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CURVATURES OF SEMI-SYMMETRIC METRIC CONNECTIONS ON STATISTICAL MANIFOLDS

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Abstract. By using a statistical connection, we define a semi-symmetric metric connection on statistical manifolds and study the geometry of these manifolds and their submanifolds. We show the symmetry properties of the curvature tensor with respect to the semi-symmetric metric connections. Also, we prove the induced connection on a submanifold with respect to a semi-symmetric metric connection is a semi-symmetric metric connection and the second fundamental form coincides with the second fundamental form of the Levi-Civita connection. Furthermore, we obtain the Gauss, Codazzi and Ricci equations with respect to the new connection. Finally, we construct non-trivial examples of statistical manifolds admitting a semi-symmetric metric connection.

1. Introduction

As a generalization of the Riemannian connection, the notion of a semi-symmetric connection was introduced in [7]. This type of connections is a linear connection whose torsion tensor does not vanish, and for a 1-form η, satisfies $T(X,Y)=\eta(Y)X-\eta(X)Y$. In [13], K. Yano studied a semi-symmetric metric connection and proved some interesting results. In [1], the authors defined a semi-symmetric non-metric connection and investigated the curvature tensor of the manifold with respect to the semi-symmetric non-metric connection. Many authors studied manifolds endowed with the semi-symmetric, quarter-symmetric non-metric connections equipped with the complex and contact structures [6, 8–11].

On the other hand, statistical manifolds were studied in terms of information geometry by Amari [2]. Statistical manifolds are equipped with dual affine and torsion free connections which are related to each other with respect to the Riemannian metric g. Many authors initiated the study of geometry of submanifolds of statistical manifolds [4, 5, 12].

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In this paper, we consider a statistical manifold endowed with a semi-symmetric metric connection. First, we give a brief information about the statistical manifolds and their submanifolds. In Section 3, we study a semi-symmetric metric connection on a statistical manifold. Also, the curvature tensor with respect to the semi-symmetric metric connection and its symmetry properties are obtained. In Section 4, we deduce the Gauss and Weingarten formulas with respect to a semi-symmetric metric connection. Moreover, we prove that the induced connection on a submanifold is also semi-symmetric metric and the corresponding second fundamental form coincides with the second fundamental form with respect to the Levi-Civita connection. In Section 5, the Gauss, Codazzi and Ricci equations with respect to a semi-symmetric metric connection are obtained. Furthermore, we give some examples of semi-symmetric connections on statistical manifolds.

2. Preliminaries

Let $(\bar{M}, g)$ be an m-dimensional Riemannian manifold and $\hat{\nabla}$ be the Levi-Civita connection on $\bar{M}$.

Definition ([2])**.** A Riemannian manifold $(\bar{M}, g, \bar{\nabla})$ is said to be a statistical manifold if $\bar{\nabla}$ is an affine and torsion free connection and $\bar{\nabla}g$ satisfies in Codazzi equation, that is, for all $X, Y, Z \in \Gamma(T\bar{M})$

$$(\bar{\nabla}_X g)(Y, Z) = (\bar{\nabla}_Y g)(X, Z). \tag{1}$$

It is well-known that there exists an affine connection $\bar{\nabla}^*$ dual of $\bar{\nabla}$ with respect to g such that

$$Xg(Y, Z) = g(\bar{\nabla}_X Y, Z) + g(Y, \bar{\nabla}^*_X Z). \tag{2}$$

Also $\bar{\nabla}^*$ satisfies in (1) and $(\bar{\nabla}^*)^* = \bar{\nabla}$. From compatibility of $\hat{\nabla}$ with g and Equation (2), we obtain [5]

$$\hat{\nabla} = \frac{1}{2}(\bar{\nabla} + \bar{\nabla}^*). \tag{3}$$

A tensor field $\bar{K}$ of type $(1, 2)$ on $\bar{M}$ is defined

$$\bar{K}_X Y = \bar{\nabla}_X Y - \hat{\nabla}_X Y, \qquad \bar{K}_X Y = \frac{1}{2}(\bar{\nabla}_X Y - \bar{\nabla}^*_X Y). \tag{4}$$

$\bar{K}$ is symmetric and we have

$$g(\bar{K}_X Y, Z) = g(\bar{K}_X Z, Y), \qquad \bar{K}_X Y = \bar{K}_Y X. \tag{5}$$

The statistical curvature tensor field with respect to $\bar{\nabla}$ is defined [4]

$$\bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z. \tag{6}$$

By changing $\bar{\nabla}$ to $\bar{\nabla}^*$ we obtain the statistical curvature tensor field $\bar{R}^*$. The curvature tensor fields $\bar{R}$ and $\bar{R}^*$ satisfy

$$\bar{R}(X, Y)Z = -\bar{R}(Y, X)Z, \qquad \bar{R}^*(X, Y)Z = -\bar{R}^*(Y, X)Z,$$

$$g(\bar{R}(X,Y)Z,W) = -g(\bar{R}^*(X,Y)W,Z),$$

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0.$$

Let N be a submanifold of statistical manifold $\bar{M}$ with the induced metric g. The Gauss and Weingarten formulas for the Levi-Civita connection are

$$\hat{\nabla}_X Y = \nabla^\circ_X Y + h^\circ(X,Y), \qquad \hat{\nabla}_X V = -A^\circ_V X + D^\circ_X V, \tag{7}$$

for all $X, Y \in \Gamma(TN)$ and $V \in \Gamma(TN^\perp)$, where ∇° and h° are the induced connection and the second fundamental form on N, respectively. A° is the shape operator and D° is the normal connection on $TN^\perp$. Now, the Gauss and Weingarten formulas for submanifold N of statistical manifold $\bar{M}$ with respect to the statistical connections $\bar{\nabla}$ and $\bar{\nabla}^*$ are given by [4]

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \qquad \bar{\nabla}_X V = -A_V X + D_X V, \tag{8}$$

$$\bar{\nabla}^*_X Y = \nabla^*_X Y + h^*(X,Y), \qquad \bar{\nabla}^*_X V = -A^*_V X + D^*_X V, \tag{9}$$

for all $X, Y \in \Gamma(TN)$ and $V \in \Gamma(TN^\perp)$, where ∇, ∇^* and h, h^* are induced statistical connections and second fundamental forms on N, respectively. A, A^* are the shape operators and D, D^* are the normal connections on $TN^\perp$. It is well-known that ∇ and ∇^* are dual and statistical connections [5]. From (8) and (9) we have

$$g(A_V X, Y) = g(h(X,Y), V), \qquad g(A^*_V X, Y) = g(h^*(X,Y), V). \tag{10}$$

N is called a totally geodesic submanifold with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$ if the second fundamental forms h and h^* vanish. The submanifold is called a totally umbilical submanifold if

$$h(X,Y) = Hg(X,Y), \qquad h^*(X,Y) = H^*g(X,Y),$$

where H and H^* are the mean curvature vectors with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$, respectively.

3. Semi-symmetric metric connections on statistical manifolds

A linear connection $\tilde{\nabla}$ on $(\bar{M}, g)$ is called a semi-symmetric connection if for all $X, Y \in \Gamma(T\bar{M})$, its torsion tensor $\tilde{T}$ satisfies

$$\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] = \eta(Y)X - \eta(X)Y, \tag{11}$$

where η is a 1-form and for a vector field U

$$g(X,U) = \eta(X).$$

Moreover, if the semi-symmetric connection $\tilde{\nabla}$ satisfies $\tilde{\nabla} g = 0$, then $\tilde{\nabla}$ is said to be a semi-symmetric metric connection.

By using the approach of [9], we give the following definition.

Definition. Let $(\bar{M}, \bar{\nabla}, g)$ be a statistical manifold and U be a vector field on $\bar{M}$. For any $X, Y \in \Gamma(T\bar{M})$, we define the linear connection $\tilde{\nabla}$ on $\bar{M}$ by

$$\tilde{\nabla}_X Y = \bar{\nabla}_X Y + \eta(Y)X - g(X,Y)U - \bar{K}_X Y, \tag{12}$$

where $g(X, U) = \eta(X)$.

By taking (4) in (12) we get

$$\tilde{\nabla}_X Y = \bar{\nabla}^*_X Y + \eta(Y)X - g(X,Y)U + \bar{K}_X Y. \tag{13}$$

It is easy to see that the torsion tensor $\tilde{T}$ with respect to the linear connection $\tilde{\nabla}$ satisfies in (11).

Proposition 3.1. *Let $(\bar{M}, \bar{\nabla}, g)$ be a statistical manifold admitting a semi-symmetric linear connection $\tilde{\nabla}$ which is defined in* (12). *Then $\tilde{\nabla}$ is a metric connection.*

Proof. For all X, Y, Z on $\bar{M}$ from (2), (5) and (12) we have

$$\begin{aligned}
(\tilde{\nabla}_X g)(Y,Z) &= Xg(Y,Z) - g(\tilde{\nabla}_X Y, Z) - g(Y, \tilde{\nabla}_X Z) \\
&= Xg(Y,Z) - g(\bar{\nabla}_X Y + \eta(Y)X - g(X,Y)U - \bar{K}_X Y, Z) \\
&\quad - g(\bar{\nabla}_X Z + \eta(Z)X - g(X,Z)U - \bar{K}_X Z, Y) \\
&= g(Y, \bar{\nabla}^*_X Z) - g(Y, \bar{\nabla}_X Z) + 2g(K_X Z, Y) \\
&= -2g(K_X Z, Y) + 2g(K_X Z, Y) = 0.
\end{aligned}$$

It gives the assertion. □

The previous proposition shows that $\tilde{\nabla}$ is a semi-symmetric metric connection. Now, we prove any semi-symmetric metric connection on a statistical manifold satisfies in (12).

Proposition 3.2. *Let $(\bar{M}, \bar{\nabla}, g)$ be a statistical manifold which admits a semi-symmetric metric connection $\tilde{\nabla}$. Then $\tilde{\nabla}$ satisfies in* (12) *and* (13).

Proof. Let $\tilde{\nabla}$ be a metric connection satisfying (11) on a statistical manifold $\bar{M}$ defined by

$$\tilde{\nabla}_X Y = \bar{\nabla}_X Y + m(X,Y), \tag{14}$$

where $\bar{\nabla}$ is a statistical connection and m is a $(1,2)$-tensor field on $\bar{M}$. From (2) and (14) we get

$$\begin{aligned}
0 = (\tilde{\nabla}_X g)(Y,Z) &= Xg(Y,Z) - g(\tilde{\nabla}_X Y, Z) - g(Y, \tilde{\nabla}_X Z) \\
&= Xg(Y,Z) - g(\bar{\nabla}_X Y + m(X,Y), Z) - g(Y, \bar{\nabla}_X Z + m(X,Z)) \\
&= -2g(K_X Z, Y) - g(m(X,Y), Z) - g(m(X,Z), Y).
\end{aligned}$$

So

$$g(m(X,Y), Z) + g(m(X,Z), Y) = -2g(K_X Z, Y).$$

Now, from (14) we have

$$\tilde{T}(X,Y) = m(X,Y) - m(Y,X).$$

By using (11) we obtain

$$\begin{aligned}
&g(\tilde{T}(X,Y),Z) + g(\tilde{T}(Z,X),Y) + g(\tilde{T}(Z,Y),X) \\
&= g(m(X,Y) - m(Y,X),Z) + g(m(Z,X),Y) - m(X,Z),Y) \\
&\quad + g(m(Z,Y) - m(Y,Z),X) \\
&= 2(g(m(X,Y),Z) + g(\bar{K}_X Z, Y)).
\end{aligned}$$

Substituting (11) in the last equation implies

$$\begin{aligned}
g(m(X,Y),Z) = \frac{1}{2}\{&g(\eta(Y)X - \eta(X)Y, Z) + g(\eta(X)Z - \eta(Z)X, Y) \\
&+ g(\eta(Y)Z - \eta(Z)Y, X)\} - g(\bar{K}_X Z, Y).
\end{aligned}$$

Thus we get

$$m(X,Y) = \eta(Y)X - g(X,Y)U - \bar{K}_X Y.$$

By taking the Equations (4) and (12), we get (13). □

Example 3.3. We recall Example 2.2 in [12] for a 5-dimensional statistical manifold $\bar{M}$ with standard coordinate (x_1, x_2, y_1, y_2, z). We consider the metric g and the conjugate connections $\bar{\nabla}$ and $\bar{\nabla}^*$ defined in that example. Assume $U = \partial z$ and $\eta(X) = g(X,U)$ for all $X \in \Gamma(T\bar{M})$. We define an affine connection $\tilde{\nabla}$ as follows

$$\tilde{\nabla}_{\partial x_1}\partial x_1 = y_1\partial y_1 - (2+y_1^2)\partial z - y_1\partial x_1, \tilde{\nabla}_{\partial x_2}\partial x_2 = y_2\partial y_2 - (2+y_2^2)\partial z - y_2\partial x_2,$$

$$\tilde{\nabla}_{\partial x_2}\partial x_1 = \frac{1}{2}y_1\partial y_2 + \frac{1}{2}y_2\partial y_1 - y_1y_2\partial z - y_1\partial x_2,$$

$$\tilde{\nabla}_{\partial x_1}\partial x_2 = \frac{1}{2}y_1\partial y_2 + \frac{1}{2}y_2\partial y_1 - y_1y_2\partial z - y_2\partial x_1,$$

$$\tilde{\nabla}_{\partial x_1}\partial y_1 = y_1\partial x_1 - \frac{3}{4}y_1\partial x_2 + \frac{1}{4}(y_1^2 - 2)\partial z - y_1\partial y_1,$$

$$\tilde{\nabla}_{\partial y_1}\partial x_1 = y_1\partial x_1 - \frac{3}{4}y_1\partial x_2 + \frac{1}{4}(y_1^2 - 2)\partial z,$$

$$\tilde{\nabla}_{\partial y_2}\partial x_1 = \frac{1}{4}(y_1\partial x_2 + y_1y_2\partial z) - y_1\partial y_2, \tilde{\nabla}_{\partial x_1}\partial y_2 = \frac{1}{4}(y_1\partial x_2 + y_1y_2\partial z),$$

$$\tilde{\nabla}_{\partial y_1}\partial x_2 = \frac{1}{4}(y_2\partial x_1 + y_1y_2\partial z) - y_2\partial y_1, \tilde{\nabla}_{\partial x_2}\partial y_1 = \frac{1}{4}(y_2\partial x_1 + y_1y_2\partial z),$$

$$\tilde{\nabla}_{\partial y_2}\partial x_2 = \frac{1}{4}(\partial x_2 + (y_2^2 - 2)\partial z) - y_2\partial y_2, \tilde{\nabla}_{\partial x_2}\partial y_2 = \frac{1}{4}(\partial x_2 + (y_2^2 - 2)\partial z),$$

$$\tilde{\nabla}_{\partial z}\partial x_1 = \frac{-1}{2}\partial y_1, \qquad \tilde{\nabla}_{\partial x_1}\partial z = \frac{-1}{2}\partial y_1 + \partial x_1 + y_1\partial z,$$

$$\tilde{\nabla}_{\partial z}\partial x_2 = \frac{-1}{2}\partial y_2, \qquad \tilde{\nabla}_{\partial x_1}\partial z = \frac{-1}{2}\partial y_2 + \partial x_2 + y_2\partial z,$$

$$\tilde{\nabla}_{\partial z}\partial y_1 = \frac{-1}{4}(\partial x_1 + y_1\partial z), \quad \tilde{\nabla}_{\partial y_1}\partial z = \frac{-1}{4}(\partial x_1 + y_1\partial z) + \partial y_1,$$

$$\tilde{\nabla}_{\partial z}\partial y_2 = \frac{-1}{4}(\partial x_2 + y_2\partial z), \quad \tilde{\nabla}_{\partial y_2}\partial z = \frac{-1}{4}(\partial x_2 + y_2\partial z) + \partial y_2,$$
$$\tilde{\nabla}_{\partial y_1}\partial y_1 = \tilde{\nabla}_{\partial y_2}\partial y_2 = \partial z, \qquad \tilde{\nabla}_{\partial y_2}\partial y_1 = \tilde{\nabla}_{\partial y_1}\partial y_2 = \tilde{\nabla}_{\partial z}\partial z = 0.$$

So, we get the torsion tensor $\tilde{T}$ with respect to the connection $\tilde{\nabla}$ as follows:

$$\tilde{T}(\partial x_1, \partial x_2) = -y_2\partial x_1 + y_1\partial x_2, \quad \tilde{T}(\partial x_1, \partial y_1) = y_1\partial y_1, \quad \tilde{T}(\partial x_2, \partial y_1) = y_2\partial y_1,$$
$$\tilde{T}(\partial x_1, \partial y_2) = y_1\partial y_2, \qquad \tilde{T}(\partial x_2, \partial y_1) = y_2\partial y_1, \qquad \tilde{T}(\partial x_2, \partial y_2) = y_2\partial y_2,$$
$$\tilde{T}(\partial x_1, \partial z) = \partial x_1 + y_1\partial z, \qquad \tilde{T}(\partial x_2, \partial z) = \partial x_2 + y_2\partial z,$$
$$\tilde{T}(\partial y_1, \partial z) = \partial y_1, \qquad \tilde{T}(\partial y_2, \partial z) = \partial y_2.$$

Hence $\tilde{\nabla}$ is a semi-symmetric connection and the relation (11) is satisfied. Moreover, it is easy to see $\tilde{\nabla}g = 0$. So, $\tilde{\nabla}$ is a semi-symmetric metric connection on the statistical manifold $\bar{M}$.

We denote the curvature tensor associated with the semi-symmetric metric connection $\tilde{\nabla}$ by $\tilde{R}$. From [9], the curvature tensor $\tilde{R}$ is related to statistical connections $\bar{\nabla}$ and $\bar{\nabla}^*$ by the following relations

$$\begin{aligned}\tilde{R}(X,Y)Z = {} & \bar{R}(X,Y)Z + \{\eta(X)U - X - \bar{\nabla}_X U + \bar{K}_X U\}g(Y,Z)\\ & - \{\eta(Y)U - Y - \bar{\nabla}_Y U + \bar{K}_Y U\}g(X,Z)\\ & - (\bar{\nabla}_X\bar{K})(Y,Z) + (\bar{\nabla}_Y\bar{K})(X,Z) + \bar{K}_X\bar{K}(Y,Z) - \bar{K}_Y\bar{K}(X,Z)\\ & - g(\eta(X)U - \bar{\nabla}_X U + \bar{K}_X U, Z)Y + g(\eta(Y)U - \bar{\nabla}_Y U + \bar{K}_Y U, Z)X, \end{aligned} \tag{15}$$

and

$$\begin{aligned}\tilde{R}(X,Y)Z = {} & \bar{R}^*(X,Y)Z + \{\eta(X)U - X - \bar{\nabla}^*_X U - \bar{K}_X U\}g(Y,Z)\\ & - \{\eta(Y)U - Y - \bar{\nabla}^*_Y U - \bar{K}_Y U\}g(X,Z)\\ & + (\bar{\nabla}^*_X\bar{K})(Y,Z) - (\bar{\nabla}^*_Y\bar{K})(X,Z) + \bar{K}_X\bar{K}(Y,Z) - \bar{K}_Y\bar{K}(X,Z)\\ & - g(\eta(X)U - \bar{\nabla}^*_X U - \bar{K}_X U, Z)Y + g(\eta(Y)U - \bar{\nabla}^*_Y U - \bar{K}_Y U, Z)X. \end{aligned} \tag{16}$$

For investigating the symmetry of curvature tensor $\tilde{R}$, we need the following proposition.

Proposition 3.4. *Let $(\bar{M}, \bar{\nabla}, g)$ be a statistical manifold. The following relations hold*:

1) $-g(\bar{\nabla}_X U, W) + g(\bar{K}_X U, W) = -g(\bar{\nabla}^*_X U, W) - g(\bar{K}_X U, W)$,
2) $g((\bar{\nabla}_Y\bar{K})(X,Z), W) - g((\bar{\nabla}_X\bar{K})(Y,Z), W) = g((\bar{\nabla}^*_Y\bar{K})(X,W), Z) - g((\bar{\nabla}^*_X\bar{K})(Y,W), Z)$,
3) $g(\bar{K}_X\bar{K}_Y Z, W) = g(\bar{K}_Y\bar{K}_X W, Z)$.

Proof. From (4), by direct computations, we get 1).

2) From (5), we prove that

$$\begin{aligned} & g(\bar{K}_{\bar{\nabla}_Y X}Z, W) - g(\bar{K}_{\bar{\nabla}_X Y}Z, W) = g(\bar{K}_{[Y,X]}Z, W)\\ & = g(\bar{K}_{[Y,X]}W, Z) = g(\bar{K}_{\bar{\nabla}^*_Y X}W, Z) - g(\bar{K}_{\bar{\nabla}^*_X Y}W, Z). \end{aligned} \tag{17}$$

Now, by using (2) and (5), we get

$$\begin{aligned}
& g((\bar{\nabla}_Y \bar{K})(X,Z),W) \\
&= g(\bar{\nabla}_Y \bar{K}_X Z, W) - g(\bar{K}_X \bar{\nabla}_Y Z, W) - g(\bar{K}_{\bar{\nabla}_Y X} Z, W) \\
&= Y g(\bar{K}_X Z, W) - g(\bar{K}_X Z, \bar{\nabla}^*_Y W) - g(\bar{K}_X W, \bar{\nabla}_Y Z) - g(\bar{K}_{\bar{\nabla}_Y X} Z, W) \\
&= Y g(\bar{K}_X Z, W) - g(\bar{K}_X \bar{\nabla}^*_Y W, Z) - Y g(\bar{K}_X W, Z) + g(Z, \bar{\nabla}^*_Y \bar{K}_X W) \\
&\quad - g(\bar{K}_{\bar{\nabla}_Y X} Z, W) \\
&= g(Z, \bar{\nabla}^*_Y \bar{K}_X W) - g(\bar{K}_X \bar{\nabla}^*_Y W, Z) - g(\bar{K}_{\bar{\nabla}_Y X} Z, W), \qquad (18)
\end{aligned}$$

from (17) and (18), we obtain

$$\begin{aligned}
& g((\bar{\nabla}_Y \bar{K})(X,Z),W) - g((\bar{\nabla}_X \bar{K})(Y,Z),W) \\
&= g(Z, \bar{\nabla}^*_Y \bar{K}_X W) - g(\bar{K}_X \bar{\nabla}^*_Y W, Z) - g(\bar{K}_{\bar{\nabla}^*_Y X} W, Z) - g(Z, \bar{\nabla}^*_X \bar{K}_Y W) \\
&\quad + g(\bar{K}_Y \bar{\nabla}^*_X W, Z) + g(\bar{K}_{\bar{\nabla}^*_X Y} W, Z) \\
&= g((\bar{\nabla}^*_Y \bar{K})(X,W),Z) - g((\bar{\nabla}^*_X \bar{K})(Y,W),Z).
\end{aligned}$$

3) From symmetry property of $\bar{K}$, we deduce

$$g(\bar{K}_X \bar{K}_Y Z, W) = g(\bar{K}_X W, \bar{K}_Y Z) = g(\bar{K}_Y \bar{K}_X W, Z). \qquad \square$$

The next proposition expresses the symmetry of curvature tensor $\tilde{R}$ associated with the semi-symmetric metric connection $\tilde{\bar{\nabla}}$.

Proposition 3.5. *Let $(\bar{M}, \bar{\nabla}, g)$ be a statistical manifold admitting the semi-symmetric metric connection $\tilde{\bar{\nabla}}$. Then the curvature tensor $\tilde{R}$ associated with $\tilde{\bar{\nabla}}$ satisfies the following conditions*:

1) $\tilde{R}(X,Y)Z = -\tilde{R}(Y,X)Z$,
2) $g(\tilde{R}(X,Y)Z,W) = -g(\tilde{R}(X,Y)W,Z)$,
3) $$\begin{aligned}\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y &= g(\bar{\nabla}_X U, Z)Y - g(\bar{\nabla}_Y U, Z)X \\ &\quad + g(\bar{\nabla}_Y U, X)Z - g(\bar{\nabla}_Z U, X)Y \\ &\quad + g(\bar{\nabla}_Z U, Y)X - g(\bar{\nabla}_X U, Y)Z.\end{aligned}$$

Proof. By a simple computation and using (15), we get 1). In view of (15), (16) and Proposition 3.4, we get the relation 2). For 3), by cycling $\tilde{R}$ on X, Y, Z and direct calculating we get the result. $\square$

Corollary 3.6. *Let $(\bar{M}, \bar{\nabla}, g)$ be a statistical manifold admitting the semi-symmetric metric connection $\tilde{\bar{\nabla}}$. If η is closed, then*

$$\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 0.$$

Proof. By using (2), we derive

$$\begin{aligned}
g(\bar{\nabla}_X U, Z)Y - g(\bar{\nabla}_Z U, X)Y &= X g(U,Z)Y - g(U, \bar{\nabla}^*_X Z)Y \\
&\quad - Z g(U,X)Y + g(U, \bar{\nabla}^*_Z X)Y \\
&= (X\eta(Z) - Z\eta(X) - \eta[X,Z])Y = d\eta(X,Z)Y.
\end{aligned}$$

Now, from 3) in Proposition 3.5 we conclude

$$\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = d\eta(X,Z)Y + d\eta(Y,X)Z + d\eta(Z,Y)X,$$

which this equation gives the result. □

4. Induced connection of a semi-symmetric metric connection on submanifolds of statistical manifolds

In this section we obtain the Gauss and Weingarten formulas for a semi-symmetric metric connection.

We consider ∇' and h' as the induced connection and the second fundamental form on the submanifold N with respect to the semi-symmetric metric connection $\tilde{\nabla}$, respectively. So, the Gauss formula with respect to semi-symmetric metric connection $\tilde{\nabla}$ is

$$\tilde{\nabla}_X Y = \nabla'_X Y + h'(X,Y). \tag{19}$$

The submanifold N is called totally geodesic with respect to $\tilde{\nabla}$ if the second fundamental form h' vanishes and N is called totally umbilical if we have

$$h'(X,Y) = H' g(X,Y),$$

where, H' is the mean curvature vector with respect to $\tilde{\nabla}$.

In the next theorem we assume $U \in \Gamma(TN)$ and obtain the relations between ∇' and ∇.

Theorem 4.1. *Let N be a submanifold of statistical manifold $\bar{M}$ such that $\bar{M}$ admits a semi-symmetric metric connection $\tilde{\nabla}$ and $U \in \Gamma(TN)$. Then we have*

$$\nabla'_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)U - K_X Y, \quad \forall X,Y \in \Gamma(TN) \tag{20}$$

$$h'(X,Y) = \frac{1}{2}(h(X,Y) + h^*(X,Y)), \tag{21}$$

where $K_X Y = \frac{1}{2}(\nabla - \nabla^*)$.

Proof. Applying (12) and Gauss formula in (8) we get

$$\begin{aligned}
\tilde{\nabla}_X Y &= \bar{\nabla}_X Y + \eta(Y)X - g(X,Y)U - \bar{K}_X Y \\
&= \nabla_X Y + h(X,Y) + \eta(Y)X - g(X,Y)U \\
&\quad - \frac{1}{2}(\nabla_X Y + h(X,Y) - \nabla^*_X Y - h^*(X,Y)) \\
&= \nabla_X Y + \eta(Y)X - g(X,Y)U - K_X Y + \frac{1}{2}(h(X,Y) + h^*(X,Y)).
\end{aligned} \tag{22}$$

By separating the tangential and normal parts we get the result. □

Remark 4.2. By similar proof of Theorem 4.1, we can show

$$\nabla'_X Y = \nabla^*_X Y + \eta(Y)X - g(X,Y)U + K_X Y, \qquad \forall X,Y \in \Gamma(TN).$$

By using (3), (19), (20) and (21), we have the following corollaries.

Corollary 4.3. *Let N be a submanifold of statistical manifold $\bar{M}$ such that $\bar{M}$ admits a semi-symmetric metric connection $\tilde{\nabla}$. Then the induced connection ∇' of the semi-symmetric metric connection $\tilde{\nabla}$ is also semi-symmetric connection and*

$$(\nabla'_X g)(Y, Z) = (\tilde{\nabla}_X g)(Y, Z).$$

Corollary 4.4. *Let N be a submanifold of statistical manifold $\bar{M}$ such that $\bar{M}$ admits a semi-symmetric metric connection $\tilde{\nabla}$. Then the second fundamental form with respect to the semi-symmetric metric connection $\tilde{\nabla}$ coincides with the second fundamental form of the Levi-Civita connection.*

Proposition 4.5. *Let N be a submanifold of statistical manifold $\bar{M}$ admitting a semi-symmetric metric connection $\tilde{\nabla}$. If N is totally umbilical with respect to the statistical connections then N is totally umbilical with respect to the semi-symmetric metric connection.*

Proof. Since N is totally umbilical with respect to the statistical connections, we have

$$h(X, Y) = Hg(X, Y), \qquad h^*(X, Y) = H^*g(X, Y),$$

from (3) and (20), we get

$$h'(X, Y) = \frac{1}{2}(h(X, Y) + h^*(X, Y)) = \frac{1}{2}(H + H^*)g(X, Y),$$

so, N is totally umbilical with respect to the semi-symmetric metric connection and the mean curvature vector with respect to $\tilde{\nabla}$ is

$$H'(X, Y) = \frac{1}{2}(H + H^*).$$

□

Theorem 4.6. *Let N be a submanifold of statistical manifold $\bar{M}$ such that $\bar{M}$ admits a semi-symmetric metric connection $\tilde{\nabla}$ and $U \in \Gamma(TN)$. Then for all $X, Y \in \Gamma(TN)$ and $V \in \Gamma(TN^{\perp})$, we have*

$$\tilde{\nabla}_X V = \frac{-1}{2}(A_V X + A^*_V X) + \frac{1}{2}(D_X V + D^*_X V).$$

Proof. Since $\eta(V) = g(U, V) = 0$, the Equations (12) and (8) imply

$$\begin{aligned} \tilde{\nabla}_X V &= \bar{\nabla}_X V + \eta(V)X - g(X, V)U - \bar{K}_X V \\ &= -A_V X + D_X V - \frac{1}{2}(\bar{\nabla}_X V - \bar{\nabla}^*_X V) \\ &= \frac{-1}{2}(A_V X + A^*_V X) + \frac{1}{2}(D_X V + D^*_X V). \qquad (23) \end{aligned}$$

□

Proposition 4.7. *Let N be a submanifold of statistical manifold $\bar{M}$ such that $\bar{M}$ admits a semi-symmetric metric connection $\tilde{\nabla}$ and $U \in \Gamma(TN)$. Then the shape operator and the normal connection associated with the semi-symmetric metric connection coincide with the shape operator and the normal connection of the Levi-Civita connection, respectively.*

Proof. If we take the Weingarten formula associated with the semi-symmetric metric connection by

$$\tilde{\nabla}_X V = -A'_V X + D'_X V, \tag{24}$$

equating the tangential and normal parts of (23) implies

$$A'_V X = \frac{1}{2}(A_V X + A^*_V X), \qquad D'_X V = \frac{1}{2}(D_X V + D^*_X V). \tag{25}$$

The assertions follows from (3) and (25). □

From (10), (21) and (25), we find

$$\begin{aligned} g(A'_V X, Y) = \frac{1}{2} g((A_V X + A^*_V X), Y) &= \frac{1}{2} g(h(X,Y) + h^*(X,Y), V) \\ &= g(h'(X,Y), V). \end{aligned} \tag{26}$$

Remark 4.8. In Theorem 4.1, if we take $U \in \Gamma(TN^{\perp})$, by the same proof we obtain the Gauss formula associated with the semi-symmetric metric connection $\tilde{\nabla}$ as follows

$$\tilde{\nabla}_X Y = \nabla_X Y - K_X Y - g(X,Y)U + \frac{1}{2}(h(X,Y) + h^*(X,Y)), \tag{27}$$

thus, $\nabla'_X Y = \nabla_X Y - K_X Y$. In view of (4) and (7), we find that

$$\nabla^{\circ}_X Y = \nabla_X Y - K_X Y.$$

So, when $U \in \Gamma(TN^{\perp})$ the induced connection ∇' on the submanifold coincides with the induced connection of the Levi-Civita connection.

Proposition 4.9. *Let N be a submanifold of statistical manifold $\bar{M}$ such that $\bar{M}$ admits a semi-symmetric metric connection $\tilde{\nabla}$ and $U \in \Gamma(TN^{\perp})$. Then h' is totally geodesic if and only if $h^{\circ}(X,Y) = g(X,Y)U$.*

Proof. By equating the normal part of (27) we get

$$h'(X,Y) = \frac{1}{2}(h(X,Y) + h^*(X,Y)) - g(X,Y)U, \ \forall X, Y \in \Gamma(TN)$$

from (7) we obtain

$$h'(X,Y) = h^{\circ}(X,Y) - g(X,Y)U,$$

this implies the result. □

5. The Gauss, Codazzi and Ricci equations with respect to the semi-symmetric metric connection

We denote the tangent and normal parts of the curvature tensor $\tilde{R}$ by R' and $R^{\perp}$, respectively. By direct computations from (19) and (24), we get

$$\tilde{\nabla}_X \tilde{\nabla}_Y Z = \nabla'_X \nabla'_Y Z + h'(X, \nabla'_Y Z) - A'_{h'(Y,Z)} X + D'_X h'(Y,Z), \tag{28}$$

by changing the role of X and Y in (28) we obtain $\tilde{\nabla}_Y \tilde{\nabla}_X Z$. So,

$$\tilde{R}(X,Y)Z = \nabla'_X \nabla'_Y Z + h'(X, \nabla'_Y Z) - A'_{h'(Y,Z)} X + D'_X h'(Y,Z)$$

$$
\begin{aligned}
&- \nabla'_Y \nabla'_X Z - h'(Y, \nabla'_X Z) + A'_{h'(X,Z)} Y - D'_Y h'(X, Z) \\
&- \nabla'_{[X,Y]} Z - h'([X, Y], Z). \qquad (29)
\end{aligned}
$$

Let $W \in \Gamma(TN)$, from (29) we obtain the Gauss equation with respect to the semi-symmetric metric connection as follows:

$$
\begin{aligned}
g(\tilde{R}(X, Y)Z, W) = {}& g(R'(X, Y)Z, W) - g(h'(X, W), h'(Y, Z)) \\
&+ g(h'(X, Z), h'(Y, W)). \qquad (30)
\end{aligned}
$$

Proposition 5.1. *Let N be a submanifold of statistical manifold $\bar{M}$ such that $\bar{M}$ admits a semi-symmetric metric connection $\tilde{\nabla}$ and $U \in \Gamma(TN)$. Then we have*

$$
\begin{aligned}
&g(\tilde{R}(X, Y)Z, W) \\
={}& g(R(X, Y)Z, W) \\
&+ \{\eta(X)\eta(W) - g(X, W) - g(\nabla_X U, W) + g(K_X U, W)\} g(Y, Z) \\
&- \{\eta(Y)\eta(W) - g(Y, W) - g(\nabla_Y U, W) + g(K_Y U, W)\} g(X, Z) \\
&- g((\nabla_X K)(Y, Z), W) + g((\nabla_Y K)(X, Z), W) + g(K_X K(Y, Z), W) \\
&- g(K_Y K(X, Z), W) - g(\eta(X)U - \nabla_X U + K_X U, Z) g(Y, W) \\
&+ g(\eta(Y)U - \nabla_Y U + K_Y U, Z) g(X, W) \\
&- \frac{1}{4} g(h(X, W) + h^*(X, W), h(Y, Z) + h^*(Y, Z)) \\
&+ \frac{1}{4} g(h(X, Z) + h^*(X, Z), h(Y, W) + h^*(Y, W)),
\end{aligned}
$$

where R is the curvature tensor of the induced statistical connection ∇ on N.

Proof. The assertion follows from (20), (21) and (30). □

Proposition 5.2. *Let N be a submanifold of statistical manifold $\bar{M}$ such that $\bar{M}$ admits a semi-symmetric metric connection $\tilde{\nabla}$ and $U \in \Gamma(TN)$. Then the Codazzi equation with respect to the semi-symmetric metric connection $\tilde{\nabla}$ is given by*

$$
\begin{aligned}
g(\tilde{R}(X, Y)Z, \xi) = {}& g((\nabla'_X h')(Y, Z), \xi) - g((\nabla'_Y h')(X, Z), \xi) \\
&+ \eta(Y) g(h'(X, Z), \xi) - \eta(X) g(h'(Y, Z), \xi), \qquad (31)
\end{aligned}
$$

where

$$(\nabla'_X h')(Y, Z) = D'_X h'(Y, Z) - h'(\nabla'_X Y, Z) - h'(Y, \nabla'_X Z).$$

Proof. Inner product of Equation (29) and $\xi \in \Gamma(TN^{\perp})$ implies

$$
\begin{aligned}
g(\tilde{R}(X, Y)Z, \xi) = {}& g(h'(X, \nabla'_Y Z), \xi)) + g(D'_X h'(Y, Z), \xi) \\
&- g(h'(Y, \nabla'_X Z), \xi)) + g(D'_Y h'(X, Z), \xi) \\
&- g(h'([X, Y], Z), \xi). \qquad (32)
\end{aligned}
$$

In the last term, we have

$$-[X,Y] = \eta(Y)X - \eta(X)Y + \nabla'_Y X - \nabla'_X Y,$$

thus (31) holds. □

Theorem 5.3. *Let N be a submanifold of statistical manifold $\bar{M}$ such that $\bar{M}$ admits a semi-symmetric metric connection $\tilde{\nabla}$ and $U \in \Gamma(TN)$. Then the Ricci equation with respect to the semi-symmetric metric connection $\tilde{\nabla}$ is given by*

$$g(\tilde{R}(X,Y)V,W) = g(R^\perp(X,Y)V,W) + g([A'_V, A'_W]X,Y),$$

where $R^\perp(X,Y)V$ is defined

$$R^\perp(X,Y)V = D'_X D'_Y V - D'_Y D'_X V - D'_{[X,Y]}V$$

for all $X, Y \in \Gamma(TN)$ and $V, W \in \Gamma(TN^\perp)$ and

$$[A'_V, A'_W] = A'_V A'_W - A'_W A'_V.$$

Proof. For all $X, Y \in \Gamma(TN)$ and $V, W \in \Gamma(TN^\perp)$ from (19) and (24) we obtain

$$\begin{aligned} g(\tilde{R}(X,Y)V,W) &= g(\tilde{\nabla}_X\tilde{\nabla}_Y V,W) - g(\tilde{\nabla}_Y\tilde{\nabla}_X V,W) - g(\tilde{\nabla}_{[X,Y]}V,W) \\ &= g(R^\perp(X,Y)V,W) - g(h'(X,A'_V Y),W) + g(h'(Y,A'_V X),W). \end{aligned}$$

By using (26), we get

$$\begin{aligned} g(\tilde{R}(X,Y)V,W) &= g(R^\perp(X,Y)V,W) - g(A'_W A'_V Y,X) + g(A'_V A'_W X,Y) \\ &= g(R^\perp(X,Y)V,W) + g([A'_V, A'_W]X,Y). \end{aligned}$$

□

Example 5.4. Let $\bar{M}$ be the 5-dimensional statistical manifold as in Example 3.3. Assume N be a 3-dimensional submanifold with the coordinate (u, v, w) given by

$$i : N \longrightarrow \bar{M}$$
$$i(u,v,w) = (\frac{1}{2}u, \frac{1}{2}v, \frac{-1}{2}v, \frac{1}{2}u, w).$$

Then the tangent bundle TN and normal bundle $TN^\perp$ are spanned by

$$TN = \{Z_1 = \frac{1}{2}(\partial x_1 + \partial y_2),\ Z_2 = \frac{1}{2}(\partial x_2 - \partial y_1),\ Z_3 = \partial z\},$$

$$TN^\perp = \{N_1 = \frac{1}{2}(\partial x_1 - \partial y_2),\ N_2 = \frac{1}{2}(\partial x_2 + \partial y_1)\}.$$

We obtain

$$\tilde{\nabla}_{Z_1} Z_1 = \frac{-y_1}{2} Z_1 - \frac{y_1}{8} Z_2 - \frac{1}{4}(1 + y_1^2 - \frac{1}{2}y_1 y_2)Z_3 + \frac{3}{8}y_1 N_2,$$

$$\tilde{\nabla}_{Z_2} Z_2 = \frac{y_2}{8} Z_1 - \frac{y_2}{2} Z_2 - \frac{1}{4}(1 + y_2^2 + \frac{1}{2}y_1 y_2)Z_3 - \frac{3}{8}y_2 N_1,$$

$$\tilde{\nabla}_{Z_2} Z_1 = \frac{y_1}{8} Z_1 - \frac{1}{16}(5y_1 + 2y_2 - 1)Z_2 - \frac{1}{16}(y_1^2 - y_2^2 + 4y_1 y_2)Z_3 - \frac{y_1}{8} N_1$$

$$+\frac{1}{16}(3y_1+2y_2+1)N_2,$$

$$\tilde{\nabla}_{Z_1}Z_2=\frac{-1}{8}(y_1+4y_2)Z_1+\frac{1}{16}(1-2y_2)Z_2-\frac{1}{16}(y_1^2-y_2^2+4y_1y_2)Z_3$$
$$-\frac{3y_1}{8}N_1+(\frac{y_2}{8}+\frac{1}{16})N_2,$$

$$\tilde{\nabla}_{Z_3}Z_1=\frac{1}{8}Z_2-\frac{y_2}{8}Z_3-\frac{3}{8}N_2,\qquad \tilde{\nabla}_{Z_1}Z_3=Z_1+\frac{1}{8}Z_2+\frac{1}{2}(y_1-\frac{1}{4}y_2)Z_3-\frac{3}{8}N_2,$$

$$\tilde{\nabla}_{Z_3}Z_2=\frac{-1}{8}Z_1+\frac{y_1}{8}Z_3+\frac{3}{8}N_1,\qquad \tilde{\nabla}_{Z_2}Z_3=Z_2-\frac{1}{8}Z_1+\frac{1}{2}(y_2+\frac{1}{4}y_1)Z_3+\frac{3}{8}N_1.$$

By direct computations, we obtain some components of the connection as follows

$$\nabla_{Z_1}Z_1=y_1Z_2+\frac{1}{2}y_1y_2Z_3,\qquad \nabla^*_{Z_1}Z_1=\frac{-5y_1}{4}Z_2-\frac{y_1y_2}{4}Z_3,$$

$$\nabla_{Z_2}Z_2=-y_2Z_1-\frac{y_1y_2}{2}Z_3,\qquad \nabla^*_{Z_2}Z_2=\frac{5y_2}{4}Z_1+\frac{y_1y_2}{4}Z_3,$$

$$h(Z_1,Z_1)=0,\ h^*(Z_1,Z_1)=\frac{3y_1}{4}N_2,\ h(Z_2,Z_2)=0,\ h^*(Z_2,Z_2)=-\frac{3y_2}{4}N_1.$$

We see that in this example, Theorem 4.1 is satisfied.

Example 5.5. Let $\bar{M}$ be the statistical manifold of Gaussian density functions given by (see [3, p. 45] and [5, p. 51])

$$M=\{n(z;\mu,\sigma)\mid \mu\in\mathbb{R},\sigma\in\mathbb{R}^+\},\quad z\in\mathbb{R},$$

where

$$n(z;\mu,\sigma)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\dfrac{(z-\mu)^2}{2\sigma^2}},$$

μ and σ^2 are mean and variance, respectively. The Fisher metric for this Gaussian manifold with parameters $(x,y)=(\mu,\sigma)$ is given by

$$g=\begin{pmatrix}\frac{1}{y^2} & 0\\ 0 & \frac{2}{y^2}\end{pmatrix}.$$

For components of the affine and dual connections $\bar{\nabla}$ and $\bar{\nabla}^*$, we have

$$\bar{\nabla}_{\partial x}\partial x=0,\quad \bar{\nabla}^*_{\partial x}\partial x=\frac{1}{y}\partial y,\quad \bar{\nabla}_{\partial y}\partial y=\frac{-3}{y}\partial y,\quad \bar{\nabla}^*_{\partial y}\partial y=\frac{1}{y}\partial y,$$

$$\bar{\nabla}_{\partial x}\partial y=\frac{-2}{y}\partial x,\quad \bar{\nabla}_{\partial y}\partial x=\frac{-2}{y}\partial x,\quad \bar{\nabla}^*_{\partial x}\partial y=0,\quad \bar{\nabla}^*_{\partial y}\partial x=0.$$

Then $(\bar{M},g,\nabla,\nabla^*)$ is a 2-dimensional statistical manifold. By taking $U=\partial x$, we obtain the semi-symmetric metric connection $\tilde{\nabla}$

$$\tilde{\nabla}_{\partial x}\partial x=\frac{1}{2y}\partial y,\quad \tilde{\nabla}_{\partial y}\partial y=\frac{-2}{y^2}\partial x-\frac{1}{y}\partial y,$$

$$\tilde{\nabla}_{\partial x}\partial y = \frac{-1}{y}\partial x, \qquad \tilde{\nabla}_{\partial y}\partial x = \frac{1}{y^2}\partial y - \frac{1}{y}\partial x.$$

The non-zero component of the torsion tensor is $\tilde{T}(\partial x, \partial y) = \frac{-1}{y^2}\partial y$. By direct calculation, we have

$$\tilde{R}(\partial x, \partial y)\partial x = \frac{1}{2y^2}\partial y, \qquad \tilde{R}(\partial x, \partial y)\partial y = \frac{-1}{y^2}\partial x.$$

Example 5.6. Let 4-manifold $\bar{M}$ be the set of Freund bivariate mixture exponential density function $\mathcal{F}$ such as [3]

$$M = \{\mathcal{F}(x, y; \alpha_1, \beta_1, \alpha_2, \beta_2) \mid \alpha_i, \beta_i > 0,\ i = 1, 2\}, \qquad x, y > 0,$$

$$\mathcal{F}(x, y) = \begin{cases} \alpha_1\beta_2 e^{-\beta_2 y - (\alpha_1+\alpha_2-\beta_2)x}, & 0 < x < y, \\ \alpha_2\beta_1 e^{-\beta_1 y - (\alpha_1+\alpha_2-\beta_1)x}, & 0 < y < x. \end{cases}$$

The Fisher information metric g is given by

$$g = \frac{1d\alpha_1 d\alpha_1}{\alpha_1^2 + \alpha_1\alpha_2} + \frac{\alpha_2}{\beta_1^2(\alpha_1+\alpha_2)}d\beta_1 d\beta_1 + \frac{1d\alpha_2 d\alpha_2}{\alpha_2^2 + \alpha_1\alpha_2} + \frac{\alpha_1}{\beta_2^2(\alpha_1+\alpha_2)}d\beta_2 d\beta_2.$$

We obtain the dual statistical connections $\bar{\nabla}$ and $\bar{\nabla}^*$ as follows:

$$\bar{\nabla}_{\partial\alpha_1}\partial\alpha_1 = \frac{-\alpha_2}{\alpha_1(\alpha_1+\alpha_2)}\partial\alpha_1 + \frac{\alpha_2}{\alpha_1(\alpha_1+\alpha_2)}\partial\alpha_2, \qquad \bar{\nabla}^*_{\partial\alpha_1}\partial\alpha_1 = \frac{-2}{\alpha_1+\alpha_2}\partial\alpha_1,$$

$$\bar{\nabla}_{\partial\beta_1}\partial\beta_1 = \frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)\beta_1^2}\partial\alpha_1 - \frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)\beta_1^2}\partial\alpha_2, \qquad \bar{\nabla}^*_{\partial\beta_1}\partial\beta_1 = \frac{-2}{\beta_1}\partial\beta_1,$$

$$\bar{\nabla}_{\partial\alpha_2}\partial\alpha_2 = \frac{\alpha_1}{\alpha_2(\alpha_1+\alpha_2)}\partial\alpha_1 - \frac{\alpha_1}{\alpha_2(\alpha_1+\alpha_2)}\partial\alpha_2, \qquad \bar{\nabla}^*_{\partial\alpha_2}\partial\alpha_2 = \frac{-2}{\alpha_1+\alpha_2}\partial\alpha_2,$$

$$\bar{\nabla}_{\partial\beta_2}\partial\beta_2 = \frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)\beta_2^2}\partial\alpha_2, \qquad \bar{\nabla}^*_{\partial\beta_2}\partial\beta_2 = \frac{-2}{\beta_2}\partial\beta_2 - \frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)\beta_2^2}\partial\alpha_1,$$

$$\bar{\nabla}_{\partial\beta_1}\partial\alpha_1 = 0, \qquad \bar{\nabla}^*_{\partial\beta_1}\partial\alpha_1 = \frac{-1}{\alpha_1+\alpha_2}\partial\beta_1,$$

$$\bar{\nabla}_{\partial\alpha_2}\partial\alpha_1 = 0, \qquad \bar{\nabla}^*_{\partial\alpha_2}\partial\alpha_1 = \frac{-1}{\alpha_1+\alpha_2}\partial\alpha_1 - \frac{1}{\alpha_1+\alpha_2}\partial\alpha_2,$$

$$\bar{\nabla}_{\partial\beta_2}\partial\alpha_1 = \frac{\alpha_2}{\alpha_1(\alpha_1+\alpha_2)}\partial\beta_2, \qquad \bar{\nabla}^*_{\partial\beta_2}\partial\alpha_1 = 0,$$

$$\bar{\nabla}_{\partial\alpha_2}\partial\beta_1 = 0, \qquad \bar{\nabla}^*_{\partial\alpha_2}\partial\beta_1 = \frac{\alpha_1}{\alpha_2(\alpha_1+\alpha_2)}\partial\beta_1,$$

$$\bar{\nabla}_{\partial\beta_2}\partial\beta_1 = 0, \quad \bar{\nabla}^*_{\partial\beta_2}\partial\beta_1 = 0, \quad \bar{\nabla}_{\partial\beta_2}\partial\alpha_2 = 0, \quad \bar{\nabla}^*_{\partial\beta_2}\partial\alpha_2 = \frac{-1}{\alpha_1+\alpha_2}\partial\beta_2.$$

By taking $U = \partial\beta_2$, we obtain the semi-symmetric metric connection $\tilde{\nabla}$ as follows:

$$\tilde{\nabla}_{\partial\alpha_1}\partial\alpha_1 = \frac{-\alpha_2 - 2\alpha_1}{2\alpha_1(\alpha_1+\alpha_2)}\partial\alpha_1 + \frac{\alpha_2}{2\alpha_1(\alpha_1+\alpha_2)}\partial\alpha_2 - \frac{1}{\alpha_1(\alpha_1+\alpha_2)}\partial\beta_2,$$

$$\tilde{\nabla}_{\partial\beta_1}\partial\beta_1 = \frac{\alpha_1\alpha_2}{2\beta_1^2(\alpha_1+\alpha_2)}\partial\alpha_1 - \frac{\alpha_1\alpha_2}{2\beta_1^2(\alpha_1+\alpha_2)}\partial\alpha_2 - \frac{\alpha_2}{\beta_1^2(\alpha_1+\alpha_2)}\partial\beta_2 - \frac{1}{\beta_1}\partial\beta_1,$$

$$\tilde{\nabla}_{\partial\alpha_2}\partial\alpha_2 = \frac{\alpha_1}{2\alpha_2(\alpha_1+\alpha_2)}\partial\alpha_1 - \frac{\alpha_1+2\alpha_2}{2\alpha_2(\alpha_1+\alpha_2)}\partial\alpha_2 - \frac{1}{\alpha_2(\alpha_1+\alpha_2)}\partial\beta_2,$$

$$\tilde{\nabla}_{\partial\beta_2}\partial\beta_2 = \frac{\alpha_1\alpha_2}{2\beta_2^2(\alpha_1+\alpha_2)}\partial\alpha_2 - \frac{\alpha_1\alpha_2}{2\beta_2^2(\alpha_1+\alpha_2)}\partial\alpha_1 - \frac{1}{\beta_2}\partial\beta_2,$$

$$\tilde{\nabla}_{\partial\alpha_1}\partial\beta_1 = \tilde{\nabla}_{\partial\beta_1}\partial\alpha_1 = \frac{-1}{2(\alpha_1+\alpha_2)}\partial\beta_1,$$

$$\tilde{\nabla}_{\partial\alpha_2}\partial\alpha_1 = \tilde{\nabla}_{\partial\alpha_1}\partial\alpha_2 = \frac{-1}{2(\alpha_1+\alpha_2)}\partial\alpha_1 - \frac{1}{2(\alpha_1+\alpha_2)}\partial\alpha_2,$$

$$\tilde{\nabla}_{\partial\beta_2}\partial\alpha_1 = \frac{\alpha_2}{2\alpha_1(\alpha_1+\alpha_2)}\partial\beta_2,$$

$$\tilde{\nabla}_{\partial\alpha_1}\partial\beta_2 = \frac{\alpha_1}{\beta_2^2(\alpha_1+\alpha_2)}\partial\alpha_1 + \frac{\alpha_2}{2\alpha_1(\alpha_1+\alpha_2)}\partial\beta_2,$$

$$\tilde{\nabla}_{\partial\beta_1}\partial\alpha_2 = \tilde{\nabla}_{\partial\alpha_2}\partial\beta_1 = \frac{\alpha_1}{2\alpha_2(\alpha_1+\alpha_2)}\partial\beta_1,$$

$$\tilde{\nabla}_{\partial\beta_1}\partial\beta_2 = \frac{\alpha_1}{\beta_2^2(\alpha_1+\alpha_2)}\partial\beta_1, \qquad \tilde{\nabla}_{\partial\beta_2}\partial\beta_1 = 0,$$

$$\tilde{\nabla}_{\partial\beta_2}\partial\alpha_2 = \frac{-1}{2(\alpha_1+\alpha_2)}\partial\beta_2,$$

$$\tilde{\nabla}_{\partial\alpha_2}\partial\beta_2 = \frac{-1}{2(\alpha_1+\alpha_2)}\partial\beta_2 + \frac{\alpha_1}{\beta_2^2(\alpha_1+\alpha_2)}\partial\alpha_2.$$

The non-zero components of the torsion tensor $\tilde{T}$ are

$$\tilde{T}(\partial\alpha_1,\partial\beta_2) = \frac{\alpha_1}{\beta_2^2(\alpha_1+\alpha_2)}\partial\alpha_1,$$

$$\tilde{T}(\partial\beta_1,\partial\beta_2) = \frac{\alpha_1}{\beta_2^2(\alpha_1+\alpha_2)}\partial\beta_1,$$

$$\tilde{T}(\partial\alpha_2,\partial\beta_2) = \frac{\alpha_1}{\beta_2^2(\alpha_1+\alpha_2)}\partial\alpha_2.$$

By direct calculation, we get some components of the curvature tensor of the semi-symmetric metric connection $\tilde{\nabla}$:

$$\tilde{R}(\partial\alpha_1,\partial\beta_1)\partial\alpha_1 = \frac{4\alpha_1-\alpha_2\beta_2^2}{4\alpha_1\beta_2^2(\alpha_1+\alpha_2)^2}\partial\beta_1, \qquad \tilde{R}(\partial\alpha_1,\partial\alpha_2)\partial\beta_1 = 0,$$

$$\tilde{R}(\partial\alpha_1,\partial\beta_1)\partial\beta_2 = \frac{\alpha_2}{2\beta_2^2(\alpha_1+\alpha_2)^2}\partial\beta_1, \quad \tilde{R}(\partial\beta_1,\partial\beta_2)\partial\alpha_2 = \frac{-\alpha_1}{2\beta_2^2(\alpha_1+\alpha_2)^2}\partial\beta_1,$$

$$\tilde{R}(\partial\beta_1,\partial\alpha_2)\partial\beta_2 = \frac{\alpha_1}{2\beta_2^2(\alpha_1+\alpha_2)^2}\partial\beta_1,$$

$$\tilde{R}(\partial\beta_1,\partial\beta_2)\partial\beta_2 = (\frac{\alpha_1^2+\alpha_1\alpha_2}{4\beta_2^2(\alpha_1+\alpha_2)^2} + \frac{\alpha_1}{\beta_2^3(\alpha_1+\alpha_2)})\partial\beta_1.$$

Now, we can see that this example verifies Proposition 3.5.

References

[1] N. S. Agashe and M. R. Chafle, *A semi-symmetric nonmetric connection on a Riemannian manifold*, Indian J. Pure Appl. Math. **23** (1992), no. 6, 399–409.

[2] S. Amari and H. Nagaoka, *Methods of information geometry*, translated from the 1993 Japanese original by Daishi Harada, Translations of Mathematical Monographs, 191, American Mathematical Society, Providence, RI, 2000. https://doi.org/10.1090/mmono/191

[3] K. A. Arwini and C. T. J. Dodson, *Information geometry*, Lecture Notes in Mathematics, 1953, Springer-Verlag, Berlin, 2008. https://doi.org/10.1007/978-3-540-69393-2

[4] M. E. Aydin, A. Mihai, and I. Mihai, *Generalized Wintgen inequality for statistical submanifolds in statistical manifolds of constant curvature*, Bull. Math. Sci. **7** (2017), no. 1, 155–166. https://doi.org/10.1007/s13373-016-0086-1

[5] O. Calin and C. Udrişte, *Geometric Modeling in Probability and Statistics*, Springer, Cham, 2014. https://doi.org/10.1007/978-3-319-07779-6

[6] U. C. De and A. Barman, *On a type of semisymmetric metric connection on a Riemannian manifold*, Publ. Inst. Math. (Beograd) (N.S.) **98(112)** (2015), 211–218. https://doi.org/10.2298/PIM150317025D

[7] A. Friedmann and J. A. Schouten, *Über die Geometrie der halbsymmetrischen Übertragungen*, Math. Z. **21** (1924), no. 1, 211–223. https://doi.org/10.1007/BF01187468

[8] A. Haseeb and R. Prasad, *Certain curvature conditions in Kenmotsu manifolds with respect to the semi-symmetric metric connection*, Commun. Korean Math. Soc. **32** (2017), no. 4, 1033–1045. https://doi.org/10.4134/CKMS.c160266

[9] S. Kazan and A. Kazan, *Sasakian statistical manifolds with semi-symmetric metric connection*, Univers. J. Math. Appl. **1** (2018), 226–232.

[10] C. Murathan and C. Özgür, *Riemannian manifolds with a semi-symmetric metric connection satisfying some semisymmetry conditions*, Proc. Est. Acad. Sci. **57** (2008), no. 4, 210–216. https://doi.org/10.3176/proc.2008.4.02

[11] R. N. Singh, S. K. Pandey, G. Pandey, and K. Tiwari, *On a semi-symmetric metric connection in an* (ε)*-Kenmotsu manifold*, Commun. Korean Math. Soc. **29** (2014), no. 2, 331–343. https://doi.org/10.4134/CKMS.2014.29.2.331

[12] K. Takano, *Statistical manifolds with almost contact structures and its statistical submersions*, J. Geom. **85** (2006), no. 1-2, 171–187. https://doi.org/10.1007/s00022-006-0052-2

[13] K. Yano, *On semi-symmetric metric connection*, Rev. Roumaine Math. Pures Appl. **15** (1970), 1579–1586.

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