamma$.

The immediate difficulty we encounter when we try to generalize this procedure to the ZMC surfaces in $\mathbb{I}^3$ is that we do not have an appropriate cross product. Fortunately, as in [8] and [9], we can overcome this difficulty by realizing that in Björling representation formula, the Björling data can be considered in fact as an analytic curve γ and an analytic vector field V along γ such that

(1) $\dot\gamma$ and V are orthogonal, i.e., $\langle\dot\gamma, V\rangle = 0$,

(2) $\dot\gamma$ and V are of the same length, i.e., $\langle\dot\gamma,\dot\gamma\rangle = \langle V, V\rangle$.

Definition 31. By Björling data we mean a pair (γ, V) which satisfy the above two conditions.

Lemma 32. *In* $\mathbb{I}^3$*, suppose that an analytic curve* $\gamma(u) = (x(u), y(u), \ell(u))$ *and an analytic vector field* $V(u)$ *along* γ *are Björling data. Then*

$$V(u) = \pm(-\dot y(u), \dot x(u), m(u))$$

for some analytic function $m(u)$*.*

Proof. Let $V = (a(u), b(u), c(u))$. Then the two conditions imply

$$\dot x a + \dot y b = 0, \qquad \dot x^2 + \dot y^2 = a^2 + b^2.$$

By solving these equations, we get the conclusion. □

So we arrive at the following:

Theorem 33 (Björling representation formula)**.** *Given an analytic curve* $\gamma(u) := (x(u), y(u), \ell(u))$ *and an analytic function* $m(u)$*, the following* X *is a ZMC surface which contains* γ *and which is tangent to* $V(u) := (-\dot y(u), \dot x(u), m(u))$*.*

$$X := \operatorname{Re}\Phi(z), \quad \Phi(z) := \textit{the analytic extension of } \gamma(u) - i\int V(u)du.$$

As an application we derive the formula for the isotropic helicoid from Björling formula. Consider

$$\gamma(\theta) := (\cos\theta, \sin\theta, \theta), \qquad V(\theta) := (-\cos\theta, -\sin\theta, 0).$$

Then

$$\dot\gamma(\theta) - iV(\theta) = (-\sin\theta + i\cos\theta, \cos\theta + i\sin\theta, 1) = (ie^{i\theta}, e^{i\theta}, 1),$$
$$\phi(z) := \text{the analytic extension of } \dot\gamma(u) - iV(u) = (ie^{iz}, e^{iz}, 1),$$
$$\Phi(z) = (e^{iz}, -ie^{iz}, z),$$
$$X = \operatorname{Re}\Phi(z) = \operatorname{Re}(e^{iz}, -ie^{iz}, z) = (e^{-r}\cos\theta, e^{-r}\sin\theta, \theta), \quad \text{where } z = \theta + ir.$$

We can derive the isotropic helicoid from another Björling data as follows. Consider

$$\gamma(u) = (0, u, 0), \qquad V(u) = (1, 0, \frac{1}{u}).$$

Then, with $z = u + iv = re^{i\theta}$,

$$\begin{aligned}
\phi(z) &:= \text{the analytic extension of } \dot\gamma(u) - iV(u) \\
&= (0,1,0) - i(1, 0, \frac{1}{z}) = (-i, 1, -\frac{i}{z}),\\
\Phi(z) &:= \int \phi(z)dz = (-iz, z, -i\ln z) = (-i(u+iv), u+iv, -i(\ln r + i\theta)),\\
X &:= \operatorname{Re}\Phi = (v, u, \theta) = (v, u, \arctan\frac{v}{u}).
\end{aligned}$$

9. Surfaces with constant H/K in $\mathbb{I}^3$

It is well know that the surfaces in $\mathbb{E}^3$ parallel to minimal surfaces and the surfaces in $\mathbb{L}^3$ parallel to maximal surfaces have constant H/K. See for example the references in [7]. We observe in this section that the same phenomenon is true in $\mathbb{I}^3$.

Lemma 34. *For an arbitrary constant d, suppose that $Y := X + d\sigma$ is an immersion. Then σ is still the Gauss map of Y.*

Proof. Without loss of generality we may assume that $X(u,v) = (u, v, f(u,v))$. Let N_1^Y, N_2^Y, N_3^Y be the N_1, N_2, N_3 in Lemma 8 for Y. Then straightforward calculations show that $N_3^Y = 1 + 2dH_X + d^2K_X$ and

$$N_1^Y = -f_1(1 + 2dH_X + d^2K_X), \quad N_2^Y = -f_2(1 + 2dH_X + d^2K_X).$$

Hence the conclusion follows. □

Lemma 35. *For any constant d, if $Y := X + d\sigma$ is an immersion, then*

$$K_Y = \frac{K_X}{1 + 2H_Xd + K_Xd^2}, \quad H_Y = \frac{H_X + K_Xd}{1 + 2H_Xd + K_Xd^2}, \quad \frac{H_Y}{K_Y} = \frac{H_X}{K_X} + d.$$

Proof. We can derive by direct computations. □

Corollary 36. *Suppose that X is a minimal immersion. Then, for any constant d, if $Y := X + d\sigma$ is an immersion, then*

$$H_Y/K_Y = d.$$

The following lemma is motivated by the results in [6], and are found by trial and error with Mathematica.

Lemma 37. *Given a harmonic function f, both of*

$$Y(u,v) := (u, v, f(u,v)) + \lambda(-f_v(u,v), -f_u(u,v), 0),$$
$$Z(u,v) := (u, v, f(u,v)) + \lambda(-f_u(u,v), f_v(u,v), 0)$$

are of ZMC for all λ.

Proof. We have

$$Y_u = (1 - \lambda f_{vu}, -\lambda f_{uu}, f_u), \quad Y_v = (-\lambda f_{vv}, 1 - \lambda f_{uv}, f_v).$$

Because $f_{uu} + f_{vv} = 0$, we see that

$$g_{11} := \langle Y_u, Y_u\rangle = 1 - 2\lambda f_{vu} + \lambda^2(f_{vu}^2 + f_{uu}^2) = \langle Y_v, Y_v\rangle =: g_{22}, \quad g_{12} = 0.$$

Then, we can see that Y is of ZMC by directly computing σ, its derivatives, and $A_{ii} := \langle Y_i, \sigma_i\rangle$ and by showing that $A_{11} + A_{22} = 0$. We may use the fact that $f_{11} = -f_{22}$ at a couple of places.

The proof for Z is similar and left to the reader. □

Remark 38. Given a harmonic function f, the following

$$Y(u,v) := (u,v,f(u,v)) + \lambda(-f_u,-f_v,0)$$

may not be of zero mean curvature nor of constant H/K. $f(u,v) := e^u \cos v$ is an example. Direct calculations show

$$H = -\frac{4\lambda e^{2u}\left(\lambda e^u\cos(v)-1\right)}{\left(\lambda^2e^{2u}-1\right)^3}, \qquad \frac{H}{K} = \frac{4\lambda\left(\lambda^2e^{2u}-1\right)\left(\lambda e^u\cos(v)-1\right)}{\lambda^2e^{2u}-2\lambda e^u\cos(v)+1}.$$

10. Weierstrass representation formula for ZMC surfaces in $\mathbb{Q}^3$

It is well known that CMC-c surfaces in the hyperbolic three-space $\mathbb{H}^3(-c^2)$ and in the de Sitter three-space $\mathbb{S}^3_1(c^2)$ admit holomorphic representation formula. See [1, 3, 15, 17] for example. As c approaches zero, both $\mathbb{H}^3(-c^2)$ and in $\mathbb{S}^3_1(c^2)$ converge to the light cone $\mathbb{Q}^3 := \{\vec{x} \in \mathbb{L}^4 : \langle \vec{x}, \vec{x}\rangle = 0\}$, hence it is natural to expect that a holomorphic representation formula for ZMC surfaces in $\mathbb{Q}^3$ exists. In fact, Liu in [12] already presented a Weierstrass representation formula for ZMC surfaces in $\mathbb{Q}^3$ in terms of a holomorphic function and a complex function without integration. Pember states in [15] that given an arbitrary meromorphic function g and a holomorphic one form ω, the map $X := F\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)F^*$ where F is a solution of $F^{-1}dF = \left(\begin{smallmatrix}1&-g\\g^{-1}&-1\end{smallmatrix}\right)g\omega$ is intrinsically flat in $\mathbb{Q}^3$.

In this section, we show the following:

Theorem 39. *Given a holomorphic function G and one form Ω on a Riemann surface M, if F satisfies*

$$dFF^{-1} = \begin{pmatrix}1 & -G\\ G^{-1} & -1\end{pmatrix} G\Omega,$$

then

$$X := F\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}F^*$$

is a map from the universal cover of M into $\mathbb{Q}^3$, and, whenever it is an immersion, is conformal and has zero mean curvature.

As a preparation for the proof, we recall a few facts from [11]. Let X be a surface in $\mathbb{Q}^3$ with isothermal parameters u, v. Let

$$ds^2 = \langle\langle dX, dX\rangle\rangle = 2e^w(du^2+dv^2), \qquad \Delta = (2e^w)^{-1}(\partial_{uu}+\partial_{vv}),$$

where $\langle\langle\ ,\ \rangle\rangle$ denotes the Lorentzian inner product of $\mathbb{L}^4$.

Definition 40 ([11])**.** Let

$$\mathcal{G} := -\frac{1}{2}\Delta X - \frac{1}{8}\langle\langle \Delta X, \Delta X\rangle\rangle X.$$

Lemma 41 ([11])**.** *We have*

$$\langle\langle \mathcal{G},\mathcal{G}\rangle\rangle = 0, \qquad \langle\langle \mathcal{G}, X\rangle\rangle = 1, \qquad \langle\langle \mathcal{G}, X_u\rangle\rangle = 0, \qquad \langle\langle \mathcal{G}, X_v\rangle\rangle = 0.$$

In fact, this property characterizes $\mathcal{G}$ uniquely. See [13].

Definition 42 ([11])**.** The mean curvature of X in $\mathbb{Q}^3$ is

$$H := \frac{1}{2}\langle\!\langle \Delta X, \mathcal{G}\rangle\!\rangle.$$

If $H \equiv 0$, X is called a ZMC surface.

In [13] a surface with $H \equiv 0$ is called a maximal (hyper)surface.

Proof of Theorem 39. Let z be a local coordinate and $dFF^{-1} = \mathcal{F}dz$ in this coordinate. Note that $\det(\mathcal{F}) = 0$. It is immediate to see that

$$X_z = F_z \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} F^* = F_z F^{-1} F \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} F^* = \mathcal{F}X,$$

hence

$$\langle\!\langle X_z, X_z\rangle\!\rangle = \langle\!\langle \mathcal{F}X, \mathcal{F}X\rangle\!\rangle = -\det(\mathcal{F}X) = 0$$

hence X is conformal. Now we observe

$$X_{z\bar{z}} = F_z \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (F_z)^* = F_z F^{-1} F \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} F^*(F_z F^{-1})^* = \mathcal{F}X\mathcal{F}^*.$$

Therefore

$$\langle\!\langle \Delta X, \Delta X\rangle\!\rangle = -\det(\Delta X) = -\det(2e^{-w}\mathcal{F}X\mathcal{F}^*) = 0.$$

Hence

$$\begin{aligned} H &= \frac{1}{2}\langle\!\langle \Delta X, \mathcal{G}\rangle\!\rangle = \frac{1}{2}\langle\!\langle \Delta X, -\frac{1}{2}\Delta X - \frac{1}{8}\langle\!\langle \Delta X, \Delta X\rangle\!\rangle X\rangle\!\rangle \\ &= -\frac{1}{4}\langle\!\langle \Delta X, \Delta X\rangle\!\rangle - \frac{1}{16}\langle\!\langle \Delta X, \Delta X\rangle\!\rangle\langle\!\langle \Delta X, X\rangle\!\rangle = 0. \end{aligned}$$

□

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