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ON THE V-SEMI-SLANT SUBMERSIONS FROM ALMOST HERMITIAN MANIFOLDS

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Abstract. In this paper, we deal with the notion of a v-semi-slant submersion from an almost Hermitian manifold onto a Riemannian manifold. We investigate the integrability of distributions, the geometry of foliations, and a decomposition theorem. Given such a map with totally umbilical fibers, we have a condition for the fibers of the map to be minimal. We also obtain an inequality of a proper v-semi-slant submersion in terms of squared mean curvature, scalar curvature, and a v-semi-slant angle. Moreover, we give some examples of such maps and some open problems.

1. Introduction

Let F be a C^∞-submersion from a (semi-)Riemannian manifold (M, g_M) onto a (semi-)Riemannian manifold (N, g_N). Then according to the conditions on the map $F : (M, g_M) \mapsto (N, g_N)$, we have the following submersions: a semi-Riemannian submersion and a Lorentzian submersion [6], a Riemannian submersion ([8, 15]), an invariant submersion [24], an anti-invariant submersion [20], a slant submersion ([5, 22]), an almost Hermitian submersion [25], a contact-complex submersion [9], a quaternionic submersion [10], an almost h-slant submersion [16], a semi-invariant submersion [23], an almost h-semi-invariant submersion [17], a semi-slant submersions [19], an almost h-semi-slant submersions [18], etc. The theory of isometric immersions was begun with the work of Gauss [7] on surfaces in the Euclidean space $\mathbb{R}^3$ in 1827. On the other hand, the study of Riemannian submersions was independently initiated by B. O'Neill [15] in 1966 and A. Gray [8] in 1967 as the counterpart of the theory of isometric immersions. Using the notion of almost Hermitian submersions, B. Watson [25] obtained a classification theorem among fibers, base manifolds, and total manifolds in 1976. As we know, Riemannian submersions are related with physics and have their applications in the Yang-Mills theory ([3, 26]), Kaluza-Klein theory ([2, 11]), Supergravity and superstring theories

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([12, 14]), etc. And any C^∞-maps between Riemannian manifolds are useful and important in several areas ([21], references therein).

The paper is organized as follows. In Section 2 we remind some notions which are needed at the following sections. In Section 3 we give the definition of a v-semi-slant submersion and obtain some properties on it. In Section 4 we deal with an inequality of a proper v-semi-slant submersion in terms of squared mean curvature, scalar curvature, and a v-semi-slant angle. In Section 5 we give some examples of a v-semi-slant submersion. In Section 6 we give some open problems.

2. Preliminaries

Let (M, g_M) and (N, g_N) be Riemannian manifolds, where M, N are C^∞-manifolds and g_M, g_N are Riemannian metrics on M, N, respectively. Let $F : M \mapsto N$ be a C^∞-map. We call the map F a *C^∞-submersion* if F is surjective and the differential $(F_*)_p$ of F at any $p \in M$ has a maximal rank. The map F is said to be a *Riemannian submersion* [6] if F is a C^∞-submersion and the differential F_* preserves the lengths of horizontal vectors.

Let (M, g_M, J) be an almost Hermitian manifold, where J is an almost complex structure. A Riemannian submersion $F : (M, g_M, J) \mapsto (N, g_N)$ is called a *slant submersion* [22] if the angle $\theta = \theta(X)$ between JX and the space $\ker(F_*)_p$ is constant for any nonzero $X \in T_pM$ and $p \in M$.

We call the angle θ a *slant angle*.

Let $F : (M, g_M, J) \mapsto (N, g_N)$ be a slant submersion with the slant angle θ. If $\theta = 0$, then we call the map F an *invariant submersion* [24]. If $\theta = \frac{\pi}{2}$, then we call the map F an *anti-invariant submersion* [20].

A Riemannian submersion $F : (M, g_M, J) \mapsto (N, g_N)$ is called a *semi-invariant submersion* [23] if there is a distribution $\mathcal{D}_1 \subset \ker F_*$ such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2,\ J(\mathcal{D}_1) = \mathcal{D}_1,\ J(\mathcal{D}_2) \subset (\ker F_*)^\perp,$$

where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$.

A Riemannian submersion $F : (M, g_M, J) \mapsto (N, g_N)$ is called a *semi-slant submersion* [19] if there is a distribution $\mathcal{D}_1 \subset \ker F_*$ such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2,\ J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta = \theta(X)$ between JX and the space $(\mathcal{D}_2)_p$ is constant for nonzero $X \in (\mathcal{D}_2)_p$ and $p \in M$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$.

We call the angle θ a *semi-slant angle*.

As we know, a semi-slant submersion is a generalization of a slant submersion and a semi-invariant submersion.

Let (M, g_M) and (N, g_N) be Riemannian manifolds and $F : (M, g_M) \mapsto (N, g_N)$ a C^∞-map. The second fundamental form of F is given by

$$(\nabla F_*)(X, Y) := \nabla^F_X F_*Y - F_*(\nabla_X Y) \quad \text{for } X, Y \in \Gamma(TM),$$

where ∇^F is the pullback connection and we denote conveniently by ∇ the Levi-Civita connections of the metrics g_M and g_N [1]. Recall that F is said to be *harmonic* if $trace(\nabla F_*) = 0$ and F is called a *totally geodesic* map if $(\nabla F_*)(X, Y) = 0$ for $X, Y \in \Gamma(TM)$ [1].

3. v-semi-slant submersions

Definition. Let (M, g_M, J) be an almost Hermitian manifold and (N, g_N) a Riemannian manifold. A Riemannian submersion $F : (M, g_M, J) \mapsto (N, g_N)$ is called a *v-semi-slant submersion* if there is a distribution $\mathcal{D}_1 \subset (\ker F_*)^\perp$ such that

$$(\ker F_*)^\perp = \mathcal{D}_1 \oplus \mathcal{D}_2, \ J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta = \theta(X)$ between JX and the space $(\mathcal{D}_2)_p$ is constant for nonzero $X \in (\mathcal{D}_2)_p$ and $p \in M$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $(\ker F_*)^\perp$.

We call the angle θ a *v-semi-slant angle.*

Remark 3.1. Let F be a v-semi-slant submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). If $\theta \in (0, \frac{\pi}{2})$, then we call the map F *proper.* And if $\theta = \frac{\pi}{2}$, then we call the map F a *v-semi-invariant submersion* [23]. On the other hand, if $\mathcal{D}_2 = (\ker F_*)^\perp$, then we call the map F a *v-slant submersion* and the angle θ a *v-slant angle* [22].

Let $F : (M, g_M, J) \mapsto (N, g_N)$ be a v-semi-slant submersion. Then there is a distribution $\mathcal{D}_1 \subset (\ker F_*)^\perp$ such that

$$(\ker F_*)^\perp = \mathcal{D}_1 \oplus \mathcal{D}_2, \ J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta = \theta(X)$ between JX and the space $(\mathcal{D}_2)_p$ is constant for nonzero $X \in (\mathcal{D}_2)_p$ and $p \in M$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $(\ker F_*)^\perp$.

Then for $X \in \Gamma((\ker F_*)^\perp)$, we write

$$X = PX + QX,$$

where $PX \in \Gamma(\mathcal{D}_1)$ and $QX \in \Gamma(\mathcal{D}_2)$.

For $X \in \Gamma(\ker F_*)$, we get

$$JX = \phi X + \omega X,$$

where $\phi X \in \Gamma(\ker F_*)$ and $\omega X \in \Gamma((\ker F_*)^\perp)$.

For $Z \in \Gamma((\ker F_*)^\perp)$, we obtain

$$JZ = BZ + CZ,$$

where $BZ \in \Gamma(\ker F_*)$ and $CZ \in \Gamma((\ker F_*)^\perp)$.

For $U \in \Gamma(TM)$, we have

$$U = \mathcal{V}U + \mathcal{H}U,$$

where $\mathcal{V}U \in \Gamma(\ker F_*)$ and $\mathcal{H}U \in \Gamma((\ker F_*)^\perp)$.

Then

$$\ker F_* = B\mathcal{D}_2 \oplus \mu, \tag{3.1}$$

where μ is the orthogonal complement of $B\mathcal{D}_2$ in $\ker F_*$ and is invariant by J. Furthermore,

$$\begin{aligned} &C\mathcal{D}_1 = \mathcal{D}_1,\ B\mathcal{D}_1 = 0,\ C\mathcal{D}_2 \subset \mathcal{D}_2,\ \omega(\ker F_*) = \mathcal{D}_2, \\ &\phi^2 + B\omega = -id,\ C^2 + \omega B = -id,\ \omega\phi + C\omega = 0,\ BC + \phi B = 0. \end{aligned} \tag{3.2}$$

Define the (O'Neill) tensors $\mathcal{T}$ and $\mathcal{A}$ by

$$\begin{aligned} \mathcal{A}_E F &= \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F, \\ \mathcal{T}_E F &= \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F \end{aligned}$$

for vector fields E, F on M, where ∇ is the Levi-Civita connection of g_M.

Define

$$\widehat{\nabla}_X Y := \mathcal{V}\nabla_X Y \quad \text{for } X, Y \in \Gamma(\ker F_*).$$

We also define

$$\begin{aligned} (\nabla_Z B)W &:= \mathcal{V}\nabla_Z BW - B\mathcal{H}\nabla_Z W, \\ (\nabla_Z C)W &:= \mathcal{H}\nabla_Z CW - C\mathcal{H}\nabla_Z W \end{aligned}$$

for $Z, W \in \Gamma((\ker F_*)^\perp)$.

We call the tensors B and C *parallel* if $\nabla B = 0$ and $\nabla C = 0$, respectively.

Remark 3.2. Let F be a v-semi-slant submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Since $\ker F_* = B\mathcal{D}_2 \oplus \mu$ and $J(\mu) = \mu$, each fiber $F^{-1}(y)$ is a generic submanifold of M for $y \in N$ [4].

Then we easily have:

Lemma 3.3. *Let (M, g_M, J) be a Kähler manifold and (N, g_N) a Riemannian manifold. Let $F : (M, g_M, J) \mapsto (N, g_N)$ be a v-semi-slant submersion. Then we get*

(1)

$$\begin{aligned} \widehat{\nabla}_X \phi Y + \mathcal{T}_X \omega Y &= \phi \widehat{\nabla}_X Y + B\mathcal{T}_X Y, \\ \mathcal{T}_X \phi Y + \mathcal{H}\nabla_X \omega Y &= \omega \widehat{\nabla}_X Y + C\mathcal{T}_X Y \end{aligned}$$

for $X, Y \in \Gamma(\ker F_*)$.

(2)

$$\begin{aligned} \mathcal{V}\nabla_Z BW + \mathcal{A}_Z CW &= \phi\mathcal{A}_Z W + B\mathcal{H}\nabla_Z W, \\ \mathcal{A}_Z BW + \mathcal{H}\nabla_Z CW &= \omega\mathcal{A}_Z W + C\mathcal{H}\nabla_Z W \end{aligned}$$

for $Z, W \in \Gamma((\ker F_*)^\perp)$.

(3)

$$\widehat{\nabla}_X BZ + \mathcal{T}_X CZ = \phi \mathcal{T}_X Z + B\mathcal{H}\nabla_X Z,$$
$$\mathcal{T}_X BZ + \mathcal{H}\nabla_X CZ = \omega \mathcal{T}_X Z + C\mathcal{H}\nabla_X Z$$

for $X \in \Gamma(\ker F_*)$ *and* $Z \in \Gamma((\ker F_*)^{\perp})$.

Corollary 3.4. *Let* (M, g_M, J) *be a Kähler manifold and* (N, g_N) *a Riemannian manifold. Let* $F : (M, g_M, J) \mapsto (N, g_N)$ *be a v-semi-slant submersion. Then we obtain*

$$(\nabla_Z B)W = \phi \mathcal{A}_Z W - \mathcal{A}_Z CW,$$
$$(\nabla_Z C)W = \omega \mathcal{A}_Z W - \mathcal{A}_Z BW$$

for $Z, W \in \Gamma((\ker F_*)^{\perp})$.

Proposition 3.5. *Let* F *be a v-semi-slant submersion from an almost Hermitian manifold* (M, g_M, J) *onto a Riemannian manifold* (N, g_N) *with the v-semi-slant angle* θ. *Then we obtain*

$$C^2 X = -\cos^2 \theta X \quad \textit{for } X \in \Gamma(\mathcal{D}_2).$$

Proof. Since

$$\cos\theta = \frac{g_M(JX, CX)}{||JX|| \cdot ||CX||} = \frac{-g_M(X, C^2X)}{||X|| \cdot ||CX||}$$

and $\cos\theta = \frac{||CX||}{||JX||}$, we have

$$\cos^2\theta = -\frac{g_M(X, C^2X)}{||X||^2} \quad \text{for } X \in \Gamma(\mathcal{D}_2).$$

Hence,

$$C^2X = -\cos^2\theta X \quad \text{for } X \in \Gamma(\mathcal{D}_2).$$ □

Remark 3.6. Let F be a v-semi-slant submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N) with the v-semi-slant angle θ. Using Proposition 3.5, we easily get

$$g_M(CX, CY) = \cos^2\theta g_M(X, Y),$$
$$g_M(BX, BY) = \sin^2\theta g_M(X, Y)$$

for $X, Y \in \Gamma(\mathcal{D}_2)$ so that given $\theta \in [0, \frac{\pi}{2})$, there exists a local orthonormal frame $\{X_1, \sec\theta CX_1, \ldots, X_k, \sec\theta CX_k\}$ of $\mathcal{D}_2$.

Theorem 3.7. *Let* F *be a v-semi-slant submersion from an almost Hermitian manifold* (M, g_M, J) *onto a Riemannian manifold* (N, g_N). *Then the slant distribution* $\mathcal{D}_2$ *is integrable if and only if we obtain*

$$\mathcal{A}_X Y = 0 \quad \textit{and} \quad PC(\nabla_X Y - \nabla_Y X) = 0$$

for $X, Y \in \Gamma(\mathcal{D}_2)$.

Proof. Given $X, Y \in \Gamma(\mathcal{D}_2)$ and $Z \in \Gamma(\mathcal{D}_1)$, assuming that $\mathcal{A}_X Y = \frac{1}{2}\mathcal{V}[X,Y] = 0$ [6], we obtain

$$\begin{aligned} g_M([X,Y], JZ) &= -g_M(J(\nabla_X Y - \nabla_Y X), Z) \\ &= -g_M(B\nabla_X Y + C\nabla_X Y - B\nabla_Y X - C\nabla_Y X, Z) \\ &= -g_M(C(\nabla_X Y - \nabla_Y X), Z). \end{aligned}$$

Since the integrability of $\mathcal{D}_2$ implies that $\mathcal{A}_X Y = 0$ for $X, Y \in \Gamma(\mathcal{D}_2)$, we have the result. □

Similarly, we get:

Theorem 3.8. *Let F be a v-semi-slant submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then the complex distribution $\mathcal{D}_1$ is integrable if and only if we have*

$$\mathcal{A}_X Y = 0 \quad \textit{and} \quad B(\nabla_X Y - \nabla_Y X) = 0$$

for $X, Y \in \Gamma(\mathcal{D}_1)$.

Lemma 3.9. *Let (M, g_M, J) be a Kähler manifold and (N, g_N) a Riemannian manifold. Let $F : (M, g_M, J) \mapsto (N, g_N)$ be a v-semi-slant submersion. Then the complex distribution $\mathcal{D}_1$ is integrable if and only if we get*

$$\mathcal{A}_X Y = 0 \quad \textit{for } X, Y \in \Gamma(\mathcal{D}_1).$$

Proof. Given $X, Y \in \Gamma(\mathcal{D}_1)$ and $Z \in \Gamma(\ker F_*)$, assuming that $\mathcal{A}_X Y = 0$, we have

$$\begin{aligned} g_M([X,Y], \omega Z) = g_M([X,Y], JZ) &= -g_M(J(\nabla_X Y - \nabla_Y X), Z) \\ &= -g_M(\mathcal{A}_X JY + \mathcal{H}\nabla_X JY - \mathcal{A}_Y JX - \mathcal{H}\nabla_Y JX, Z) \\ &= -g_M(\mathcal{A}_X JY - \mathcal{A}_Y JX, Z). \end{aligned}$$

Since $\omega(\ker F_*) = \mathcal{D}_2$, the result follows. □

In a similar way, we have:

Lemma 3.10. *Let (M, g_M, J) be a Kähler manifold and (N, g_N) a Riemannian manifold. Let $F : (M, g_M, J) \mapsto (N, g_N)$ be a v-semi-slant submersion. Then the slant distribution $\mathcal{D}_2$ is integrable if and only if we obtain*

$$\mathcal{A}_X Y = 0 \quad \textit{and} \quad P((\mathcal{A}_X BY - \mathcal{A}_Y BX) + \mathcal{H}(\nabla_X CY - \nabla_Y CX)) = 0$$

for $X, Y \in \Gamma(\mathcal{D}_2)$.

Lemma 3.11. *Let (M, g_M, J) be a Kähler manifold and (N, g_N) a Riemannian manifold. Let $F : (M, g_M, J) \mapsto (N, g_N)$ be a v-semi-slant submersion with the v-semi-slant angle θ. Assume that the tensor B is parallel. Given $Z \in \Gamma(\mathcal{D}_2)$ and $W \in \Gamma((\ker F_*)^\perp)$, we get*

$$\mathcal{A}_{CZ} CW = -\cos^2\theta \mathcal{A}_Z W.$$

Proof. Since the tensor B is parallel, from Corollary 3.4, we have

$$\mathcal{A}_Z CW = \phi \mathcal{A}_Z W \quad \text{for } Z, W \in \Gamma((\ker F_*)^\perp).$$

So,

$$\begin{aligned}\mathcal{A}_{CZ} CW &= \phi \mathcal{A}_{CZ} W = -\phi \mathcal{A}_W CZ = -\mathcal{A}_W C^2 Z \\ &= \cos^2\theta \mathcal{A}_W Z = -\cos^2\theta \mathcal{A}_Z W.\end{aligned}$$

□

Using Lemma 3.11 and Remark 3.6, we obtain:

Corollary 3.12. *Let (M, g_M, J) be a Kähler manifold and (N, g_N) a Riemannian manifold. Let $F : (M, g_M, J) \mapsto (N, g_N)$ be a proper v-slant submersion with the v-slant angle θ. Assume that the tensor B is parallel. Then we have*

$$trace \mathcal{A} = 0 \quad on\ (\ker F_*)^\perp.$$

Assume that the v-semi-slant angle θ is not equal to $\frac{\pi}{2}$ and define an endomorphism $\widehat{J}$ of $(\ker F_*)^\perp$ by

$$\widehat{J} := JP + \sec\theta CQ.$$

Then,

$$\widehat{J}^2 = -id \quad \text{on } (\ker F_*)^\perp. \tag{3.3}$$

From (3.3), we have:

Theorem 3.13. *Let F be a v-semi-slant submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N) with the v-semi-slant angle $\theta \in [0, \frac{\pi}{2})$. Then N is an even-dimensional manifold.*

Now we deal with the conditions for distributions to be totally geodesic foliations.

Proposition 3.14. *Let F be a v-semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then the distribution $\mathcal{D}_1$ defines a totally geodesic foliation if and only if*

$$\phi \mathcal{A}_X JY + B\mathcal{H}\nabla_X JY = 0 \ \textit{and}\ Q(\omega \mathcal{A}_X JY + C\mathcal{H}\nabla_X JY) = 0$$

for $X, Y \in \Gamma(\mathcal{D}_1)$.

Proof. Given $X, Y \in \Gamma(\mathcal{D}_1)$, we get

$$\begin{aligned}\nabla_X Y &= -J\nabla_X JY = -J(\mathcal{A}_X JY + \mathcal{H}\nabla_X JY) \\ &= -(\phi \mathcal{A}_X JY + \omega \mathcal{A}_X JY + B\mathcal{H}\nabla_X JY + C\mathcal{H}\nabla_X JY).\end{aligned}$$

Hence,

$$\begin{aligned}&\nabla_X Y \in \Gamma(\mathcal{D}_1) \\ \Leftrightarrow\ &\phi \mathcal{A}_X JY + B\mathcal{H}\nabla_X JY = 0 \text{ and } Q(\omega \mathcal{A}_X JY + C\mathcal{H}\nabla_X JY) = 0.\end{aligned}$$

□

In a similar way, we obtain:

Proposition 3.15. *Let F be a v-semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then the distribution $\mathcal{D}_2$ defines a totally geodesic foliation if and only if*

$$\phi(\mathcal{V}\nabla_X BY + \mathcal{A}_X CY) + B(\mathcal{A}_X BY + \mathcal{H}\nabla_X CY) = 0,$$
$$P(\omega(\mathcal{V}\nabla_X BY + \mathcal{A}_X CY) + C(\mathcal{A}_X BY + \mathcal{H}\nabla_X CY)) = 0$$

for $X, Y \in \Gamma(\mathcal{D}_2)$.

We also have the same results with the case of a semi-slant submersion [19]. We can prove them in the same way.

Theorem 3.16. *Let F be a v-semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then M is locally a Riemannian product manifold if and only if*

$$\omega(\widehat{\nabla}_X \phi Y + \mathcal{T}_X \omega Y) + C(\mathcal{T}_X \phi Y + \mathcal{H}\nabla_X \omega Y) = 0 \quad \textit{for } X, Y \in \Gamma(\ker F_*),$$

$$\phi(\mathcal{V}\nabla_Z BW + \mathcal{A}_Z CW) + B(\mathcal{A}_Z BW + \mathcal{H}\nabla_Z CW) = 0 \quad \textit{for } Z, W \in \Gamma((\ker F_*)^\perp).$$

Theorem 3.17. *Let F be a v-semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then F is a totally geodesic map if and only if*

$$\omega(\widehat{\nabla}_X \phi Y + \mathcal{T}_X \omega Y) + C(\mathcal{T}_X \phi Y + \mathcal{H}\nabla_X \omega Y) = 0,$$
$$\omega(\widehat{\nabla}_X BZ + \mathcal{T}_X CZ) + C(\mathcal{T}_X BZ + \mathcal{H}\nabla_X CZ) = 0$$

for $X, Y \in \Gamma(\ker F_*)$ and $Z \in \Gamma((\ker F_*)^\perp)$.

Remark 3.18. Let F be a Riemannian submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). By the properties of Riemannian submersion, the conditions for F to be totally geodesic are the same among a v-semi-slant submersion, a v-semi-invariant submersion, and a v-slant submersion.

Let $F : (M, g_M) \mapsto (N, g_N)$ be a Riemannian submersion. Then the map F is called a Riemannian submersion *with totally umbilical fibers* if

$$\mathcal{T}_X Y = g_M(X, Y)H \quad \text{for } X, Y \in \Gamma(\ker F_*), \tag{3.4}$$

where H is the mean curvature vector field of any fiber.

Then we obtain:

Lemma 3.19. *Let F be a v-semi-slant submersion with totally umbilical fibers from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then we have*

$$H \in \Gamma(\mathcal{D}_2).$$

Proof. Given $X, Y \in \Gamma(\mu)$ and $W \in \Gamma(\mathcal{D}_1)$, we get

$$\mathcal{T}_X JY + \widehat{\nabla}_X JY = \nabla_X JY = J\nabla_X Y = B\mathcal{T}_X Y + C\mathcal{T}_X Y + \phi\widehat{\nabla}_X Y + \omega\widehat{\nabla}_X Y.$$

Using (3.4), we easily obtain

$$g_M(X, JY)g_M(H, W) = -g_M(X, Y)g_M(H, JW).$$

Interchanging the role of X and Y, we get

$$g_M(Y, JX)g_M(H, W) = -g_M(Y, X)g_M(H, JW)$$

so that combining the above two equations, we have

$$g_M(X, Y)g_M(H, JW) = 0,$$

which means $H \in \Gamma(\mathcal{D}_2)$. □

Corollary 3.20. *Let F be a v-semi-slant submersion with totally umbilical fibers from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N) such that $\mathcal{D}_1 = (\ker F_*)^\perp$. Then the fibers of F are minimal submanifolds of M.*

Remark 3.21. Let F be a v-semi-slant submersion with totally umbilical fibers from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N) such that $\mathcal{D}_1 = (\ker F_*)^\perp$. Then we get a family $\{F^{-1}(y) \,|\, y \in N\}$ of minimal submanifolds of M.

4. Curvature tensors

Let F be a v-semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then we can take a distribution $\mathcal{D}_1 \subset (\ker F_*)^\perp$ such that

$$(\ker F_*)^\perp = \mathcal{D}_1 \oplus \mathcal{D}_2, \; J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta = \theta(X)$ between JX and the space $(\mathcal{D}_2)_p$ is constant for nonzero $X \in (\mathcal{D}_2)_p$ and $p \in M$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $(\ker F_*)^\perp$.

Moreover,

$$C\mathcal{D}_2 \subset \mathcal{D}_2, \quad B\mathcal{D}_2 \subset \ker F_*, \quad \ker F_* = B\mathcal{D}_2 \oplus \mu,$$

where μ is the orthogonal complement of $B\mathcal{D}_2$ in $\ker F_*$ and is J-invariant. For the curvature tensor in a Kähler manifold, it is sufficient to deal with only the holomorphic sectional curvatures.

Given a J-invariant plane P in T_pM, $p \in M$, there is an orthonormal basis $\{X, JX\}$ of P. Denote by $K(P)$, $K_*(P)$, and $\widehat{K}(P)$ the sectional curvatures of the plane P in M, N, and the fiber $F^{-1}(F(p))$, respectively, where $K_*(P)$ denotes the sectional curvature of the plane $P_* = \langle F_*X, F_*JX \rangle$ in N. Denote by $K(X \wedge Y)$ the sectional curvature of the plane spanned by the tangent vectors $X, Y \in T_pM$, $p \in M$. Using both Corollary 1 of ([15], p. 465) and (1.28) of ([6], p. 13), we obtain

(1) If $P \subset (\mu)_p$, then we have

$$K(P) = \widehat{K}(P) + ||\mathcal{T}_X X||^2 - ||\mathcal{T}_X JX||^2 - g_M(\mathcal{T}_X X, J[JX, X]).$$

(2) If $P \subset (\mathcal{D}_2 \oplus B\mathcal{D}_2)_p$ with $X \in (\mathcal{D}_2)_p$, then we get

$$\begin{aligned} K(P) = & \sin^2\theta \cdot K(X \wedge BX) + 2(g_M((\nabla_X \mathcal{A})(X, CX), BX) \\ & + g_M(\mathcal{A}_X CX, \mathcal{T}_{BX} X) - g_M(\mathcal{A}_{CX} X, \mathcal{T}_{BX} X) \\ & - g_M(\mathcal{A}_X X, \mathcal{T}_{BX} CX)) + \cos^2\theta \cdot K(X \wedge CX). \end{aligned}$$

(3) If $P \subset (\mathcal{D}_1)_p$, then we obtain

$$K(P) = K_*(P) - 3||\mathcal{V} J \nabla_X X||^2. \tag{4.1}$$

Using (4.1), we have:

Theorem 4.1. *Let F be a v-semi-slant submersion from a Kähler manifold (M, g_M, J) onto a space $(N(c), g_N)$ of constant holomorphic sectional curvature c with $\dim \mathcal{D}_1 > 0$. Assume that the complex distribution $\mathcal{D}_1$ defines a totally geodesic foliation. Then we get*

$$K(P) = c \quad \textit{for any } J\textit{-invariant plane } P \subset \mathcal{D}_1.$$

Remark 4.2. By using Theorem 4.1, there does not exist a v-semi-slant submersion F from a Kähler manifold (M, g_M, J) onto a space $(N(c), g_N)$ of constant sectional curvature c such that the complex distribution $\mathcal{D}_1$ is a totally geodesic foliation, $\dim \mathcal{D}_1 > 0$, and $K(P) < c$ for some J-invariant plane $P \subset \mathcal{D}_1$.

We will introduce an inequality of a proper v-semi-slant submersion in terms of squared mean curvature, scalar curvature, and v-semi-slant angle.

Let $(M^n(c), g, J)$ be a space of constant holomorphic sectional curvature c with $\dim M^n(c) = 2n$ and $n \geq 2$ [13]. Then its Riemannian curvature tensor R is given by [13]

$$\begin{aligned} R(X, Y)Z = \frac{c}{4}\{ & g(Z, Y)X - g(Z, X)Y + g(Z, JY)JX \\ & - g(Z, JX)JY + 2g(X, JY)JZ\} \end{aligned}$$

for any vector fields X, Y, Z on $M^n(c)$.

Let F be a proper v-semi-slant submersion from a space $(M^n(c), g, J)$ of constant holomorphic sectional curvature c onto a Riemannian manifold (N^{2n-2}, g_N) with $\dim N^{2n-2} = 2n - 2$. Then since F is proper (i.e., $\theta \in (0, \frac{\pi}{2})$), we get

$$(\ker F_*)^\perp = \mathcal{D}_1 \oplus \mathcal{D}_2, \quad \ker F_* = B\mathcal{D}_2, \quad \dim(\ker F_*) = \dim \mathcal{D}_2 = 2$$

so that by Remark 3.6, there is a local orthonormal frame

$$\{X_1, JX_1, \ldots, X_{n-2}, JX_{n-2}, Y, \sec\theta CY\}$$

of $(\ker F_*)^\perp$ such that $\{X_1, JX_1, \ldots, X_{n-2}, JX_{n-2}\} \subset \Gamma(\mathcal{D}_1)$, $\{Y, \sec\theta CY\} \subset \Gamma(\mathcal{D}_2)$, and $\{\csc\theta BY, \csc\theta\sec\theta BCY\}$ is a local orthonormal frame of $\ker F_*$.

Denote by $\hat{\tau}$ and H the scalar curvature of any fiber and the mean curvature vector field of any fiber, respectively, i.e.,

$$\hat{\tau} = \widehat{K}(\ker F_*) = \csc^4\theta \sec^2\theta g(\widehat{R}(BY, BCY)BCY, BY)$$

and

$$H = \frac{1}{2}\csc^2\theta(\mathcal{T}_{BY}BY + \sec^2\theta\mathcal{T}_{BCY}BCY),$$

where $\widehat{R}$ is the Riemannian curvature tensor of any fiber.

Denote also by $||H||^2$ the squared mean curvature, i.e., $||H||^2 = g(H, H)$.

Theorem 4.3. *Let F be a proper v-semi-slant submersion from a space $(M^n(c), g, J)$ of constant holomorphic sectional curvature c onto a Riemannian manifold (N^{2n-2}, g_N) with $\dim N^{2n-2} = 2n-2$ and $n \geq 2$. Then we obtain*

$$||H||^2 \geq \frac{1}{2}\hat{\tau} - \frac{c}{8}(1 + 3\cos^2\theta)$$

with equality holding if and only if all the fibers are totally geodesic.

Proof. We will use the above notations.

Conveniently, let $e_1 := \csc\theta BY$ and $e_2 := \csc\theta\sec\theta BCY$.

Then we have

$$||H||^2 = \frac{1}{4}\{g(\mathcal{T}_{e_1}e_1, \mathcal{T}_{e_1}e_1) + g(\mathcal{T}_{e_2}e_2, \mathcal{T}_{e_2}e_2) + 2g(\mathcal{T}_{e_1}e_1, \mathcal{T}_{e_2}e_2)\}$$

and

$$\hat{\tau} = g(\widehat{R}(e_1, e_2)e_2, e_1) = \frac{c}{4}(1 + 3g(e_1, Je_2)^2) + g(\mathcal{T}_{e_1}e_1, \mathcal{T}_{e_2}e_2) - g(\mathcal{T}_{e_1}e_2, \mathcal{T}_{e_1}e_2).$$

Moreover, since $BC + \phi B = 0$ on $(\ker F_*)^\perp$, using Remark 3.6, we get

$$\begin{aligned} g(e_1, Je_2)^2 &= \csc^4\theta \cdot \sec^2\theta g(JBY, BCY)^2 \\ &= \csc^4\theta \cdot \sec^2\theta g(\phi BY, BCY)^2 \\ &= \csc^4\theta \cdot \sec^2\theta g(BCY, BCY)^2 \\ &= \cos^2\theta. \end{aligned}$$

Using the above equations, we obtain

$$||H||^2 = \frac{1}{2}\hat{\tau} - \frac{c}{8}(1 + 3\cos^2\theta) + \frac{1}{4}||\mathcal{T}_{e_1}e_1||^2 + \frac{1}{4}||\mathcal{T}_{e_2}e_2||^2 + \frac{1}{2}||\mathcal{T}_{e_1}e_2||^2.$$

Hence,

$$||H||^2 \geq \frac{1}{2}\hat{\tau} - \frac{c}{8}(1 + 3\cos^2\theta)$$

with equality holding if and only if $\mathcal{T} = 0$.

Therefore, the result follows. □

5. Examples

Example 5.1. Let (M, g_M, J) be an almost Hermitian manifold. Let $\pi : TM \mapsto M$ be the natural projection. Then the map π is a v-semi-slant submersion such that $\mathcal{D}_1 = (\ker\pi_*)^\perp$ [6].

Example 5.2. Let (M, g_M, J) be a $2m$-dimensional almost Hermitian manifold and (N, g_N) a $(2m-1)$-dimensional Riemannian manifold. Let F be a Riemannian submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then the map F is a v-semi-slant submersion such that

$$\mathcal{D}_1 = ((\ker F_*) \oplus J(\ker F_*))^\perp \quad \text{and} \quad \mathcal{D}_2 = J(\ker F_*)$$

with the v-semi-slant angle $\theta = \frac{\pi}{2}$.

Example 5.3. Define a map $F : \mathbb{R}^6 \mapsto \mathbb{R}^4$ by

$$F(x_1, x_2, \dots, x_6) = (x_1, x_3 \sin\alpha - x_5 \cos\alpha, x_6, x_2),$$

where $\alpha \in (0, \frac{\pi}{2})$. Then the map F is a v-semi-slant submersion such that

$$\mathcal{D}_1 = \langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \rangle \text{ and } \mathcal{D}_2 = \langle \frac{\partial}{\partial x_6}, \sin\alpha \frac{\partial}{\partial x_3} - \cos\alpha \frac{\partial}{\partial x_5} \rangle$$

with the v-semi-slant angle $\theta = \alpha$.

Furthermore, $\ker F_* = \langle \frac{\partial}{\partial x_4}, \cos\alpha \frac{\partial}{\partial x_3} + \sin\alpha \frac{\partial}{\partial x_5} \rangle$ and the map F is a slant submersion with the slant angle $\theta = \alpha$.

Example 5.4. Define a map $F : \mathbb{R}^8 \mapsto \mathbb{R}^4$ by

$$F(x_1, x_2, \dots, x_8) = (x_4, x_3, \frac{x_5 - x_8}{\sqrt{2}}, x_6).$$

Then the map F is a v-semi-slant submersion such that

$$\mathcal{D}_1 = \langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \rangle \text{ and } \mathcal{D}_2 = \langle \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_8} \rangle$$

with the v-semi-slant angle $\theta = \frac{\pi}{4}$.

Example 5.5. Define a map $F : \mathbb{R}^{12} \mapsto \mathbb{R}^5$ by

$$F(x_1, x_2, \dots, x_{12}) = (x_2, \frac{x_5 + x_6}{\sqrt{2}}, \frac{x_7 + x_9}{\sqrt{2}}, \frac{x_8 + x_{10}}{\sqrt{2}}, x_1).$$

Then the map F is a v-semi-slant submersion such that

$$\mathcal{D}_1 = \langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_8} + \frac{\partial}{\partial x_{10}} \rangle \text{ and } \mathcal{D}_2 = \langle \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6} \rangle$$

with the v-semi-slant angle $\theta = \frac{\pi}{2}$.

Example 5.6. Define a map $F : \mathbb{R}^{10} \mapsto \mathbb{R}^6$ by

$$F(x_1, x_2, \dots, x_{10}) = (\frac{x_3 - x_5}{\sqrt{2}}, x_6, \frac{x_7 + x_9}{\sqrt{2}}, x_8, x_1, x_2).$$

Then the map F is a v-semi-slant submersion such that

$$\mathcal{D}_1 = \langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \rangle \text{ and } \mathcal{D}_2 = \langle \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_9} \rangle$$

with the v-semi-slant angle $\theta = \frac{\pi}{4}$.

Example 5.7. Define a map $F : \mathbb{R}^8 \mapsto \mathbb{R}^4$ by

$$F(x_1, x_2, \dots, x_8) = (x_1, x_3 \cos\alpha - x_5 \sin\alpha, x_2, x_4 \sin\beta + x_6 \cos\beta),$$

where α and β are constants. Then the map F is a v-semi-slant submersion such that

$$\mathcal{D}_1 = \langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \rangle \text{ and } \mathcal{D}_2 = \langle \cos\alpha \frac{\partial}{\partial x_3} - \sin\alpha \frac{\partial}{\partial x_5}, \sin\beta \frac{\partial}{\partial x_4} + \cos\beta \frac{\partial}{\partial x_6} \rangle$$

with the v-semi-slant angle θ satisfying $\cos\theta = |\sin(\alpha - \beta)|$.

6. Open questions

We investigated some properties on a v-semi-slant submersion

$$F : (M, g_M, J) \mapsto (N, g_N).$$

In particular, we studied the integrabilities of distributions and the totally geodesicness of distributions.

As future projects, we have:

Question.

(1) Let $F : (M, g_M, J) \mapsto (N, g_N)$ be a v-semi-slant submersion with the v-semi-slant angle θ.

Then

(a) Can we give a characterization of a semi-slant angle θ?

(b) What kind of rigidity problems can we do on the map F?

(c) Using the map F, what are the advantages for studying complex geometry?

(2) In this paper, we only studied the properties of v-semi-slant submersions $F : (M, g_M, J) \mapsto (N, g_N)$.

So, as future works, we need to investigate the properties of v-semi-invariant submersions, v-slant submersion, and v-anti-invariant submersions (i.e., $\mathcal{D}_2 = (\ker F_*)^\perp$ and $J((\ker F_*)^\perp) \subset \ker F_*$) (See Definition 3 and Remark 3.1).

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