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g-NATURAL METRIC AND HARMONICITY ON THE COTANGENT BUNDLE

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Abstract. In this paper, we introduce the harmonicity of a covector field on a Riemannian manifold (M, g) to its cotangent bundle T^*M equipped with g-natural metric. Afterward we also construct some examples of harmonic covector fields.

1. Introduction

The geometry of the cotangent bundle T^*M has been studied by many authors, for example, A. A. Salimov and F. Agca [2,10], K. Yano and S. Ishihara [11], F. Agca [1], F. Ocak and S. Kazimova [8], A. Gezer and M. Altunbas [5] (see [12, 13]) etc.

We will study harmonicity on cotangent bundle equipped with g-natural metric [1]. We establish necessary and sufficient conditions under which a covector field is harmonic with respect to the g-natural metrics. Next we also construct some examples of harmonic covector fields.

Consider a smooth map $\phi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds. Then the second fundamental form of ϕ is defined by

$$(\nabla d\phi)(X, Y) = \nabla^{\phi}_X d\phi(Y) - d\phi(\nabla_X Y) = \nabla'_X d\phi(Y) - d\phi(\nabla_X Y). \tag{1.1}$$

Here ∇, ∇', ∇^{ϕ} are the Riemannian connections on M, N, $\phi^{-1}TN$ (the pull-back bundle) respectively, and

$$\tau(\phi) = trace_g \nabla d\phi, \tag{1.2}$$

is the tension field of ϕ.

The energy functional of ϕ is defined by

$$E(\phi) = \int_K e(\phi) dv_g, \tag{1.3}$$

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such that K is any compact of M, where

$$e(\phi)=\frac{1}{2}trace_g h(d\phi,d\phi), \tag{1.4}$$

is the energy density of ϕ.

A map is called harmonic if it is a critical point of the energy functional E. For any smooth variation $\{\phi_t\}_{t\in I}$ of ϕ with $\phi_0=\phi$ and $V=\frac{d}{dt}\phi_t\big|_{t=0}$, we have

$$\frac{d}{dt}E(\phi_t)\Big|_{t=0}=-\int_K h(\tau(\phi),V)dv_g. \tag{1.5}$$

Then ϕ is harmonic if and only if $\tau(\phi)=0$.

One can refer to $[3,4,6,9]$ for background on harmonic maps.

2. Cotangent bundles T^*M

Let (M^m,g) be an m-dimensional Riemannian manifold, T^*M be its cotangent bundle and $\pi: T^*M\to M$ the natural projection. A local chart $(U,x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U),x^i,x^{\bar{i}}=p_i)_{i=\overline{1,m},\bar{i}=m+i}$ on T^*M, where p_i is the component of covector p in each cotangent space T^*_xM, $x\in U$ with respect to the natural coframe dx^i. Let $C^\infty(M)$ (resp. $C^\infty(T^*M)$) be the ring of real-valued C^∞ functions on M(resp. T^*M) and $\Im^r_s(M)$ (resp. $\Im^r_s(T^*M)$) be the module over $C^\infty(M)$ (resp. $C^\infty(T^*M)$) of C^∞ tensor fields of type (r,s).

Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

Let $X=X^i\frac{\partial}{\partial x^i}$ and $\omega=\omega_i dx^i$ be a local expressions in $U\subset M$ of a vector and covector field $X\in\Im^1_0(M)$ and $\omega\in\Im^0_1(M)$, respectively. Then the horizontal and the vertical lifts of X and ω are defined, respectively by

$$X^H=X^i\frac{\partial}{\partial x^i}+p_h\Gamma^h_{ij}X^j\frac{\partial}{\partial p_i}, \tag{2.1}$$

$$\omega^V=\omega_i\frac{\partial}{\partial p_i} \tag{2.2}$$

with respect to the natural frame $\{\frac{\partial}{\partial x^i},\frac{\partial}{\partial p_i}\}$, where Γ^h_{ij} are components of the Levi-Civita connection ∇ on M. (See [11] for more details.)

Lemma 2.1 ([11])**.** *Let (M,g) be a Riemannian manifold, ∇ be the Levi-Civita connection and R be the Riemannian curvature tensor. Then the Lie bracket of the cotangent bundle T^*M of M satisfies the following*

(1) $[\omega^V,\theta^V]=0,$
(2) $[X^H,\theta^V]=(\nabla_X\theta)^V,$
(3) $[X^H,Y^H]=[X,Y]^H-(pR(X,Y))^V$

for all vector fields $X,Y\in\Im^1_0(M)$ and $\omega,\theta\in\Im^0_1(M)$.

For a Riemannian manifold (M, g), we define the map

$$\begin{array}{rcl} \sharp : \Im^0_1(M) & \longrightarrow & \Im^1_0(M) \\ \omega & \mapsto & \sharp\omega \end{array}$$

by $g(\sharp\omega, X) = \omega(X)$ for all $X \in \Im^1_0(M)$, where the map $\sharp$ is a $C^\infty(M)$-isomorphism.

Locally for all $\omega = \omega_i dx^i \in \Im^0_1(M)$, we have $\sharp\omega = g^{ij}\omega_i \frac{\partial}{\partial x^j}$, where (g^{ij}) is the inverse matrix of the matrix (g_{ij}).

For each $x \in M$ the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space $T^*_x M$ by $g^{-1}(\omega, \theta) = g(\sharp\omega, \sharp\theta) = g^{ij}\omega_i\theta_j$.

If ∇ is the Levi-Civita connection of (M, g), we have

$$\nabla_X(\sharp\omega) = \sharp(\nabla_X\omega), \tag{2.3}$$

$$Xg^{-1}(\omega, \theta) = g^{-1}(\nabla_X\omega, \theta) + g^{-1}(\omega, \nabla_X\theta) \tag{2.4}$$

for all $X \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$.

From now on, we noted $\sharp\omega$ by $\tilde{\omega}$ for all $\omega \in \Im^0_1(M)$.

3. *g*-natural metric

3.1. *g*-natural metric

Definition ([1])**.** Let (M, g) be a Riemannian manifold. On the cotangent bundle T^*M, we define a g-natural metric noted $\tilde{g}$ by

$$\tilde{g}(X^H, Y^H) = g(X, Y)^V = g(X, Y) \circ \pi, \tag{3.1}$$

$$\tilde{g}(X^H, \theta^V) = 0, \tag{3.2}$$

$$\tilde{g}(\omega^V, \theta^V) = \varphi(z)g^{-1}(\omega, \theta) + \psi(z)g^{-1}(\omega, p)g^{-1}(\theta, p)) \tag{3.3}$$

for all $X, Y \in \Im^1_0(M)$, $\omega, \theta \in \Im^0_1(M)$, where φ and ψ are some functions of argument $z = \frac{1}{2}g^{-1}(p, p)$ such that $\varphi > 0$ and $\varphi + 2z\psi > 0$.

Remark 3.1. Since $z = \frac{1}{2}g^{-1}(p, p) > 0$, $\varphi > 0$ and $\varphi + 2z\psi > 0$ then just propose $\psi > 0$.

Theorem 3.2 ([1])**.** *Let (M, g) be a Riemannian manifold and $(T^*M, \tilde{g})$ its cotangent bundle equipped with the g-natural metric. If ∇ (resp $\widetilde{\nabla}$) denote the Levi-Civita connection of (M, g) (resp $(T^*M, \tilde{g})$), we have*:

(1) $\widetilde{\nabla}_{X^H} Y^H = (\nabla_X Y)^H + (pR(X, Y))^V$,

(2) $\widetilde{\nabla}_{X^H} \theta^V = (\nabla_X \theta)^V + \dfrac{\varphi(z)}{2}(R(\tilde{p}, \tilde{\theta})X)^H$,

(3) $\widetilde{\nabla}_{\omega^V} Y^H = \dfrac{\varphi(z)}{2}(R(\tilde{p}, \tilde{\omega})Y)^H$,

(4) $$\begin{aligned} \widetilde{\nabla}_{\omega^V} \theta^V = & -A\left[\tilde{g}(\omega^V, \mathcal{P}^V)\theta^V + \tilde{g}(\theta^V, \mathcal{P}^V)\omega^V\right] \\ & + B\tilde{g}(\omega^V, \theta^V)\mathcal{P}^V + C\,\tilde{g}(\omega^V, \mathcal{P}^V)\tilde{g}(\theta^V, \mathcal{P}^V)\mathcal{P}^V \end{aligned}$$

for all $X, Y \in \Im^1_0(M)$ *and* $\omega, \theta \in \Im^0_1(M)$, *where* $\mathcal{P}^V$ *denotes the canonical vertical vector field on* T^*M, R *denotes the curvature tensor of* (M, g) *and*

$$A = \frac{\varphi'(z)}{2\varphi(z)(\varphi(z) + 2z\psi(z))},\ B = \frac{2\psi(z) + \varphi'(z)}{2\varphi(z)(\varphi(z) + 2z\psi(z))},$$

$$C = \frac{\psi'(z)\varphi(z) - \varphi'(z)\psi(z) - 2\psi^2(z)}{2\varphi(z)(\varphi(z) + 2z\psi(z))}.$$

4. g-natural metric and harmonicity

4.1. Harmonic sections $\omega : (M, g) \to (T^*M, \tilde{g})$

Now we study the harmonicity of section $\omega : (M, g) \to (T^*M, \tilde{g})$, i.e., covector field ω on M, and we give the necessary and sufficient conditions under which a covector field is harmonic with respect to the g-natural metrics (for tangent bundle version, see [7]).

Lemma 4.1 ([7, 14]). *Let* (M, g) *be a Riemannian manifold. If* $\omega \in \Im^0_1(M)$ *is a covector field* (1*-form*) *on* M *and* $(x, p) \in T^*M$ *such that* $\omega_x = p$, *then we have*:

$$d_x\omega(X_x) = X^H_{(x,p)} + (\nabla_X \omega)^V_{(x,p)},$$

where $X \in \Im^1_0(M)$.

Proof. Let (U, x^i) be a local chart on M with $x \in M$ and $(\pi^{-1}(U), x^i, p_i)$ the induced chart on T^*M. If $X_x = X^i(x)\frac{\partial}{\partial x^i}|_x$ and $\omega_x = \omega_i(x)dx^i|_x = p$, then

$$\begin{aligned}
d_x\omega(X_x) &= X^i(x)\frac{\partial}{\partial x^i}|_{(x,p)} + X^i(x)\frac{\partial \omega_j}{\partial x^i}(x)\frac{\partial}{\partial p_j}|_{(x,p)} \\
&= X^i(x)\frac{\partial}{\partial x^i}|_{(x,p)} + \omega_k(x)\Gamma^k_{ji}(x)X^j(x)\frac{\partial}{\partial p_i}|_{(x,p)} \\
&\quad - \omega_k(x)\Gamma^k_{ji}(x)X^j(x)\frac{\partial}{\partial p_i}|_{(x,p)} + X^i(x)\frac{\partial \omega_j}{\partial x^i}(x)\frac{\partial}{\partial p_j}|_{(x,p)} \\
&= X^i(x)\frac{\partial}{\partial x^i}|_{(x,p)} + p_k\Gamma^k_{ji}(x)X^j(x)\frac{\partial}{\partial p_i}|_{(x,p)} \\
&\quad + X^i(x)\frac{\partial \omega_j}{\partial x^i}(x)\frac{\partial}{\partial p_j}|_{(x,p)} - \omega_k(x)\Gamma^k_{ij}(x)X^i(x)\frac{\partial}{\partial p_j}|_{(x,p)} \\
&= X^H_{(x,p)} + X^i(x)\big[\frac{\partial \omega_j}{\partial x^i}(x) - \omega_k(x)\Gamma^k_{ij}(x)X^i(x)\big](dx^i)^V_{(x,p)} \\
&= X^H_{(x,p)} + (\nabla_X \omega)^V_{(x,p)}.
\end{aligned}$$

□

Hence we have the following lemma.

Lemma 4.2. *Let* (M^m, g) *be a Riemannian* m*-dimensional manifold and* $(T^*M, \tilde{g})$ *its cotangent bundle equipped with the* g*-natural metric. If* $\omega \in$

$\Im^0_1(M)$, *then the energy density associated to* ω *is given by*:

$$e(\omega) = \frac{m}{2} + \frac{1}{2} trace_g\big[\varphi(z) g^{-1}(\nabla\omega, \nabla\omega) + \psi(z) g^{-1}(\nabla\omega, \omega)^2\big], \tag{4.1}$$

where φ *and* ψ *are some functions of argument* $z = \frac{1}{2}g^{-1}(p,p)$ *such that* $\varphi > 0$ *and* $\psi > 0$.

Proof. Let $(x,p) \in T^*M$, $\omega \in \Im^0_1(M)$, $\omega_x = p$ and $(E_1, \dots, E_m)$ a local orthonormal frame on M. Then:

$$\begin{aligned} e(\omega)_x &= \frac{1}{2} trace_g \tilde{g}(d\omega, d\omega)_{(x,p)} \\ &= \frac{1}{2}\sum_{i=1}^m \tilde{g}(d\omega(E_i), d\omega(E_i))_{(x,p)}. \end{aligned}$$

Using Lemma 4.1, we obtain:

$$\begin{aligned} e(\omega) &= \frac{1}{2}\sum_{i=1}^m \tilde{g}(E_i^H + (\nabla_{E_i}\omega)^V, E_i^H + (\nabla_{E_i}\omega)^V) \\ &= \frac{1}{2}\sum_{i=1}^m \big[(\tilde{g}(E_i^H, E_i^H) + \tilde{g}((\nabla_{E_i}\omega)^V, (\nabla_{E_i}\omega)^V))\big] \\ &= \frac{1}{2}\sum_{i=1}^m \big[g(E_i, E_i) + \varphi(z) g^{-1}(\nabla_{E_i}\omega, \nabla_{E_i}\omega) + \psi(z) g^{-1}(\nabla_{E_i}\omega, \omega)^2)\big] \\ &= \frac{m}{2} + \frac{1}{2} trace_g\big[\varphi(z) g^{-1}(\nabla\omega, \nabla\omega) + \psi(z) g^{-1}(\nabla\omega, \omega)^2\big]. \end{aligned}$$

□

A direct consequence of usual calculations using Lemma 4.2 gives the following result.

Theorem 4.3. *Let* (M^m, g) *be a Riemannian m-dimensional manifold and* $(T^*M, \tilde{g})$ *its cotangent bundle equipped with the g-natural metric. If* $\omega \in \Im^0_1(M)$, *then the tension field associated to* ω *is given by*:

$$\tau(\omega) = \Big[trace_g\big[\varphi(z) R(\tilde{\omega}, \widetilde{\nabla\omega})*\big]\Big]^H + \Big[trace_g\big[T(\omega)\big]\Big]^V, \tag{4.2}$$

and $T(\omega)$ *is a bilinear map defined by*

$$\begin{aligned} T(\omega) = \nabla^2\omega &- 2A\,\tilde{g}((\nabla\omega)^V, \omega^V)\nabla\omega + B\,\tilde{g}((\nabla\omega)^V, (\nabla\omega)^V)\omega \\ &+ C\,\tilde{g}((\nabla\omega)^V, \omega^V)^2\omega, \end{aligned}$$

where A, B *and* C *are as in Theorem* 3.2.

Proof. Let $(x,p) \in T^*M$, $\omega \in \Im^0_1(M)$, $\omega_x = p$ and $\{E_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on M such that $(\nabla_{E_i}E_i)_x = 0$. Then

$$\tau(\omega)_x = trace_g(\nabla d\omega)_x$$

$$\begin{aligned}
&= \sum_{i=1}^{m}\{\nabla^{\omega}_{E_i} d\omega(E_i) - d\omega(\nabla_{E_i}E_i)\}_x \\
&= \sum_{i=1}^{m}\{\widetilde{\nabla}_{d\omega(E_i)} d\omega(E_i)\}_{(x,p)} \\
&= \sum_{i=1}^{m}\{\widetilde{\nabla}_{(E_i^H+(\nabla_{E_i}\omega)^V)}(E_i^H + (\nabla_{E_i}\omega)^V)\}_{(x,p)} \\
&= \sum_{i=1}^{m}\{\widetilde{\nabla}_{E_i^H}E_i^H + \widetilde{\nabla}_{E_i^H}(\nabla_{E_i}\omega)^V + \widetilde{\nabla}_{(\nabla_{E_i}\omega)^V}(E_i)^H \\
&\qquad + \widetilde{\nabla}_{(\nabla_{E_i}\omega)^V}(\nabla_{E_i}\omega)^V\}_{(x,p)}.
\end{aligned}$$

Using Theorem 3.2, we obtain

$$\begin{aligned}
\tau(\omega) = &\sum_{i=1}^{m}\Big[(\nabla_{E_i}E_i)^H + (pR(E_i,E_i))^V + (\nabla_{E_i}\nabla_{E_i}\omega)^V \\
&+ \frac{\varphi(z)}{2}(R(\tilde{\omega},\widetilde{\nabla_{E_i}\omega})E_i)^H + \frac{\varphi(z)}{2}(R(\tilde{\omega},\widetilde{\nabla_{E_i}\omega})E_i)^H \\
&- A\left[\tilde{g}((\nabla_{E_i}\omega)^V,\omega^V)(\nabla_{E_i}\omega)^V + \tilde{g}((\nabla_{E_i}\omega)^V,\omega^V)(\nabla_{E_i}\omega)^V\right] \\
&+ B\,\tilde{g}((\nabla_{E_i}\omega)^V,(\nabla_{E_i}\omega)^V)\omega^V \\
&+ C\,\tilde{g}((\nabla_{E_i}\omega)^V,\omega^V)\tilde{g}((\nabla_{E_i}\omega)^V,\omega^V)\omega^V\Big] \\
= &\Big[trace_g\big[\varphi(z)R(\tilde{\omega},\widetilde{\nabla\omega})\,*\,\big]\Big]^H \\
&+ \Big[trace_g\big[\nabla^2\omega - 2A\,\tilde{g}((\nabla\omega)^V,\omega^V)\nabla\omega \\
&+ B\,\tilde{g}((\nabla\omega)^V,(\nabla\omega)^V)\omega + C\,\tilde{g}((\nabla\omega)^V,\omega^V)^2\omega\big]\Big]^V.
\end{aligned}$$

□

From that, we have the following result.

Theorem 4.4. *Let (M^m, g) be a Riemannian m-dimensional manifold and $(T^*M, \tilde{g})$ its cotangent bundle equipped with the g-natural metric. If $\omega \in \Im^0_1(M)$, then ω is a harmonic covector field if and only if the following conditions are verified*

$$trace_g\big[R(\tilde{\omega},\widetilde{\nabla\omega})\,*\,\big] = 0, \tag{4.3}$$

and

$$\begin{aligned}
trace_g\big[&\nabla^2\omega - 2A\,\tilde{g}((\nabla\omega)^V,\omega^V)\nabla\omega + B\,\tilde{g}((\nabla\omega)^V,(\nabla\omega)^V)\omega \\
&+C\,\tilde{g}((\nabla\omega)^V,\omega^V)^2\omega\big] = 0,
\end{aligned} \tag{4.4}$$

where A, B and C are as in Theorem 3.2.

Proof. The statement is a direct consequence of Theorem 4.3. □

The direct consequence of Theorem 4.4 is the following corollary.

Corollary 4.5. *Let (M^m, g) be a Riemannian m-dimensional manifold and $(T^*M, \tilde{g})$ its cotangent bundle equipped with the g-natural metric. If $\omega \in \Im_0^1(M)$ is a parallel covector field (i.e., $\nabla\omega = 0$), then ω is harmonic.*

The necessary and sufficient condition under which a covector field is harmonic with respect to the g-natural metrics is given in the following theorem.

Theorem 4.6. *Let (M^m, g) be a Riemannian compact m-dimensional manifold and $(T^*M, \tilde{g})$ its cotangent bundle equipped with the g-natural metric.*

If $\omega \in \Im_1^0(M)$, then ω is a harmonic covector field if and only if ω is parallel.

Proof. If ω is parallel from Corollary 4.5, we deduce that ω is a harmonic covector field.

Conversely, let φ_t be a compactly supported variation of ω defined by:

$$\begin{array}{rcl} \mathbb{R} \times M & \longrightarrow & T_x^*M \\ (t, x) & \longmapsto & \varphi_t(x) = (1+t)\omega_x \end{array}$$

From Lemma 4.2 we have:

$$\begin{aligned} e(\varphi_t) &= \frac{m}{2} + \frac{(1+t)^2}{2}\varphi(z)trace_g g^{-1}(\nabla\omega, \nabla\omega) \\ &\quad + \frac{(1+t)^4}{4}\psi(z)trace_g g^{-1}(\nabla\omega, \omega)^2, \\ E(\varphi_t) &= \frac{m}{2}Vol(M) + \frac{(1+t)^2}{2}\int_M \varphi(z)\, trace_g g^{-1}(\nabla\omega, \nabla\omega)dv_g \\ &\quad + \frac{(1+t)^4}{2}\int_M \psi(z)\, trace_g g^{-1}(\nabla\omega, \omega)^2 dv_g, \end{aligned}$$

ω is harmonic, then we have:

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}E(\varphi_t)|_{t=0} \\ &= \frac{\partial}{\partial t}\Big[\frac{m}{2}Vol(M)\Big]_{t=0} + \frac{\partial}{\partial t}\Big[\frac{(1+t)^2}{2}\int_M \varphi(z)\, trace_g g^{-1}(\nabla\omega, \nabla\omega)dv_g\Big]_{t=0} \\ &\quad + \frac{\partial}{\partial t}\Big[\frac{(1+t)^4}{2}\int_M \psi(z)\, trace_g g^{-1}(\nabla\omega, \omega)^2 dv_g\Big]_{t=0} \\ &= \int_M \varphi(z)\, trace_g g^{-1}(\nabla\omega, \nabla\omega)dv_g + 2\int_M \psi(z)\, trace_g g^{-1}(\nabla\omega, \omega)^2 dv_g \\ &= \int_M trace_g\big[\varphi(z)g^{-1}(\nabla\omega, \nabla\omega) + 2\psi(z)g^{-1}(\nabla\omega, \omega)^2\big]dv_g, \end{aligned}$$

since $\varphi(z) > 0$ and $\psi(z) > 0$ then

$$\varphi(z)g^{-1}(\nabla\omega, \nabla\omega) + 2\psi(z)g^{-1}(\nabla\omega, \omega)^2 = 0,$$

which gives

$$g^{-1}(\nabla\omega, \nabla\omega) = g^{-1}(\nabla\omega, \omega)^2 = 0,$$

hence $\nabla\omega = 0$. □

As an application to the above, we give the following two examples.

Example 4.7. Let $\mathbb{S}^1$ (Riemannian compact manifold) be equipped with the metric:

$$g_{\mathbb{S}^1} = e^x dx^2.$$

The Christoffel symbols of the Levi-cita connection are given by:

$$\Gamma^1_{11} = \frac{1}{2} g^{11} (\frac{\partial g_{11}}{\partial x_1} + \frac{\partial g_{11}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_1}) = \frac{1}{2}.$$

The 1-form $\omega = f(x)dx$, $f \in \mathcal{C}^\infty(\mathbb{S}^1)$ is harmonic if and only if ω is parallel,

$$\begin{aligned} \nabla\omega = 0 \quad &\Leftrightarrow \quad f'(x) - \frac{1}{2} f(x) = 0 \\ &\Leftrightarrow \quad f(x) = k \exp(\frac{x}{2}) \ , \ k \in \mathbb{R} \\ &\Leftrightarrow \quad \omega = k \exp(\frac{x}{2}) \frac{d}{dx} \ , \ k \in \mathbb{R}. \end{aligned}$$

Example 4.8. Let $\mathbb{R}^3$ be equipped with the Riemannian metric in cylindrical coordinates defined by:

$$g_{\mathbb{R}^3} = dr^2 + r^2 d\theta + dt^2.$$

The non-null Christoffel symbols of the Riemannian connection are:

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, \ \Gamma^1_{22} = -r,$$

then we have,

$$\nabla_{\frac{\partial}{\partial r}} dr = 0, \ \nabla_{\frac{\partial}{\partial r}} d\theta = -\frac{1}{r} d\theta, \ \nabla_{\frac{\partial}{\partial r}} dt = 0, \ \nabla_{\frac{\partial}{\partial \theta}} dr = r d\theta, \ \nabla_{\frac{\partial}{\partial \theta}} d\theta = -\frac{1}{r} dr,$$

$$\nabla_{\frac{\partial}{\partial \theta}} dt = 0, \ \nabla_{\frac{\partial}{\partial t}} dr = 0, \ \nabla_{\frac{\partial}{\partial t}} d\theta = 0, \ \nabla_{\frac{\partial}{\partial t}} dt = 0,$$

the covector field $\omega = \cos\theta dr - r \sin\theta d\theta + dt$ is harmonic because ω is parallel, indeed,

$$\begin{aligned} &\nabla_{\frac{\partial}{\partial r}} \omega = \cos\theta \nabla_{\frac{\partial}{\partial r}} dr - \sin\theta d\theta - r \sin\theta \nabla_{\frac{\partial}{\partial r}} d\theta + \nabla_{\frac{\partial}{\partial r}} dt = 0, \\ &\nabla_{\frac{\partial}{\partial \theta}} \omega = -\sin\theta dr + \cos\theta \nabla_{\frac{\partial}{\partial \theta}} dr - r \cos\theta d\theta - r \sin\theta \nabla_{\frac{\partial}{\partial \theta}} d\theta + \nabla_{\frac{\partial}{\partial \theta}} dt = 0, \\ &\nabla_{\frac{\partial}{\partial t}} \omega = \cos\theta \nabla_{\frac{\partial}{\partial t}} dr - r \sin\theta \nabla_{\frac{\partial}{\partial t}} d\theta + \nabla_{\frac{\partial}{\partial t}} dt = 0, \end{aligned}$$

i.e., $\nabla\omega = 0$, then ω is harmonic.

Remark 4.9. In general , using Corollary 4.5 and Theorem 4.6, we can construct many examples for harmonic covector fields.

Now we study a special case on the flat Riemannian manifold which is the real Euclidean space $(\mathbb{R}^m, g_0)$.

Theorem 4.10. *Let $(\mathbb{R}^m, g_0)$ be the real Euclidean space and $(T^*\mathbb{R}^m, \tilde{g}_0)$ its cotangent bundle equipped with the g_0-natural metric. If $\omega = (\omega_1, \dots, \omega_m) \in \Im^0_1(\mathbb{R}^m)$, then ω is a harmonic covector field if and only if the following conditions are verified*

$$(4.5)\quad \sum_{i=1}^m \Big\{ \frac{\partial^2 \omega_k}{\partial (x^i)^2} + \sum_{j=1}^m \Big[-2A(\varphi(z) + 2z\psi(z)) \frac{\partial \omega_j}{\partial x^i} \omega_j \frac{\partial \omega_k}{\partial x^i} + B(\varphi(z) + \psi(z)\omega_j^2)(\frac{\partial \omega_j}{\partial x^i})^2 \omega_k + C(\varphi(z) + 2z\psi(z))^2 (\frac{\partial \omega_j}{\partial x^i})^2 \omega_j^2 \omega_k \Big] \Big\} = 0$$

for all $k = \overline{1, m}$, where A, B and C are as in Theorem 3.2.

Proof. Let $\{\frac{\partial}{\partial x^i}\}_{i=\overline{1,m}}$ be a canonical frame on $\mathbb{R}^m$. Using Theorem 4.4, we have:

$\tau(\omega) = 0$ equivalent the following conditions (4.3) and (4.4) are verified.

Since $(\mathbb{R}^m, g_0)$ is flat i.e., $R = 0$, then the equation (4.3) is trivial.

$$(4.4) \Leftrightarrow trace_g \big[\nabla^2 \omega - 2A\, \tilde{g}((\nabla\omega)^V, \omega^V) \nabla\omega + B\, \tilde{g}((\nabla\omega)^V, (\nabla\omega)^V)\omega + C\, \tilde{g}((\nabla\omega)^V, \omega^V)^2 \omega \big] = 0$$

$$\Leftrightarrow \sum_{i=1}^m \Big\{ \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^i}} \omega - 2A\, \tilde{g}((\nabla_{\frac{\partial}{\partial x^i}}\omega)^V, \omega^V) \nabla_{\frac{\partial}{\partial x^i}} \omega + B\, \tilde{g}((\nabla_{\frac{\partial}{\partial x^i}}\omega)^V, (\nabla_{\frac{\partial}{\partial x^i}}\omega)^V)\omega + C\, \tilde{g}((\nabla_{\frac{\partial}{\partial x^i}}\omega)^V, \omega^V)^2 \omega \Big\} = 0$$

$$\Leftrightarrow \sum_{i=1}^m \Big\{ \sum_{k=1}^m \frac{\partial^2 \omega_k}{\partial (x^i)^2} dx^k - 2A\,(\varphi(z) + 2z\psi(z)) \sum_{k=1}^m \big(\frac{\partial \omega_k}{\partial x^i}\omega_k\big) \sum_{j=1}^m \big(\frac{\partial \omega_j}{\partial x^i} dx^j\big) + B \sum_{k=1}^m (\varphi(z) + \psi(z)\omega_k^2)(\frac{\partial \omega_k}{\partial x^i})^2 \sum_{j=1}^m \omega_j dx^j + C\,(\varphi(z) + 2z\psi(z))^2 \sum_{k=1}^m \big(\frac{\partial \omega_k}{\partial x^i}\big)^2 \omega_k^2 \sum_{j=1}^m \omega_j dx^j \Big\} = 0$$

$$\Leftrightarrow \sum_{i=1}^m \Big\{ \sum_{k=1}^m \frac{\partial^2 \omega_k}{\partial (x^i)^2} dx^k - 2A\,(\varphi(z) + 2z\psi(z)) \sum_{j=1}^m \big(\frac{\partial \omega_j}{\partial x^i}\omega_j\big) \sum_{k=1}^m \big(\frac{\partial \omega_k}{\partial x^i} dx^k\big) + B \sum_{j=1}^m (\varphi(z) + \psi(z)\omega_j^2)(\frac{\partial \omega_j}{\partial x^i})^2 \sum_{k=1}^m \omega_k dx^k + C\,(\varphi(z) + 2z\psi(z))^2 \sum_{j=1}^m \big(\frac{\partial \omega_j}{\partial x^i}\big)^2 \omega_j^2 \sum_{k=1}^m \omega_k dx^k \Big\} = 0$$

$$\Leftrightarrow \sum_{i=1}^m \Big\{ \frac{\partial^2 \omega_k}{\partial (x^i)^2} - 2A\,(\varphi(z) + 2z\psi(z)) \sum_{j=1}^m \big(\frac{\partial \omega_j}{\partial x^i}\omega_j\big)\big(\frac{\partial \omega_k}{\partial x^i}\big)$$

$$+ B \sum_{j=1}^{m} (\varphi(z) + \psi(z)\omega_j^2)(\frac{\partial \omega_j}{\partial x^i})^2 \omega_k$$

$$+ C\,(\varphi(z) + 2z\psi(z))^2 \sum_{j=1}^{m} \big(\frac{\partial \omega_j}{\partial x^i}\big)^2 \omega_j^2 \omega_k \Big\} = 0$$

$$\Leftrightarrow \sum_{i=1}^{m} \Big\{ \frac{\partial^2 \omega_k}{\partial (x^i)^2} + \sum_{j=1}^{m} \Big[-2A\,(\varphi(z) + 2z\psi(z)) \frac{\partial \omega_j}{\partial x^i} \omega_j \frac{\partial \omega_k}{\partial x^i}$$

$$+ B\,(\varphi(z) + \psi(z)\omega_j^2)(\frac{\partial \omega_j}{\partial x^i})^2 \omega_k$$

$$+ C\,(\varphi(z) + 2z\psi(z))^2 (\frac{\partial \omega_j}{\partial x^i})^2 \omega_j^2 \omega_k \Big] \Big\} = 0$$

for all $k = \overline{1, m}$. □

Example 4.11. If $\mathbb{R}^n$ is endowed with the canonical metric and $T^*\mathbb{R}^m$ its cotangent bundle equipped with the g-natural metric such as $\varphi(z) = 1$, $\psi(z) = \frac{1}{2z}$. From Theorem 4.10, we deduce that, $\omega = (h(x_1), 0, \ldots, 0) \in \Im_1^0(\mathbb{R}^m)$ is a harmonic covector field if and only if the function h is a solution of differential equation

$$h'' - 3\frac{(h')^2}{h} = 0, \tag{4.6}$$

i.e., $h(x_1) = \pm\frac{1}{\sqrt{ax_1 + b}}$, $a, b \in \mathbb{R}$, $a \leq 0$ and $b > 0$.

4.2. Harmonicity of the map $\sigma : (M, g) \to (T^*N, \tilde{h})$

Now we study the harmonicity of the map $\sigma : (M, g) \to (T^*N, \tilde{h})$ and we give the necessary and sufficient conditions under which this map is harmonic with respect to the g-natural metrics.

Lemma 4.12. *Let (M^m, g), (N^n, h) be two Riemannian manifolds and $\phi : (M^m, g) \to (N^n, h)$ a smooth map. If σ is a map that covers ϕ, ($\phi = \pi_N \circ \sigma$) defined by*

$$\begin{array}{rcl} \sigma : M & \longrightarrow & T^*N \\ x & \longmapsto & (\phi(x), q) \end{array}$$

where $q \in T^*_{\phi(x)}N$ and $\pi_N : T^*N \to N$ is the canonical projection, then

$$d\sigma(X) = (d\phi(X))^H + (\nabla^\phi_X \sigma)^V \tag{4.7}$$

for all $X \in \Im_0^1(M)$.

Proof. Let $x \in M$, $X \in \Im_0^1(M)$, $\omega \in \Gamma(T^*N)$ such that $\omega_{\phi(x)} = q \in T^*_{\phi(x)}N$. Using Lemma 4.1, we obtain:

$$d_x\sigma(X_x) = d_x(\omega \circ \phi)(X_x)$$

$$
\begin{aligned}
&= d_{\phi(x)}\omega(d_x\phi(X_x)) \\
&= (d\phi(X))^H_{(\phi(x),q)} + (\nabla_{d\phi(X)}\omega)^V_{(\phi(x),q)} \\
&= (d\phi(X))^H_{(\phi(x),q)} + (\nabla^\phi_X\sigma)^V_{(\phi(x),q)}. \qquad \square
\end{aligned}
$$

A direct consequence of usual calculations using Lemma 4.12 gives the following theorem.

Theorem 4.13. *Let (M^m, g), (N^n, h) be two Riemannian manifolds, $(T^*N, \tilde{h})$ the cotangent bundle of N equipped with the g-natural metric and $\phi : (M^m, g) \to (N^n, h)$ a smooth map. The tension field of the map*

$$
\begin{array}{rcl}
\sigma : (M, g) & \longrightarrow & (T^*N, \tilde{h}) \\
x & \longmapsto & (\phi(x), q)
\end{array}
$$

such that $q \in T^*_{\phi(x)}N$ is given by

$$
\begin{aligned}
\tau(\sigma) = {} & \Big[\tau(\phi) + trace_g\varphi(z)R^N(\tilde{\sigma}, \widetilde{\nabla^\phi\sigma})d\phi(*)\Big]^H \\
& + \Big[trace_g\big[(\nabla^\phi)^2\sigma - 2A\,\tilde{h}((\nabla^\phi\sigma)^V, \sigma^V)\nabla^\phi\sigma \\
& + B\,\tilde{h}((\nabla^\phi\sigma)^V, (\nabla^\phi\sigma)^V)\sigma + C\,\tilde{h}((\nabla^\phi\sigma)^V, \sigma^V)^2\sigma\big]\Big]^V,
\end{aligned}
\tag{4.8}
$$

where A, B, C are as in Theorem 3.2 and R^N denote the curvature tensor of (N, h).

Proof. Let $x \in M$ and $\{E_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on M such that $(\nabla_{E_i}E_i)_x = 0$ and $\sigma(x) = (\phi(x), q), q \in T^*_{\phi(x)}N$, we have

$$
\begin{aligned}
\tau(\sigma)_x &= trace_g(\nabla d\sigma)_x \\
&= \sum_{i=1}^m \{\nabla^\sigma_{E_i} d\sigma(E_i)\}_{(\phi(x),q)} \\
&= \sum_{i=1}^m \{\nabla^{T^*N}_{d\sigma(E_i)} d\sigma(E_i)\}_{(\phi(x),q)} \\
&= \sum_{i=1}^m \{\nabla^{T^*N}_{(d\phi(E_i))^H}(d\phi(E_i))^H + \nabla^{T^*N}_{(d\phi(E_i))^H}(\nabla^\phi_{E_i}\sigma)^V \\
&\quad + \nabla^{T^*N}_{(\nabla^\phi_{E_i}\sigma)^V}(d\phi(E_i))^H + \nabla^{T^*N}_{(\nabla^\phi_{E_i}\sigma)^V}(\nabla^\phi_{E_i}\sigma)^V\}_{(\phi(x),q)}.
\end{aligned}
$$

From Theorem 3.2, we obtain:

$$
\begin{aligned}
\tau(\sigma) = \sum_{i=1}^m \Big[& (\nabla^N_{d\phi(E_i)}d\phi(E_i))^H + (\sigma R(d\phi(E_i), d\phi(E_i)))^V + (\nabla^N_{d\phi(E_i)}\nabla^\phi_{E_i}\sigma)^V \\
& + \frac{\varphi(z)}{2}(R^N(\tilde{\sigma}, \widetilde{\nabla^\phi_{E_i}\sigma})d\phi(E_i))^H + \frac{\varphi(z)}{2}(R^N(\tilde{\sigma}, \widetilde{\nabla^\phi_{E_i}\sigma})d\phi(E_i))^H
\end{aligned}
$$

$$
\begin{aligned}
&- A\big[\tilde{h}((\nabla^{\phi}_{E_i}\sigma)^V,\sigma^V)(\nabla^{\phi}_{E_i}\sigma)^V + \tilde{h}((\nabla^{\phi}_{E_i}\sigma)^V,\sigma^V)(\nabla^{\sigma}_{E_i}\sigma)^V\big]\\
&+ B\,\tilde{h}((\nabla^{\phi}_{E_i}\sigma)^V,(\nabla^{\phi}_{E_i}\sigma)^V)\sigma^V\\
&+ C\,\tilde{h}((\nabla^{\phi}_{E_i}\sigma)^V,\sigma^V)\tilde{h}((\nabla^{\phi}_{E_i}\sigma)^V,\sigma^V)\sigma^V\Big]\\
= &\sum_{i=1}^{m}\Big[(\nabla^{\phi}_{E_i}d\phi(E_i))^H + (\nabla^{\phi}_{E_i}\nabla^{\phi}_{E_i}\sigma)^V + \varphi(z)(R^N(\tilde{\sigma},\widetilde{\nabla^{\phi}_{E_i}\sigma})d\phi(E_i))^H\\
&- 2A\,\tilde{h}((\nabla^{\phi}_{E_i}\sigma)^V,\sigma^V)(\nabla^{\phi}_{E_i}\sigma)^V + B\,\tilde{h}((\nabla^{\phi}_{E_i}\sigma)^V,(\nabla^{\phi}_{E_i}\sigma)^V)\sigma^V\\
&+ C\,\tilde{h}((\nabla^{\phi}_{E_i}\sigma)^V,\sigma^V)^2\sigma^V\Big]\\
= &\Big[\tau(\phi) + trace_g\varphi(z)R^N(\tilde{\sigma},\widetilde{\nabla^{\phi}\sigma})d\phi(*)\Big]^H\\
&+ \Big[trace_g\big[(\nabla^{\phi})^2\sigma - 2A\,\tilde{h}((\nabla^{\phi}\sigma)^V,\sigma^V)\nabla^{\phi}\sigma\\
&+ B\,\tilde{h}((\nabla^{\phi}\sigma)^V,(\nabla^{\phi}\sigma)^V)\sigma + C\,\tilde{h}((\nabla^{\phi}\sigma)^V,\sigma^V)^2\sigma\big]\Big]^V.
\end{aligned}
$$

□

From Theorem 4.13 we obtain:

Theorem 4.14. *Let (M^m,g), (N^n,h) be two Riemannian manifolds, $(T^*N,\tilde{h})$ the cotangent bundle of N equipped with the g-natural metric and $\phi:(M^m,g)\to(N^n,h)$ a smooth map. The map*

$$
\begin{aligned}
\sigma:(M,g) &\longrightarrow (T^*N,\tilde{h})\\
x &\longmapsto (\phi(x),q)
\end{aligned}
$$

such that $q\in T^*_{\phi(x)}N$ is harmonic if and only if the following conditions are verified

$$\tau(\phi) = -\,trace_g\varphi(z)R^N(\tilde{\sigma},\widetilde{\nabla^{\phi}\sigma})d\phi(*),$$

and

$$
\begin{aligned}
0 = trace_g&\big[(\nabla^{\phi})^2\sigma - 2A\,\tilde{h}((\nabla^{\phi}\sigma)^V,\sigma^V)\nabla^{\phi}\sigma\\
&+ B\,\tilde{h}((\nabla^{\phi}\sigma)^V,(\nabla^{\phi}\sigma)^V)\sigma + C\,\tilde{h}((\nabla^{\phi}\sigma)^V,\sigma^V)^2\sigma\big],
\end{aligned}
$$

where A,B,C are as in Theorem 3.2 and R^N denote the curvature tensor of (N,h).

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