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# ON PARANORMED TYPE $p$-ABSOLUTELY SUMMABLE UNCERTAIN SEQUENCE SPACES DEFINED BY ORLICZ FUNCTIONS

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Abstract. In this paper we introduce the notion of paranormed $p$-absolutely convergent and paranormed Cesŕo summable sequences of complex uncertain variables with respect to measure, mean, distribution etc. defined by on Orlicz function. We have established some relationships among these notions as well as with other classes of complex uncertain variables.

## 1. Introduction

Uncertainty is an extremely important feature of the real world. How do we understand uncertainty? How do we model uncertainty? In order to answer those questions, the notion of uncertainty theory was introduced by Liu [13]. Nowadays uncertainty theory has become a branch of mathematics for modelling human uncertainty. The uncertain theory has been studied from different aspects by Chen [2], Liu ([12, 14]), You [30] and others.

## 2. Preliminaries

In this section, we procure some fundamental concepts and theorems in uncertainty theory are introduced, which will be used throughout the paper.

**Definition 2.1** ([13])**.** Let $L$ be a $\sigma$-algebra on a nonempty set $\Gamma$. A set function $M$ is called an uncertain measure if it satisfies the following axioms:
Axiom 1. (Normality Axiom) $M\{\Gamma\} = 1$;
Axiom 2. (Duality Axiom) $M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any $\Lambda \in L$;

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Axiom 3. (Subadditivity Axiom) For every countable sequence of $\{\lambda_j\} \in L$, we have

$$M\left\{\bigcup_{j=1}^{\infty} \lambda_j\right\} \leq \sum_{j=1}^{\infty} M\{\lambda_j\}.$$

The triplet $(\Gamma, L, M)$ is called an uncertainty space, and each element $\Lambda$ in $L$ is called an event. In order to obtain an uncertain measure of compound event, a product uncertain measure is define by Liu in [16] as follows:

Axiom 4. (Product Axiom) Let $(\Gamma_k, L_k, M_k)$ be an uncertainty space for $k = 1, 2, 3, \ldots$. The product uncertain measure $M$ is an measure satisfying

$$M\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} M_k\{\Lambda_k\},$$

where $\Lambda_k$ are arbitrarily chosen events from $L_k$ for $k = 1, 2, \ldots$, respectively.

**Definition 2.2** ([13])**.** An uncertain variable $\xi$ is a measurable function from an uncertainty space $(\Gamma, L, M)$ to the set of real numbers, i.e., for any Borel set $B$ of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma : \xi(\gamma) \in B\}$$

is an event.

**Definition 2.3** ([13])**.** The uncertainty distribution $\Phi$ of an uncertain variable $\xi$ is defined by

$$\Phi(x) = M\{\xi \leq x\} \text{ for all } x \in \mathbb{R}.$$

**Definition 2.4** ([15])**.** The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$M\left\{\bigcap_{j=1}^{n} (\xi_j \in B_j)\right\} = \bigwedge_{j=1}^{n} M\{\xi_j \in B_j\}$$

for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers.

**Definition 2.5** ([13])**.** Let $\xi$ be an uncertain variable. The *expected value* of $\xi$ is defined by

$$E[\xi] = \int_0^{+\infty} M\{\xi \geq r\}dr - \int_{-\infty}^{0} M\{\xi \leq r\}dr,$$

provided that at least one of the above two integrals is finite.

Considering the importance of the role of convergence of sequence in mathematics, some concepts of convergence for uncertain sequences were introduced by B. Liu (See for instance [15]) as follows:

**Definition 2.6.** The uncertain sequence $\{\xi_n\}$ is said to be *convergent almost surely* (*a.s.*) *to* $\xi$ if there exists an event $\Lambda$ with $M\{\Lambda\} = 1$ such that

$$\lim_{n\to\infty} |\xi_n(\gamma) - \xi(\gamma)| = 0$$

for every $\gamma \in \Lambda$. In that case we write $\xi_n \to \xi$, a.s., as $n \to \infty$.

**Definition 2.7.** The uncertain sequence $\{\xi_n\}$ is said to be *convergent in measure to* $\xi$ if

$$\lim_{n\to\infty} M\{|\xi_n - \xi| \geq \varepsilon\} = 0$$

for every $\varepsilon > 0$.

**Definition 2.8.** The uncertain sequence $\{\xi_n\}$ is said to be *convergent in mean to* $\xi$ if

$$\lim_{n\to\infty} E[|\xi_n - \xi|] = 0.$$

**Definition 2.9.** Let $\Phi, \Phi_1, \Phi_2, \ldots$ be the uncertainty distributions of uncertain variables $\xi, \xi_1, \xi_2, \ldots$, respectively. We say the uncertain sequence $\{\xi_n\}$ *converges in distribution to* $\xi$ if

$$\lim_{n\to\infty} \Phi_n(x) = \Phi(x)$$

for all $x$ at which $\Phi(x)$ is continuous.

**Definition 2.10.** The uncertain sequence $\{\xi_n\}$ is said to be *convergent uniformly almost surely* (*a.s.*) *to* $\xi$ if there exists an sequence of events $\{E_k\}$, $M\{E_k\} \to 0$ such that $\{\xi_n\}$ converges uniformly to $\xi$ in $\Gamma - E_k$, for any fixed $k$.

Tripathy and Nath [28] have introduced the notion of statistical convergence of sequence of complex uncertain variables and investigated some of their properties. The notion of sequences of uncertain variables has been investigated from different aspects by Chen et al. [3], Debnath and Tripathy [5], Nath and Tripathy [19], Roy et al. [21], Tripathy and Dowari [24], Datta and Tripathy [4] and others.

## 3. Complex uncertain variable

In this section, we procure some definitions, concepts and results on complex uncertain variables those can be found in Peng [20].

As a complex function on uncertainty space, complex uncertain variable is mainly used to model a complex uncertain quantity.

**Definition 3.1.** A complex uncertain variable is a measurable function $\zeta$ from an uncertainty space $(\Gamma, L, M)$ to the set of complex numbers, i.e., for any Borel set $B$ of complex numbers, the set

$$\{\zeta \in B\} = \{\gamma \in \Gamma : \zeta(\gamma) \in B\},$$

is an event.

**Definition 3.2.** The complex uncertainty distribution $\Phi(x)$ of a complex uncertain variable $\zeta$ is a function from $\mathbb{C}$ to $[0,1]$ defined by

$$\Phi(c) = M\{Re(\zeta) \leq Re(c), Im(\zeta) \leq Im(c)\}$$

for any complex number $c$.

**Lemma 3.3.** *A variable $\zeta$ from an uncertainty space $(\Gamma, L, M)$ to the set of complex numbers is a complex uncertain variable if and only if $Re(\zeta)$ and $Im(\zeta)$ are uncertain variables where $Re(\zeta)$ and $Im(\zeta)$ represent the real and the imaginary part of $\zeta$, respectively.*

**Lemma 3.4.** *A function $\Phi : \mathbb{C} \to [0,1]$ is a complex uncertainty distribution if and only if it is increasing with respect to the real part $Re(c)$ and imaginary part $Im(c)$ such that*
(i) $\lim_{x\to-\infty} \Phi(x+ib) \neq 1$, $\lim_{y\to-\infty} \Phi(a+iy) \neq 1$ *for any* $a, b \in \mathbb{R}$;
(ii) $\lim_{x\to+\infty, y\to+\infty} \Phi(x+iy) \neq 0$,
*where $i = \sqrt{-1}$ is the imaginary unit.*

Now we give some basic concepts on $p$-absolutely summable sequences of real or complex terms.

Let $1 \leq p < \infty$ be a fixed real number. The space $\ell^p$ contains the sequences $x = (\xi_i)$ of the numbers such that

$$\sum_{j=1}^{\infty} |\xi_j|^p < \infty \quad \text{for } 1 \leq p < \infty, \tag{1}$$

and the metric on $\ell_p$ is defined by

$$d(x,y) = \left( \sum_{j=1}^{\infty} |\xi_j - \eta_j|^p \right)^{\frac{1}{p}}, \tag{2}$$

where $y = (\eta_j)$ and $\sum_{j=1}^{\infty} |\eta_j|^p < \infty$.

$\ell^p$ is a Banach space with the norm given by,

$$\|x\| = \left( \sum_{j=1}^{\infty} |\xi_j|^p \right)^{\frac{1}{p}}.$$

Let $p = (p_k)$ be a sequence of positive numbers. Then the spaces $\ell(p)$ are generalized as follows:

$$\ell(p) = \{(x_k) \in \omega : \sum_k |x_k|^{p_k} < \infty\}.$$

Let $H = \sup_k p_k < \infty$ and $M = \max(1, H)$. Then the space $\ell(p)$ is paranormed by

$$f(x) = (\sum_k |x_k|^{p_k})^{\frac{1}{M}}.$$

**Definition 3.5.** Let $\omega$ be the family of all real or complex sequences. Any subspace of $\omega$ is called a sequence space. An Orlicz function is a function $\mathcal{M} : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $\mathcal{M}(0) = 0$, $\mathcal{M}(x) > 0$ for $x > 0$ and $\mathcal{M}(x) \to \infty$ as $x \to \infty$.

If convexity of Orlicz function $\mathcal{M}$ is replaced by

$$\mathcal{M}(x + y) \leq \mathcal{M}(x) + \mathcal{M}(y),$$

then this function is called a modulus function, defined and discussed by Ruckle in [22].

Lindenstrauss and Tzafriri [18] used the idea of Orlicz function to construct the sequence space

$$\ell_{\mathcal{M}} = \left\{ x \in \omega : \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}.$$

The space $\ell_{\mathcal{M}}$ with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. Lindenstrauss and Tzafriri [18] proved that every Orlicz sequence space $\ell_{\mathcal{M}}$ contains a subspace isomorphic to $c_0$ or some $\ell_p$, positively for a class of spaces. The space $\ell_{\mathcal{M}}$ is closely related to the space $\ell_p$ which is an Orlicz sequence space with $\mathcal{M}(x) = x^p; 1 \leq p \leq \infty$.

Applying the concept of Orlicz function, different classes of sequences have been introduced by Esi ([7, 8]), Esi et al. [9], Lindenstrauss and Tzafriri [18], Tripathy and Borgohain [23], Tripathy and Dutta [6], Tripathy and Dutta [25], Tripathy and Hazarika [26], Tripathy and Sarma [29], Tripathy and Mahanta [27], Altin et al. [1], Krasnoselskii and Rutitsky [11], Lindenstrauss [17], Et et al. [10] and investigated btheir different algebraic and topological properties.

## 4. Main results

In this section, we introduce the concept of $p$-absolutely summable sequences of uncertain variables for normed and paranormed spaces using an Orlicz function. Also we introduce the Cesŕo summable sequences of paranormed type defined by an Orlicz function. Let $M$ represents an uncertain measure and $\lambda$ is an event and $\xi = (\xi_n)$ be an uncertain sequence.

$$\ell_{\mathcal{M}(u.s)} = \left\{ \xi : M(\Lambda) = 1 \text{ for all } \xi \text{ and } \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|\xi_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}.$$

The space $\ell_{\mathcal{M}(u.s)}$ with the norm,

$$\|\xi\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|\xi_k|}{\rho}\right) \leq 1 \text{ with } M(\Lambda) = 1 \text{ for all } \xi \right\}.$$

Let $p = (p_k)$ be any sequence of positive real numbers. We define the following sequence spaces.

$$\ell_{\mathcal{M}(u.s)}(p) = \left\{\xi : M(\Lambda) = 1 \text{ for all } \xi \text{ and } \sum_{k=1}^{\infty}\left(\mathcal{M}\left(\frac{|\xi_k|}{\rho}\right)\right)^{p_k} < \infty \text{ for some } \rho > 0\right\},$$

$$W(\mathcal{M}(u.s), p) = \left\{\xi : M(\Lambda) = 1 \forall \xi \text{ and } \frac{1}{n}\sum_{k=1}^{\infty}\left(\mathcal{M}\left(\frac{|\xi_k - l|}{\rho}\right)\right)^{p_k} \to 0 \text{ as } n \to \infty \text{ for some } \rho > 0\right\},$$

$$W_0(\mathcal{M}(u.s), p) = \left\{\xi : M(\Lambda) = 1 \text{ for all } \xi \text{ and } \frac{1}{n}\sum_{k=1}^{\infty}\left(\mathcal{M}\left(\frac{|\xi_k|}{\rho}\right)\right)^{p_k} \to 0 \text{ as } n \to \infty \text{ for some } \rho > 0\right\},$$

$$W_\infty(\mathcal{M}(u.s), p) = \left\{\xi : M(\Lambda) = 1 \text{ for all } \xi \text{ and } \sup_n \frac{1}{n}\sum_{k=1}^{\infty}\left(\mathcal{M}\left(\frac{|\xi_k|}{\rho}\right)\right)^{p_k} < \infty \text{ for some } \rho > 0\right\}.$$

When $(\xi_n)$ is a sequence of real or complex number then $\ell_{\mathcal{M}(u.s)}(p)$ becomes $\ell_{\mathcal{M}}(p)$. We denote

$$W(\mathcal{M}(u.s), p), W_0(\mathcal{M}(u.s), p) \text{ and } W_\infty(\mathcal{M}(u.s), p)$$

as $W(\mathcal{M}(u.s))$, $W_0(\mathcal{M}(u.s))$ and $W_\infty(\mathcal{M}(u.s))$ when $p_k = 1$ for each $k$.

### 4.1. Properties of $\ell_{\mathcal{M}(u.s)}(p)$

In this section we shall establish some properties of the sequence spaces defined in the previous section. In order to discuss the properties of $\ell_{\mathcal{M}(u.s)}(p)$ we assume that $(p_k)$ is a bounded sequence of real numbers.

**Theorem 4.1.** *$\ell_{\mathcal{M}(u.s)}(p)$ is a linear set over the set of complex numbers $\mathbb{C}$.*

*Proof.* Let, $\xi, \zeta \in \ell_{\mathcal{M}(u.s)}(p)$ such that $M(\Lambda) = 1$ for $\xi, \zeta$ and $\alpha, \beta \in \mathbb{C}$. In order to proof the result we need to find some $\rho_3$ such that

$$\sum_{k=1}^{\infty}\left(\mathcal{M}\left(\frac{|\alpha\xi_k + \beta\zeta_k|}{\rho_3}\right)\right)^{p_k} < \infty.$$

Since $\xi, \zeta \in \ell_{\mathcal{M}(u.s)}(p)$, there exist some positive $\rho_1$ and $\rho_2$ such that

$$\sum_{k=1}^{\infty}\left(\mathcal{M}\left(\frac{|\xi_k|}{\rho_1}\right)\right)^{p_k} < \infty \tag{3}$$

and

$$\sum_{k=1}^{\infty}\left(\mathcal{M}\left(\frac{|\zeta_k|}{\rho_2}\right)\right)^{p_k} < \infty. \tag{4}$$

We define, $\rho_3 = \rho_1 + \rho_2$.

Since, $\mathcal{M}$ is non decreasing and convex

$$\sum_{k=1}^{\infty}\left(\mathcal{M}\left(\frac{|\alpha\xi_k + \beta\zeta_k|}{\rho_3}\right)\right)^{p_k} = \sum_{k=1}^{\infty}\left(\mathcal{M}\left(\frac{|\alpha\xi_k + \beta\zeta_k|}{\rho_1+\rho_2}\right)\right)^{p_k}$$

$$= \sum_{k=1}^{\infty}\left(\mathcal{M}\left(\frac{\rho_1}{\rho_1+\rho_2}\frac{\xi_k}{\rho_1} + \frac{\rho_2}{\rho_1+\rho_2}\frac{\zeta_k}{\rho_2}\right)\right)^{p_k}$$

$$\leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right)^{p_k}\sum_{k=1}^{\infty}\left(\mathcal{M}\left(\frac{\xi_k}{\rho_1}\right)\right)^{p_k} + \left(\frac{\rho_2}{\rho_1+\rho_2}\right)^{p_k}\sum_{k=1}^{\infty}\left(\mathcal{M}\left(\frac{\zeta_k}{\rho_2}\right)\right)^{p_k} \leq \infty,$$

by (3) and (4). □

**Theorem 4.2.** *$\ell_{\mathcal{M}(u.s)}(p)$ is a total paranormed space with*

$$g(\xi) = \inf\left\{\rho^{\frac{p_n}{H}} : \left(\sum_{k=1}^{\infty}\left[\mathcal{M}\left(\frac{|\xi_k|}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1, n = 1, 2, \ldots\right\},$$

*where* $H = \max(1, \sup_k p_k)$.

*Proof.* Clearly $g(\xi) = g(-\xi)$. By using Theorem 4.1 for $\alpha = \beta = 1$, we get $g(\xi + \zeta) \leq g(\xi) + g(\zeta)$. Since $\mathcal{M}(0) = 0$, we get $\inf\{\rho^{\frac{p_n}{H}}\} = 0$ for $\xi = 0$. Conversely, suppose $g(\xi) = 0$, then

$$\inf\left\{\rho^{\frac{p_n}{H}} : \left(\sum_{k=1}^{\infty}\left[\mathcal{M}\left(\frac{|\xi_k|}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1\right\} = 0.$$

This implies that for a given $\varepsilon > 0$, there exists some $\rho_\varepsilon(0 < \rho_\varepsilon < \varepsilon)$ such that

$$\left(\sum_{k=1}^{\infty}\left[\mathcal{M}\left(\frac{|\xi_k|}{\rho_\varepsilon}\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1.$$

Thus

$$\left(\sum_{k=1}^{\infty}\left[\mathcal{M}\left(\frac{|\xi_k|}{\varepsilon}\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq \left(\sum_{k=1}^{\infty}\left[\mathcal{M}\left(\frac{|\xi_k|}{\rho_\varepsilon}\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1.$$

Suppose $\xi_{n_m} \neq 0$ for some $m$. Let $\varepsilon \to 0$, then

$$\left(\frac{|\xi_{n_m}|}{\varepsilon}\right) \to \infty,$$

it follows that

$$\left(\sum_{m=1}^{\infty}\left[\mathcal{M}\left(\frac{|\xi_{n_m}|}{\varepsilon}\right)\right]^{p_m}\right)^{\frac{1}{H}} \to \infty,$$

which is a contradiction. Therefore $\xi_{n_m} = 0$ for each $m$.

Finally, we prove scalar multiplication is continuous. Let $\lambda$ be any number. By definition,

$$g(\lambda\xi) = \inf\left\{\rho^{\frac{p_n}{H}} : \left(\sum_{k=1}^{\infty}\left[\mathcal{M}\left(\frac{|\lambda\xi_k|}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1, n = 1, 2, 3, \ldots\right\}.$$

Then

$$g(\lambda\xi) = \inf\left\{(\lambda r)^{\frac{p_n}{H}} : \left(\sum_{k=1}^{\infty}\left[\mathcal{M}\left(\frac{|\xi_k|}{r}\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1, n = 1, 2, 3, \ldots\right\},$$

where $r = \frac{\rho}{\lambda}$.

Since $|\lambda|^{p_k} \leq \max(1, |\lambda|^H)$ therefore $|\lambda|^{\frac{p_k}{H}} \leq \max(1, |\lambda|^H)^{\frac{1}{H}}$.

Hence

$$g(\lambda\xi)$$

$$\leq \max(1, |\lambda|^H)^{\frac{1}{H}} \inf\left\{(r)^{\frac{p_n}{H}} : \left(\sum_{k=1}^{\infty}\left[\mathcal{M}\left(\frac{|\xi_k|}{r}\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1\right\} n = 1, 2, 3, \ldots,$$

which converges to zero as $g(\xi)$ converges to zero in $\ell_{\mathcal{M}(u.s)}(p)$. Now suppose $\lambda_n \to 0$ and $\xi$ is in $l_{\mathcal{M}(u.s)}(p)$. For arbitrary $\varepsilon > 0$, let $N$ be a positive integer such that

$$\sum_{k=N+1}^{\infty}\left[\mathcal{M}\left(\frac{|\xi_k|}{\rho}\right)\right]^{p_k} < \frac{\varepsilon}{2}$$

for some $\rho > 0$. This implies that

$$\left(\sum_{k=N+1}^{\infty}\left[\mathcal{M}\left(\frac{|\xi_k|}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq \frac{\varepsilon}{2}.$$

Let $0 < |\lambda| < 1$, using convexity of $\mathcal{M}$ we get

$$\sum_{k=N+1}^{\infty}\left[\mathcal{M}\left(\frac{|\xi_k|}{\rho}\right)\right]^{p_k} < \sum_{k=N+1}^{\infty}\left[|\lambda|\mathcal{M}\left(\frac{|\xi_k|}{\rho}\right)\right]^{p_k} < \left(\frac{\varepsilon}{2}\right)^H.$$

Since $\mathcal{M}$ is continuous every where in $[0, \infty)$,

$$f(t) = \sum_{k=1}^{N}\left[\mathcal{M}\left(\frac{|t\xi_k|}{\rho}\right)\right],$$

is continuous at 0. So there is $1 > \delta > 0$ such that $|f(t)| < \frac{\varepsilon}{2}$ for $0 < t < \delta$. Let $K$ be such that $|\lambda_n| < \delta$ for $n > K$, then for $n > K$,

$$\left(\sum_{k=1}^{N}\left[\mathcal{M}\left(\frac{|\lambda_n\xi_k|}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}} < \frac{\varepsilon}{2}.$$

Thus

$$\left(\sum_{k=1}^{\infty}\left[\mathcal{M}\left(\frac{|\lambda_n \xi_k|}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}} < \varepsilon \text{ for } n > K. \qquad \square$$

*Remark* 4.3. It can be easily verified that when $\mathcal{M}(\xi) = \xi$, in the paranorm defined for $\ell_{\mathcal{M}(u.s)}(p)$, we can find the paranorm defined on $\ell(p)$ are same.

**Theorem 4.4.** *Let* $1 \le p_k < \infty$. *Then* $\ell_{\mathcal{M}(u.s)}(p)$ *is a complete paranormed space with*

$$g(\xi) = \inf\left\{\rho^{\frac{p_n}{H}} : \left(\sum_{k=1}^{\infty}\left[\mathcal{M}\left(\frac{|\xi_k|}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}} \le 1,\ n = 1, 2, \dots\right\}.$$

*Proof.* Let $(\xi^i)$ be any Cauchy sequence in $\ell_{\mathcal{M}(u.s)}(p)$. Let $r$ and $\xi_0$ be fixed. Then for each $\frac{\varepsilon}{r\xi_0} > 0$ there exists a positive integer $N$ such that

$$g(\xi^i - \xi^j) < \frac{\varepsilon}{r\xi_0} \text{ for all } i, j \ge N.$$

Using definition of paranorm, we get

$$\left(\sum_{k=1}^{\infty}\left[\mathcal{M}\left(\frac{|\xi_k^i - \xi_k^j|}{g(\xi^i - \xi^j)}\right)\right]^{p_k}\right)^{\frac{1}{H}} \le 1 \text{ for all } i, j \ge N.$$

Thus

$$\sum_{k=1}^{\infty}\left[\mathcal{M}\left(\frac{|\xi_k^i - \xi_k^j|}{g(\xi^i - \xi^j)}\right)\right]^{p_k} \le 1 \text{ for all } i, j \ge N.$$

Since $1 \le p_k < \infty$ it follows that

$$\mathcal{M}\left(\frac{|\xi_k^i - \xi_k^j|}{g(\xi^i - \xi^j)}\right) \le 1 \text{ for all } k \ge 1$$

and for all $i, j \ge N$. Hence one can find $r > 0$ with

$$\left(\frac{\xi_0}{2}\right) rq \left(\frac{\xi_0}{2}\right) \ge 1,$$

where $q$ is the kernel associated with $\mathcal{M}$, such that

$$\mathcal{M}\left(\frac{|\xi_k^i - \xi_k^j|}{g(\xi^i - \xi^j)}\right) \le r\left(\frac{\xi_0}{2}\right) q\left(\frac{\xi_0}{2}\right).$$

This implies that

$$|\xi_k^i - \xi_k^j| < \frac{r\xi_0}{2}\frac{\varepsilon}{r\xi_0} = \frac{\varepsilon}{2}.$$

Hence $(\xi^i)$ is a Cauchy sequence in $\mathbb{R}$. Therefore for each $\varepsilon (0 < \varepsilon < 1)$, there exists a positive integer $N$ such that

$$|\xi_k^i - \xi| < \varepsilon \text{ for all } i, j \ge N.$$

Using continuity of $\mathcal{M}$, we find that

$$\left(\sum_{k=1}^{N}\left(\mathcal{M}\left(\frac{|\xi_k^i - \lim_{j\to\infty}\xi_k^j|}{\rho}\right)\right)^{p_k}\right)^{\frac{1}{H}} \le 1.$$

Thus

$$\left(\sum_{k=1}^{N}\left(\mathcal{M}\left(\frac{|\xi_k^i - \xi|}{\rho}\right)\right)^{p_k}\right)^{\frac{1}{H}} \le 1.$$

Taking infimum of such $\rho$'s we get

$$\inf\left\{\rho^{\frac{p_n}{H}} : \left(\sum_{k=1}^{N}\left[\mathcal{M}\left(\frac{|\xi_k - \xi|}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}} \le 1\right\} < \varepsilon$$

for all $i \ge N$ and $j \to \infty$. Since $(\xi^i) \in l_{\mathcal{M}(u.s)}(p)$ and $\mathcal{M}$ is continuous, it follows that $\xi \in l_{\mathcal{M}(u.s)}(p)$. This completes the proof of the theorem. □

**Theorem 4.5.** *Let $0 < p_k \le q_k < \infty$ for each $k$. Then*

$$\ell_{\mathcal{M}(u.s)}(p) \subseteq \ell_{\mathcal{M}(u.s)}(q).$$

*Proof.* Let $\xi \in \ell_{\mathcal{M}(u.s)(p)}$. Then there exists some $\rho > 0$ such that

$$\sum_{k=1}^{\infty}\left(\mathcal{M}\left(\frac{|\xi_k|}{\rho}\right)\right)^{p_k} \le \infty.$$

This implies that

$$\mathcal{M}\left(\frac{|\xi_k|}{\rho}\right) \le 1 \text{ for sufficiently large values of } i.$$

Since $\mathcal{M}$ is non-decreasing, we get

$$\sum\left(\mathcal{M}\left(\frac{|\xi_k|}{\rho}\right)\right)^{q_k} \le \sum\left(\mathcal{M}\left(\frac{|\xi_k|}{\rho}\right)\right)^{p_k} < \infty.$$

Hence $\xi \in l_{\mathcal{M}(u.s)}(q)$. □

### 4.2. Properties of spaces $W(\mathcal{M}(u.s), p)$, $W_0(\mathcal{M}(u.s), p)$ and $W_\infty(\mathcal{M}(u.s), p)$

In this section we study some properties of the spaces

$$W(\mathcal{M}(u.s), p), W_0(\mathcal{M}(u.s), p) \text{ and } W_\infty(\mathcal{M}(u.s), p)$$

defined in the previous section. We state the following two results without proof.

**Theorem 4.6.** *Let $(p_k)$ be bounded. Then $W(\mathcal{M}(u.s), p)$, $W_0(\mathcal{M}(u.s), p)$ and $W_\infty(\mathcal{M}(u.s), p)$ are linear spaces.*

**Theorem 4.7.** *Let* $H = \sup_k p_k$. *Then* $W_0(\mathcal{M}(u.s), p)$ *is a linear topological space paranormed by* $g'$ *defined*

$$g'(\xi) = \inf \left\{ \rho^{\frac{p_n}{H}} : \left( \frac{1}{n} \sum_{k=1}^{n} \left[ \mathcal{M}\left( \frac{|\xi_k|}{\rho} \right) \right]^{p_k} \right)^{\frac{1}{H}} \leq 1, n = 1, 2, \dots \right\}.$$

In order to discuss further result we need the following definition.

**Definition 4.8.** An Orlicz function $\mathcal{M}$ is said to satisfy $\Delta_2$-condition for all values of $u$, if there exists, constant $K > 0$, such that

$$\mathcal{M}(2u) \leq K\mathcal{M}(u), (u \geq 0).$$

The $\Delta_2$-condition is equivalent to the satisfaction of inequality

$$\mathcal{M}(lu) \leq K.l\mathcal{M}(u)$$

for all values of $u$ and for $l > 1$.

We write $[C, 1] = W_1$, $[C, 1]_0 = W_0$ and $[C, 1]_\infty = W_\infty$.

**Theorem 4.9.** *Let* $\mathcal{M}$ *be an Orlicz function which satisfies* $\Delta_2$*-condition. Then*

$$W_1 \subseteq W(\mathcal{M}(u.s)), W_0 \subseteq W_0(\mathcal{M}(u.s)) \text{ and } W_\infty \subseteq W_\infty(\mathcal{M}(u.s)).$$

*Proof.* Let $\xi \in W_1$, then

$$S_n = \frac{1}{n} \sum_{k=1}^{n} |\xi_k - l| \to 0 \quad \text{as} \quad n \to \infty.$$

Let $\varepsilon > 0$ and choose $\delta$ with $0 < \delta < 1$ such that $\mathcal{M}(t) < \varepsilon$ for $0 \leq t \leq \delta$. Write $\zeta_k = |\xi_k - l|$ and consider

$$\sum_{k=1}^{n} \mathcal{M}(|\zeta_k|) = \sum_1 + \sum_2,$$

where the first summation is over $\zeta_k \leq \delta$ and the second summation is over $\zeta_k > \delta$. Since $\mathcal{M}$ is continuous

$$\sum_1 < n\varepsilon$$

and for $\zeta_k > \delta$, we use the fact that

$$\zeta_k < \frac{\zeta_k}{\delta} < 1 + \frac{\zeta_k}{\delta}.$$

Since $\mathcal{M}$ is non decreasing and convex, it follows that

$$\mathcal{M}(\zeta_k) < \mathcal{M}(1 + \frac{\zeta_k}{\delta}) < \frac{1}{2}\mathcal{M}(2) + \frac{1}{2}\mathcal{M}(\frac{2\zeta_k}{\delta}).$$

Since $\mathcal{M}$ satisfies $\Delta_2$-condition, therefore

$$\mathcal{M}(\zeta_k) < \frac{1}{2}\frac{k\zeta_k}{\delta}\mathcal{M}(2) + \frac{1}{2}\frac{k\zeta_k}{\delta}\mathcal{M}(2) = k\zeta_k\delta^{-1}\mathcal{M}(2).$$

Hence $\sum_2 \mathcal{M}(\zeta_k) \le k\delta^{-1}\mathcal{M}(2)nS_n$, which together with $\sum_1 \le \varepsilon n$ yields $W_1 \subseteq W(\mathcal{M})$. Following similar arguments we can prove that $W_0 \subseteq W_0(\mathcal{M}(u.s))$ and $W_\infty \subseteq W_\infty(\mathcal{M}(u.s))$. □

**Theorem 4.10.** (i) *Let* $0 < \inf p_k \le p_k \le 1$. *Then*

$$W(\mathcal{M}(u.s), p) \subseteq W(\mathcal{M}(u.s)).$$

(ii) *Let* $1 < p_k \le \sup p_k < \infty$. *Then*

$$W(\mathcal{M}(u.s)) \subseteq W(\mathcal{M}(u.s), p).$$

*Proof.* (i) Let $\xi \in W(\mathcal{M}(u.s), p)$ since $0 < \inf p_k \le 1$ we get

$$\frac{1}{n}\sum_{k=1}^{\infty}\left(\mathcal{M}\left(\frac{|\xi_k - L|}{\rho}\right)\right) \le \frac{1}{n}\sum_{k=1}^{n}\left(\mathcal{M}\left(\frac{|\xi_k - L|}{\rho}\right)\right)^{p_k}$$

and hence $\xi \in W(\mathcal{M}(u.s))$.

(ii) Let $p_k \ge 1$ for each $k$, and $\sup_k p_k < \infty$. Let $\xi \in W(\mathcal{M}(u.s))$. Then for each $1 > \varepsilon > 0$, there exists a positive integer $N$ such that

$$\frac{1}{n}\sum_{k=1}^{n}\left(\mathcal{M}\left(\frac{|\xi_k - L|}{\rho}\right)\right) \le \varepsilon \le 1$$

for all $n \ge N$. This implies that

$$\frac{1}{n}\sum_{k=1}^{n}\left(\mathcal{M}\left(\frac{|\xi_k - L|}{\rho}\right)\right)^{p_k} \le \frac{1}{n}\sum_{k=1}^{n}\left(\mathcal{M}\left(\frac{|\xi_k - L|}{\rho}\right)\right).$$

Therefore $\xi \in W(\mathcal{M}(u.s), p)$. □

**Theorem 4.11.** *Let* $0 < p_k \le q_k$ *and* $\left(\frac{q_k}{p_k}\right)$ *be bounded. Then*

$$W(\mathcal{M}(u.s), q) \subseteq W(\mathcal{M}(u.s), p).$$

*Proof.* Let $\xi \in W(\mathcal{M}(u.s), q)$. Write $t_k = \left(\mathcal{M}\left(\frac{\xi_k - l}{rho}\right)\right)^{q_k}$ and $\lambda_k = \frac{p_k}{q_k}$. Since $p_k \le q_k$ therefore $0 < \lambda_k \le 1$. Take $0 < \lambda < \lambda_k$. Define $u_k = t_k(t_k \ge 1), u_k = 0(t_k < 1)$ and $v_k = 0(t_k \ge 1), v_k = t_k(t_k < 1)$. So $t_k = u_k + v_k$ and $t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$.

Now it follows that

$$u_k^{\lambda_k} \le u_k \le t_k \quad \text{and} \quad v_k^{\lambda_k} \le v_k^{\lambda}.$$

Therefore

$$\frac{1}{n}\sum_{k=1}^{n} t_k^{\lambda_k} \le \frac{1}{n}\sum_{k=1}^{n} + \left[\frac{1}{n}\sum_{k=1}^{n} v_k\right]^{\lambda}$$

and hence $\xi \in W(\mathcal{M}(u.s), p)$. □

## 5. Conclusion

In this article we introduce the paranormed absolutely summable and the paranormed Cesarosummable sequences of complex uncertain variables defined by Orlicz function and have investigated some of their algebraic and topological properties. These techniques can be applied for studying other classes of sequences.

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