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SYSTEM OF GENERALIZED SET-VALUED PARAMETRIC ORDERED VARIATIONAL INCLUSION PROBLEMS WITH OPERATOR $\oplus$ IN ORDERED BANACH SPACES

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Abstract. In this article, we study a system of generalized set-valued parametric ordered variational inclusion problems with operator $\oplus$ in ordered Banach spaces. We introduce the concept of the resolvent operator associated with (α, λ)-ANODSM set-valued mapping and establish the existence theorem of solution for the system of generalized set-valued parametric ordered variational inclusion problems in ordered Banach spaces. In order to prove the existence of solution, we suggest an iterative algorithm and discuss the convergence analysis under some suitable mild conditions.

1. Introduction

The theory of variational inequality was planted by Stampacchia [36] and this theory has appeared as a coherent tool for solving lots of problems emerging in different branches of applied and pure sciences, see; for example, [7,9,10,21,32–35,37] and references therein. Variational inclusion is an applicable and salient generalization of variational inequalities, which was instituted by Hassouni and Moudafi [20] and includes mixed variational inequalities as special cases. The concept of maximal monotonicity is widely used to solve many problems related to variational inclusions. Motivated by the applications of monotonicity to mathematics in general and particularly to variational inequality, the idea was generalized by number of researchers to H-monotonicity [14], H-accretivity [16] etc., see; for example, [2,5,11,15,22,38] and references therein. It is also noted that the mixed variational inequalities contains the nonlinear term, hence the projection method and its alternative forms are not useful to suggest numerical methods. These conclusions encouraged us to apply the concept of the resolvent

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operator. The origin of resolvent operator is mainly due to Martinet [31] and Brezis [8].

It was Amann [6] who introduced and studied plenty nonlinear equations in ordered Banach spaces. Since then, fixed point problems of various nonlinear mappings have been researched in ordered Banach spaces [12, 13, 18, 19]. Recently, Li [23] has studied general nonlinear ordered variational inequalities and ordered equations in ordered Banach spaces. In 2012, Li [24] recommended a nonlinear inclusion problem which involves (α, λ)-$NODM$ set-valued mappings. After that Li et al. [29] investigated a weak-ANODD mapping and solved some inclusion problem by using it. During last few years, a notable research work has been done by number of researchers, see, [1, 3, 4, 26, 30] to study nonlinear ordered problems in ordered Banach and Hilbert spaces.

Exhorted by the above mentioned facts and ongoing research, this work is devoted to study a system of generalized parametric set-valued ordered variational inclusions involving (α, λ)-ANODSM set-valued mapping in ordered Banach spaces. We define a resolvent operator for (α, λ)-ANODSM set-valued mapping and discuss some of its properties. We establish a fixed point formulation and the defined resolvent operator is used to study existence of unique solution. Finally, we propose an iterative scheme and discuss convergence analysis of our problem. The findings declared in this work, generalize and refine some of the results existing in the literature.

2. Preliminaries and auxiliary results

For the sake of completeness, in this section, we collect some basic notions, definitions and auxiliary results from the existing literature.

Throughout the paper, unless otherwise specified, we denote the real ordered Banach space by $\mathcal{B}$, inner product by $\langle \cdot, \cdot \rangle$ and the induced norm by $\|\cdot\|$. θ denotes the zero element and partial ordered relation " $\leq$ " defined by the normal cone $\mathcal{C}$ with a normal constant $\lambda_{\mathcal{C}}$. For any arbitrary $u, v \in \mathcal{B}$, glb$\{u, v\}$ represents the greatest lower bound and lub$\{u, v\}$ represents the least upper bound with partial ordered relation $\leq$. Assume that glb$\{u, v\}$ and lub$\{u, v\}$ exist. The operators $\wedge, \vee$ and $\oplus$ are called AND, OR and XOR operators, respectively and defined as follows:

(i) $u \wedge v = \text{glb}\{u, v\}$,
(ii) $u \vee v = \text{lub}\{u, v\}$,
(iii) $u \oplus v = (u - v) \vee (v - u)$.

Next, we recall some familiar concepts and results which are indispensable to accomplish the objective of this paper.

Definition. A closed convex subset $\mathcal{C}(\neq \phi)$ of $\mathcal{B}$ is called a cone, if

(i) $u \in \mathcal{C}$ and $\lambda > 0 \Rightarrow \lambda u \in \mathcal{C}$;
(ii) if $u \in \mathcal{C}$ and $-u \in \mathcal{C}$, then $u = \theta$.

Definition. Let $\mathcal{C}(\neq \phi)$ be a subset of $\mathcal{B}$. Then

(i) $\mathcal{C}$ is said to be a normal cone if there exists a $\lambda_{\mathcal{C}} > 0$ such that $\theta \leq u \leq v$ implies $\|u\| \leq \lambda_{\mathcal{C}} \|v\|$;
(ii) for arbitrary elements $u, v \in \mathcal{B}, u \leq v$ if $u - v \in \mathcal{C}$;
(iii) u and v are said to be comparable to each other, if $u \leq v$ or $v \leq u$ and is denoted by $u \propto v$.

A real Banach space $(\mathcal{B}, \leq)$ is called an ordered real Banach space.

Lemma 2.1 ([13]). *If $u \propto v$, then $lub\{u, v\}$ and $glb\{u, v\}$ exist, $(u-v) \propto (v-u)$ and $\theta \leq (u - v) \vee (v - u)$.*

Lemma 2.2 ([13]). *Let $\mathcal{C}$ be a normal cone with normal constant $\lambda_{\mathcal{C}}$ in real ordered Banach space $\mathcal{B}$. Then for each $u, v \in \mathcal{B}$, the following relations hold:*
(i) $\|\theta \oplus \theta\| = \|\theta\| = \theta$;
(ii) $\|u \oplus v\| \leq \|u - v\| \leq \lambda_{\mathcal{C}} \|u \oplus v\|$;
(iii) *if $u \propto v$, then $\|u \oplus v\| = \|u - v\|$.*

Lemma 2.3 ([13]). *If $u \propto v_n$ and $v_n \to v^*$ as $n \to \infty$, then $u \propto v^*$ for all $n \in \mathbb{N}$.*

Lemma 2.4 ([24]). *Let $\mathcal{C}$ be a cone in $\mathcal{B}$. Then for any $r, s, u, v \in \mathcal{B}$, we have the following:*
(i) $u \oplus v = v \oplus u$;
(ii) $u \oplus u = \theta$;
(iii) $\theta \leq u \oplus \theta$;
(iv) $(\lambda u) \oplus (\lambda v) = |\lambda|(u \oplus v)$ *for any real* λ;
(v) *if u, v and w are comparable to each other, then* $(u \oplus v) \leq (u \oplus w) + (w \oplus v)$;
(vi) *if $(r + s) \vee (u + v)$ exists with $r \propto u, v$ and $s \propto u, v$, then*
$$(r + s) \oplus (u + v) \leq ((r \oplus u) + (s \oplus v)) \wedge ((r \oplus v) + (s \oplus u));$$
(vii) *if u, v, z, w are comparable to each other, then*
$$(u \wedge v) \oplus (z \wedge w) \leq ((u \oplus z) \vee (v \oplus w)) \wedge ((u \oplus w) \vee (v \oplus z));$$
(viii) *if $r \leq s$ and $u \leq v$, then $r + u \leq s + v$;*
(ix) *if $u \propto \theta$, then $-u \oplus \theta \leq u \leq u \oplus \theta$;*
(x) *if $u \propto v$, then $(u \oplus \theta) \oplus (v \oplus \theta) \leq (u \oplus v) \oplus \theta = u \oplus v$;*
(xi) $(u \oplus \theta) - (v \oplus \theta) \leq (u - v) \oplus \theta$;
(xii) *if $\theta \leq u$ and $u \neq \theta$ and $\alpha > \theta$, then $\theta \leq \alpha u$ and $\alpha u \neq \theta$.*

Definition ([26]). Let $\Omega(\neq \phi) \subseteq \mathcal{B}$ in which parameter ϱ takes values. Let $g : \mathcal{B} \times \Omega \to \mathcal{B}$ be a single-valued mapping and $M(\cdot, g(u, \varrho), \cdot) : \mathcal{B} \times \mathcal{B} \times \Omega \to 2^{\mathcal{B}}$ be a set-valued mapping such that $M(u, \cdot, \varrho)$ is a closed subset of $\mathcal{B}$. Then
(i) M is said to be a comparison mapping, if for any $q_u \in M(u, \cdot, \cdot), u \propto q_u$ and if $u \propto v$, then for any $q_u \in M(u, \cdot, \cdot)$ and any $q_v \in M(v, \cdot, \cdot), q_u \propto q_v$ for all $u, v \in \mathcal{B}$;

(ii) M is said to be a comparison mapping with respect to g, if for any $q_u \in M(\cdot, g(u), \cdot)$, $u \propto q_u$ and if $u \propto v$, then for any $q_u \in M(\cdot, g(u), \cdot)$ and any $q_v \in M(\cdot, g(v), \cdot), q_u \propto q_v$ for all $u, v \in \mathcal{B}$.

Definition ([26])**.** Let $A : \mathcal{B} \to \mathcal{B}$ be a single-valued mapping. A comparison mapping M is said to be

(i) an α-non-ordinary difference mapping, if there exists an $\alpha > 0$, for each $u, v \in \mathcal{B}, q_u \in M(u, \cdot, \cdot)$ and $q_v \in M(v, \cdot, \cdot)$ such that

$$(q_u \oplus q_v) \oplus \alpha(u \oplus v) = \theta;$$

(ii) a λ-ordered strongly monotone increasing mapping with respect to A, if for $u \geq v$ there exists a $\lambda > 0$ such that

$$\lambda(q_u - q_v) \geq u - v, \ \forall u, v \in \mathcal{B}, q_u \in M(A(u), \cdot, \cdot), q_v \in M(A(v), \cdot, \cdot);$$

(iii) an (α, λ)-ANODSM mapping, if M is an α-non-ordinary difference mapping, λ-ordered strongly monotone increasing mapping with respect to A and $[A + \lambda M(u, \cdot, \cdot)](\mathcal{B}) = \mathcal{B}$ for every $\alpha, \lambda > 0$.

Definition ([24])**.** Let $\mathcal{C} \subseteq \mathcal{B}$ be a normal cone with normal constant $\lambda_{\mathcal{C}}$. A mapping $A : \mathcal{B} \times \Omega \to \mathcal{B}$ is said to be

(i) a γ-ordered compression mapping, if A is a comparison mapping and if there exists a $\gamma \in (0, 1)$ such that

$$(A(u, \cdot) \oplus A(v, \cdot)) \leq \gamma(u \oplus v) \text{ for all } u, v \in \mathcal{B};$$

(ii) a β-ordered non-extended mapping, if there exists a $\beta > 0$ such that

$$(A(u, \cdot) \oplus A(v, \cdot)) \geq \beta(u \oplus v) \text{ for all } u, v \in \mathcal{B};$$

(iii) a strongly comparison mapping, if A is a comparison mapping and $A(u) \propto A(v)$ if $u \propto v$ for all $u, v \in \mathcal{B}$.

Definition ([23])**.** A mapping $A : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ is said to be a restricted-accretive mapping, if A is a comparison mapping and there exist constants $0 < \beta_1, \beta_2 < 1$ such that

$$(A(u, \cdot) + I(u)) \oplus (A(v, \cdot) + I(v)) \leq \beta_1(A(u, \cdot) \oplus A(v, \cdot)) + \beta_2(u \oplus v), \ \forall u, v \in \mathcal{B};$$

where I denotes the identity mapping on $\mathcal{B}$.

Definition ([26])**.** Let $\Omega(\neq \phi)$ be an open subset of $\mathcal{B}$. An element $u = u(\varrho) \in \mathcal{B}, \varrho \in \Omega$ is said to be a comparison element, if $\varrho_1 \neq \varrho_2$ implies $u(\varrho_1) \neq u(\varrho_2)$ for any $\varrho_1, \varrho_2 \in \Omega$.

Definition ([23])**.** A mapping $F : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ is said to be an (μ, ν)-ordered mixed Lipschitz continuous, if $u \propto v, p \propto q$, then $F(p, u) \propto F(q, v)$ and there exist $\mu, \nu > 0$ such that

$$F(p, u) \oplus F(q, v) \leq \mu(p \oplus q) + \nu(u \oplus v).$$

Definition. Let $A : \mathcal{B} \to \mathcal{B}$ be a β-ordered non-extended mapping and $M : \mathcal{B} \times \mathcal{B} \times \Omega \to 2^{\mathcal{B}}$ be an α-non-ordinary difference mapping with respect to A. Then the resolvent operator $R^M_{\lambda,A} : \mathcal{B} \to \mathcal{B}$ associated with A and M is defined as

$$R^M_{\lambda,A}(u) = (A + \lambda M)^{-1}(u),\ \forall u \in \mathcal{B},\ \lambda > 0. \tag{2.1}$$

Lemma 2.5. *The resolvent operator defined by* (2.1) *is single-valued for* $\alpha, \lambda > 0$.

Proof. For any $p \in \mathcal{B}$, let $u, v \in (A + \lambda M)^{-1}(p)$. Then, we have

$$\frac{1}{\lambda}(p - A(u)) \in M(u, \cdot, \cdot)$$

and

$$\frac{1}{\lambda}(p - A(v)) \in M(v, \cdot, \cdot).$$

Thus,

$$\frac{1}{\lambda}(p - A(u)) \oplus \frac{1}{\lambda}(p - A(v)) = \Big|\frac{1}{\lambda}\Big|(A(u) \oplus A(v)). \tag{2.2}$$

Since M is an α-non-ordinary difference mapping, we have

$$\frac{1}{\lambda}(p - A(u)) \oplus \frac{1}{\lambda}(p - A(v)) + \alpha(A(u) \oplus A(v)) = \theta. \tag{2.3}$$

Thus from (2.2) and (2.3), we have

$$\Big|\frac{1}{\lambda}\Big|(A(u) \oplus A(v)) \oplus \alpha(A(u) \oplus A(v)) = \theta,$$

$$i.e.,\ \Big||\frac{1}{\lambda}| + \alpha\Big|(A(u) \oplus A(v)) = \theta.$$

Since A is a β-ordered non-extended mapping, we have

$$\Big(\Big||\frac{1}{\lambda}| + \alpha\Big|\beta\Big)(u \oplus v) \leq \theta.$$

Thus, $u \oplus v = \theta$ and so, $u = v$. Hence $R^M_{\lambda,A} = (A + \lambda M)^{-1}$ is single-valued. □

Lemma 2.6. *Let* $\mathcal{C} \subseteq \mathcal{B}$ *be a normal cone with normal constant* $\lambda_{\mathcal{C}}$ *and* $\oplus$ *be an XOR operator. Let* $A : \mathcal{B} \to \mathcal{B}$ *be a strongly comparison mapping and* $M : \mathcal{B} \times \mathcal{B} \times \Omega \to 2^{\mathcal{B}}$ *be an* (α, λ)*-ANODSM set-valued mapping with respect to* $R^M_{\lambda,A}$. *Then the resolvent operator* $R^M_{\lambda,A} : \mathcal{B} \to \mathcal{B}$ *is a comparison mapping.*

Proof. For any $u, v \in \mathcal{B}$, let $u \propto v, p_u = \frac{1}{\lambda}[u - A(R^M_{\lambda,A}(u))] \in M(R^M_{\lambda,A}(u), \cdot, \cdot)$ and $p_v = \frac{1}{\lambda}[v - A(R^M_{\lambda,A}(v))] \in M(R^M_{\lambda,A}(v), \cdot, \cdot)$. Therefore

$$p_u - p_v = \frac{1}{\lambda}[(u - v) + A(R^M_{\lambda,A}(v)) - A(R^M_{\lambda,A}(u))]. \tag{2.4}$$

Since M is a λ-ordered strongly monotone mapping, we have $\lambda(p_u - p_v) \geq u - v$. It follows from (2.4) that

$$0 \leq \lambda(p_u - p_v) - (u - v) = A(R^M_{\lambda,A}(v)) - A(R^M_{\lambda,A}(u)),$$

and $\lambda(p_u - p_v) - (u - v) \in \mathcal{C}$. Thus, we have $A(R^M_{\lambda,A}(v)) \propto A(R^M_{\lambda,A}(u))$. Since A is a strongly comparison mapping, we have $R^M_{\lambda,A}(v) \propto R^M_{\lambda,A}(u)$. This completes the proof. □

Lemma 2.7. *Let $\mathcal{C} \subseteq \mathcal{B}$ be a normal cone with normal constant $\lambda_{\mathcal{C}}$ and $\oplus$ be an XOR operator. Let $A : \mathcal{B} \to \mathcal{B}$ be a β-ordered compression mapping and $M : \mathcal{B} \times \mathcal{B} \times \Omega \to 2^{\mathcal{B}}$ be an (α, λ)-ANODSM set-valued mapping with respect to $R^M_{\lambda,A} : \mathcal{B} \to \mathcal{B}$. If $\alpha > \frac{1}{\lambda} > 0$, then the following inequality holds*:

$$R^M_{\lambda,A}(u) \oplus R^M_{\lambda,A}(v) \leq \frac{\beta}{(\alpha\lambda - 1)}(u \oplus v) \ \forall u, v \in \mathcal{B}. \tag{2.5}$$

Proof. For any $u, v \in \mathcal{B}$, let $p_u = R^M_{\lambda,A}(u) \propto p_v = R^M_{\lambda,A}(v)$ and $q_u = \frac{1}{\lambda}(A(u) - p_u) \in M(\cdot, p_u, \cdot), q_v = \frac{1}{\lambda}(A(v) - p_v) \in M(\cdot, p_v, \cdot)$, then $q_u \propto q_v$ for $u \propto v$. Since M is an (α, λ)-$ANODSM$ set-valued mapping with respect to $R^M_{\lambda,A}$, we have

$$(q_u \oplus q_v) \oplus \alpha(p_u \oplus p_v) = \theta,$$

which implies that

$$\frac{1}{\lambda}(A(u) \oplus A(v) + (p_u \oplus p_v)) \geq q_u \oplus q_v = \alpha(p_u \oplus p_v).$$

It follows that $(\lambda\alpha - 1)(p_u \oplus p_v) \leq A(u) \oplus A(v)$. Since A is a β-ordered compression mapping, we have $(\lambda\alpha - 1)(p_u \oplus p_v) \leq \beta(u \oplus v)$. Therefore

$$R^M_{\lambda,A}(u) \oplus R^M_{\lambda,A}(v) \leq \frac{\beta}{(\alpha\lambda - 1)}(u \oplus v).$$

This completes the proof. □

3. Existence results

In this section, we formulate the system of generalized set-valued parametric ordered variational inclusions and we examine the existence result.

Let $\Omega(\neq \phi)$ be an open subset of a real ordered Banach space $\mathcal{B}$ and $\lambda_{\mathcal{C}}$ denotes normal constant of a normal cone $\mathcal{C}$. For each $i = 1, 2$; let $M_i : \mathcal{B} \times \mathcal{B} \times \Omega \to 2^{\mathcal{B}}$ be set-valued mappings; $f_i, g_i : \mathcal{B} \times \Omega \to \mathcal{B}$ and $F_i : \mathcal{B} \times \mathcal{B} \times \Omega \to \mathcal{B}$ be the single-valued mappings. We consider the following system of generalized set-valued parametric ordered variational inclusions (in short $SGSPOVI$):

Find $(u, v) \in \mathcal{B} \times \mathcal{B}, u = u(\varrho), v = v(\varrho), \varrho \in \Omega$ such that

$$\begin{cases} 0 \in F_1(f_1(g_1(u, \varrho), \varrho), v, \varrho) \oplus \omega_1 M_1(g_1(u, \varrho), v, \varrho); \\ 0 \in F_2(u, f_2(g_2(v, \varrho), \varrho), \varrho) \oplus \omega_2 M_2(u, g_2(v, \varrho), \varrho). \end{cases} \tag{3.1}$$

Remark 3.1. Our problem is more general in nature and for relevant choices of mappings adopted in the system, SGSPOVI (3.1) reduces to many problems existing in the literature; see, for example, [24–28].

Lemma 3.2. *The set of elements* $(u,v)\in\mathcal{B}\times\mathcal{B}, u=u(\varrho), v=v(\varrho), \varrho\in\Omega$ *is a solution of SGSPOVI* (3.1) *if and only if* (u,v) *satisfies the following equations*:

$$g_1(u(\varrho),\varrho)=R^{M_1(\cdot,v,\varrho)}_{\lambda_1,A}[A-\frac{\lambda_1}{\omega_1}F_1(f_1(\cdot),v,\varrho)]g_1(u(\varrho),\varrho); \tag{3.2}$$

$$g_2(v(\varrho),\varrho)=R^{M_2(u,\cdot,\varrho)}_{\lambda_2,A}[A-\frac{\lambda_2}{\omega_2}F_2(u,f_2(\cdot),\varrho)]g_2(v(\varrho),\varrho), \tag{3.3}$$

where $\lambda_i,\omega_i>0$ *are constants.*

Proof. The conclusion follows immediately by using definition of resolvent operators $R^{M_1(\cdot,v,\varrho)}_{\lambda_1,A}$ and $R^{M_2(u,\cdot,\varrho)}_{\lambda_2,A}$. □

Note that the fixed point formulation (3.2)–(3.3) can be re-written as

$$u(\varrho)=u(\varrho)-g_1(u(\varrho),\varrho)+R^{M_1(\cdot,v,\varrho)}_{\lambda_1,A}\big[A-\frac{\lambda_1}{\omega_1}F_1(f_1(\cdot),v,\varrho)\big]g_1(u,\varrho); \tag{3.4}$$

$$v(\varrho)=v(\varrho)-g_2(v(\varrho),\varrho)+R^{M_2(u,\cdot,\varrho)}_{\lambda_2,A}\big[A-\frac{\lambda_2}{\omega_2}F_2(u,f_2(\cdot),\varrho)\big]g_2(v,\varrho). \tag{3.5}$$

Theorem 3.3. *Let* $\mathcal{C}\subseteq\mathcal{B}$ *be a normal cone with normal constant* $\lambda_{\mathcal{C}}$. *For each* $i=1,2$; *let* $M_i:\mathcal{B}\times\mathcal{B}\times\Omega\to 2^{\mathcal{B}}$ *be* (α_i,λ_i)*-ANODSM set-valued mappings*; $f_i,g_i:\mathcal{B}\times\Omega\to\mathcal{B}; F_i:\mathcal{B}\times\mathcal{B}\times\Omega\to\mathcal{B}$ *be the single-valued mappings such that* f_i *are* λ_{f_i}*-ordered compression mappings and* g_i *are* γ_{g_i}*-ordered compression,* (ξ_1,ξ_2)*-restricted accretive mappings*; F_1 *is* (α_1,α_2) *and* F_2 *is an* $(\alpha_1^{'},\alpha_2^{'})$*-ordered mixed Lipschitz continuous mapping and* $A:\mathcal{B}\to\mathcal{B}$ *be a single-valued* γ_A*-ordered compression mapping.*

In addition, let for any $u_i,v_i,w_i\in\mathcal{B}$ *if* $u_i\propto v_i$, $R^{M_1}_{\lambda_1,A}(u_i)\propto R^{M_1}_{\lambda_1,A}(v_i)$, $R^{M_2}_{\lambda_2,A}(u_i)\propto R^{M_2}_{\lambda_2,A}(v_i)$ *and for all* $\lambda_i,\tau_i>0$, *the following inequalities hold*:

$$R^{M_1(\cdot,v_1,\varrho)}_{\lambda_1,A}(w_2)\oplus R^{M_1(\cdot,v_2,\varrho)}_{\lambda_1,A}(w_2)\le\tau_2(v_1\oplus v_2), \tag{3.6}$$

$$R^{M_2(u_1,\cdot,\varrho)}_{\lambda_2,A}(w_1)\oplus R^{M_2(u_2,\cdot,\varrho)}_{\lambda_2,A}(w_1)\le\tau_1(u_1\oplus u_2) \tag{3.7}$$

and

$$\begin{cases}\Theta_1=L_1+\Phi_1<1;\\ \Theta_2=L_2+\Phi_2<1;\end{cases} \tag{3.8}$$

where

$$L_1=\lambda_{\mathcal{C}}(\xi_2+\xi_1\gamma_{g_1});\ \Phi_1=\lambda_{\mathcal{C}}\Big[\varphi_1\big(\gamma_A\gamma_{g_1}+\frac{\lambda_1\alpha_1\lambda_{f_1}\gamma_{g_1}}{\omega_1}\big)+\big(\frac{\varphi_2\lambda_2\alpha_1^{'}}{\omega_2}+\tau_1\big)\Big];$$

$$L_2=\lambda_{\mathcal{C}}(\xi_2^{'}+\xi_1^{'}\gamma_{g_2});\ \Phi_2=\lambda_{\mathcal{C}}\Big[\varphi_2\big(\gamma_A\gamma_{g_2}+\frac{\lambda_2\alpha_2^{'}\lambda_{f_2}\gamma_{g_2}}{\omega_2}\big)+\big(\frac{\varphi_1\lambda_1\alpha_2}{\omega_1}+\tau_2\big)\Big].$$

Then SGSPOVI (3.1) *admits unique solution* $(u,v)\in\mathcal{B}\times\mathcal{B}, u=u(\varrho), v=v(\varrho), \varrho\in\Omega$.

Proof. It is clear from Lemma 2.7 that the resolvent operators $R^{M_1(\cdot,v_1,\varrho)}_{\lambda_1,A}$ and $R^{M_2(u_1,\cdot,\varrho)}_{\lambda_2,A}$ are φ_1 and φ_2-ordered Lipschiz continuous, respectively; where

$$\varphi_1 = \frac{\beta_1}{\alpha_1\lambda_1 - 1}; \ \varphi_1 = \frac{\beta_2}{\alpha_2\lambda_2 - 1}.$$

We define a mapping $T : \mathcal{B} \times \mathcal{B} \to \mathcal{B} \times \mathcal{B}$ by

$$\begin{aligned} &T(u, v) = (G(u, v, \rho), H(u, v, \varrho)), \\ &\qquad \forall (u, v) \in \mathcal{B} \times \mathcal{B},\ u = u(\varrho),\ v = v(\varrho),\ \varrho \in \Omega; \end{aligned} \tag{3.9}$$

where $G, H : \mathcal{B} \times \mathcal{B} \times \Omega \to \mathcal{B}$ are defined by

$$\begin{aligned} G(u(\varrho), v(\varrho), \varrho) &= u(\varrho) - g_1(u(\varrho), \cdot) \\ &\quad + R^{M_1(\cdot,v,\varrho)}_{\lambda_1,A}\big[A - \frac{\lambda_1}{\omega_1}F_1(f_1(\cdot), v, \varrho)\big]g_1(u, \varrho), \end{aligned} \tag{3.10}$$

$$\begin{aligned} H(u(\varrho), v(\varrho), \varrho) &= v(\varrho) - g_2(v(\varrho), \cdot) \\ &\quad + R^{M_2(u,\cdot,\varrho)}_{\lambda_2,A}\big[A - \frac{\lambda_2}{\omega_2}F_2(u, f_2(\cdot), \varrho)\big]g_2(v, \varrho), \end{aligned} \tag{3.11}$$

where $\lambda_i, \omega_i > 0$ are constants. For any $u_i, v_j \in \mathcal{B}$ and $u_i \propto v_j, (i, j = 1, 2)$, using the fact that g_1 is a γ_{g_1}-ordered commpression, (ξ_1, ξ_2)-restricted accretive mapping, A is a γ_A-ordered compression mapping, F_1 is an (α_1, α_2)-ordered mixed Lipschitz continuous mapping and f_1 is a λ_{f_1}-ordered compression mapping and utilizing (3.6), (3.10), Lemma 2.4 and Lemma 2.7, we have

$$\begin{aligned} \theta &\le G(u_1(\varrho), v_1(\varrho), \varrho) \oplus G(u_2(\varrho), v_2(\varrho), \varrho) \\ &= \Big[u_1(\varrho) - g_1(u_1(\varrho), \varrho) + R^{M_1(\cdot,v_1,\varrho)}_{\lambda_1,A}\big[A - \frac{\lambda_1}{\omega_1}F_1(f_1(\cdot), v_1, \varrho)\big]g_1(u_1(\varrho), \varrho)\Big] \\ &\quad \oplus \Big[u_2(\varrho) - g_1(u_2(\varrho), \varrho) + R^{M_1(\cdot,v_2,\varrho)}_{\lambda_1,A}\big[A - \frac{\lambda_1}{\omega_1}F_1(f_1(\cdot), v_2, \varrho)\big]g_1(u_2(\varrho), \varrho)\Big] \\ &\le (g_1(u_1(\varrho), \varrho) - u_1(\varrho)) \oplus (g_1(u_2(\varrho), \varrho) - u_2(\varrho)) \\ &\quad + \Big[R^{M_1(\cdot,v_1,\varrho)}_{\lambda_1,A}[A - \frac{\lambda_1}{\omega_1}F_1(f_1(\cdot), v_1, \varrho)]g_1(u_1(\varrho), \varrho) \\ &\quad \oplus R^{M_1(\cdot,v_1,\varrho)}_{\lambda_1,A}[A - \frac{\lambda_1}{\omega_1}F_1(f_1(\cdot), v_2, \varrho)]g_1(u_2(\varrho), \varrho) \\ &\quad \oplus R^{M_1(\cdot,v_1,\varrho)}_{\lambda_1,A}[A - \frac{\lambda_1}{\omega_1}F_1(f_1(\cdot), v_2, \varrho)]g_1(u_2(\varrho), \varrho) \\ &\quad \oplus R^{M_1(\cdot,v_2,\varrho)}_{\lambda_1,A}[A - \frac{\lambda_1}{\omega_1}F_1(f_1(\cdot), v_2, \varrho)]g_1(u_2(\varrho), \varrho)\Big] \\ &\le \xi_2(u_1(\varrho) \oplus u_2(\varrho)) + \xi_1(g_1(u_1(\varrho) \oplus u_2(\varrho))) \\ &\quad + \varphi_1\Big[A(g_1(u_1(\varrho), \varrho)) \oplus A(g_1(u_2(\varrho), \varrho)) \\ &\quad + \frac{\lambda_1}{\omega_1}(F_1(f_1(g_1(u_1(\varrho)), \varrho), v_1(\varrho), \varrho) \oplus F_1(f_1(g_1(u_2(\varrho), \varrho)), v_2(\varrho), \varrho))\Big] \end{aligned} \tag{3.12}$$

$$
\begin{aligned}
&\oplus \tau_2(v_1(\varrho) \oplus v_2(\varrho)) \\
&\leq \xi_2(u_1(\varrho) \oplus u_2(\varrho)) + \xi_1\gamma_{g_1}(u_1(\varrho) \oplus u_2(\varrho)) \\
&\quad + \varphi_1\Big[\gamma_A(g_1(u_1(\varrho),\varrho) \oplus g_1(u_2(\varrho),\varrho)) \\
&\quad + \frac{\lambda_1}{\omega_1}(\alpha_1(f_1(g_1(u_1(\varrho),\varrho)) \oplus f_1(g_1(u_2(\varrho),\varrho))) + \alpha_2(v_1(\varrho) \oplus v_2(\varrho)))\Big] \\
&\quad + \tau_2(v_1(\varrho) \oplus v_2(\varrho)) \\
&\leq \xi_2(u_1(\varrho) \oplus u_2(\varrho)) + \xi_1\gamma_{g_1}(u_1(\varrho) \oplus u_2(\varrho)) \\
&\quad + \varphi_1\Big[\gamma_A\gamma_{g_1}(u_1(\varrho) \oplus u_2(\varrho)) \\
&\quad + \frac{\lambda_1}{\omega_1}(\alpha_1\lambda_{f_1}(g_1(u_1(\varrho),\varrho) \oplus g_1(u_2(\varrho),\varrho)) + \alpha_2(v_1(\varrho) \oplus v_2(\varrho)))\Big] \\
&\quad + \tau_2(v_1(\varrho) \oplus v_2(\varrho)) \\
&\leq \xi_2(u_1(\varrho) \oplus u_2(\varrho)) + \xi_1\gamma_{g_1}(u_1(\varrho) \oplus u_2(\varrho)) \\
&\quad + \varphi_1\Big[\gamma_A\gamma_{g_1}(u_1(\varrho) \oplus u_2(\varrho)) \\
&\quad + \frac{\lambda_1}{\omega_1}(\alpha_1\lambda_{f_1}\gamma_{g_1}(u_1(\varrho) \oplus u_2(\varrho)) + \alpha_2(v_1(\varrho) \oplus v_2(\varrho)))\Big] \\
&\quad + \tau_2(v_1(\varrho) \oplus v_2(\varrho)) \\
&= \Big[(\xi_2 + \xi_1\gamma_{g_1}) + \varphi_1\big(\gamma_A\gamma_{g_1} + \frac{\lambda_1\alpha_1\lambda_{f_1}\gamma_{g_1}}{\omega_1}\big)\Big](u_1(\varrho) \oplus u_2(\varrho)) \\
&\quad + \Big[\frac{\varphi_1\lambda_1\alpha_2}{\omega_1} + \tau_2\Big](v_1(\varrho) \oplus v_2(\varrho)).
\end{aligned}
$$

It follows from Definition 2(i) and Lemma 2.2 that

$$
\begin{aligned}
(3.13)\qquad &\|G(u_1(\varrho),v_1(\varrho),\varrho) - G(u_2(\varrho),v_2(\varrho),\varrho)\| \\
&\leq \lambda_{\mathcal{C}}\Big\|\Big[(\xi_2 + \xi_1\gamma_{g_1}) + \varphi_1\big(\gamma_A\gamma_{g_1} + \frac{\lambda_1\alpha_1\lambda_{f_1}\gamma_{g_1}}{\omega_1}\big)\Big](u_1(\varrho) \oplus u_2(\varrho)) \\
&\quad + \Big(\frac{\varphi_1\lambda_1\alpha_2}{\omega_1} + \tau_2\Big)(v_1(\varrho) \oplus v_2(\varrho))\Big\| \\
&\leq \lambda_{\mathcal{C}}\Big[(\xi_2 + \xi_1\gamma_{g_1}) + \varphi_1\big(\gamma_A\gamma_{g_1} + \frac{\lambda_1\alpha_1\lambda_{f_1}\gamma_{g_1}}{\omega_1}\big)\Big]\|u_1(\varrho) - u_2(\varrho)\| \\
&\quad + \lambda_{\mathcal{C}}\Big(\frac{\varphi_1\lambda_1\alpha_2}{\omega_1} + \tau_2\Big)\|v_1(\varrho) - v_2(\varrho)\|.
\end{aligned}
$$

Again, for any $u_i, v_j \in \mathcal{B}$ and $u_i \propto v_j, (i,j = 1,2)$, using the fact that g_2 is a γ_{g_2}-ordered commpression, $(\xi_1^{'}, \xi_2^{'})$-restricted accretive mapping, A is a γ_A-ordered compression mapping, F_2 is an $(\alpha_1^{'}, \alpha_2^{'})$-ordered mixed Lipschitz continuous mapping and f_2 is a λ_{f_2}-ordered compression mapping and utilizing (3.7), (3.11), Lemma 2.4 and Lemma 2.7, we have

$$
(3.14)\quad \theta \leq H(u_1(\varrho),v_1(\varrho),\varrho) \oplus H(u_2(\rho),v_2(\varrho),\varrho)
$$

$$
\begin{aligned}
&= \Big[v_1(\varrho) - g_2(v_1(\varrho),\varrho) + R^{M_2(u_1,\cdot,\varrho)}_{\lambda_2,A}\big[A - \frac{\lambda_2}{\omega_2}F_2(u_1,f_2(\cdot),\varrho)\big]g_2(v_1(\varrho),\varrho)\Big] \\
&\quad \oplus \Big[v_2(\varrho) - g_2(v_2(\varrho),\varrho) + R^{M_2(u_2,\cdot,\varrho)}_{\lambda_2,A}\big[A - \frac{\lambda_2}{\omega_2}F_2(u_2,f_2(\cdot),\varrho)\big]g_2(v_2(\varrho),\varrho)\Big] \\
&\le (g_2(v_1(\varrho),\varrho) - v_1(\varrho)) \oplus (g_2(v_2(\varrho),\varrho) - v_2(\varrho)) \\
&\quad + \Big[R^{M_2(u_1,\cdot,\varrho)}_{\lambda_2,A}[A - \frac{\lambda_2}{\omega_2}F_2(u_1,f_2(\cdot),\varrho)]g_2(v_1(\varrho),\varrho) \\
&\quad \oplus R^{M_2(u_1,\cdot,\varrho)}_{\lambda_2,A}[A - \frac{\lambda_2}{\omega_2}F_2(u_2,f_2(\cdot),\varrho)]g_2(v_2(\varrho),\varrho) \\
&\quad \oplus R^{M_2(u_1,\cdot,\varrho)}_{\lambda_2,A}[A - \frac{\lambda_2}{\omega_2}F_2(u_2,f_2(\cdot),\varrho)]g_2(v_2(\varrho),\varrho) \\
&\quad \oplus R^{M_2(u_2,\cdot,\varrho)}_{\lambda_2,A}[A - \frac{\lambda_2}{\omega_2}F_2(u_2,f_2(\cdot),\varrho)]g_2(v_2(\varrho),\varrho)\Big] \\
&\le \xi_2^{'}(v_1(\varrho) \oplus v_2(\varrho)) + \xi_1^{'}(g_2(v_1(\varrho) \oplus v_2(\varrho))) \\
&\quad + \varphi_2\Big[A(g_2(v_1(\varrho),\varrho)) \oplus A(g_2(v_2(\varrho),\varrho)) \\
&\quad + \frac{\lambda_2}{\omega_2}(F_2(u_1,f_2(g_2(v_1(\varrho),\varrho)),\varrho) \oplus F_2(u_2,f_2(g_2(v_2(\varrho),\varrho)),\varrho))\Big] \\
&\quad + \tau_1(u_1(\varrho) \oplus u_2(\varrho)) \\
&\le \xi_2^{'}(v_1(\varrho) \oplus v_2(\varrho)) + \xi_1^{'}\gamma_{g_2}(v_1(\varrho) \oplus v_2(\varrho)) \\
&\quad + \varphi_2\Big[\gamma_A(g_2(v_1(\varrho),\varrho) + g_2(v_2(\varrho),\varrho)) \\
&\quad + \frac{\lambda_2}{\omega_2}(\alpha_1^{'}(u_1(\varrho) \oplus u_2(\varrho)) + \alpha_2^{'}(f_2(g_2(v_1(\varrho),\varrho))) \oplus f_2(g_2(v_2(\varrho),\varrho)))\Big] \\
&\quad + \tau_1(u_1(\varrho) \oplus u_2(\varrho)) \\
&\le (\xi_2^{'} + \xi_1^{'}\gamma_{g_2})(v_1(\varrho) \oplus v_2(\varrho)) + \varphi_2\Big[\gamma_A\gamma_{g_2}(v_1(\varrho) \oplus v_2(\varrho)) \\
&\quad + \frac{\lambda_2}{\omega_2}(\alpha_1^{'}(u_1(\varrho) \oplus u_2(\varrho)) + \alpha_2^{'}\lambda_{f_2}(g_2(v_1(\varrho),\varrho)) \oplus g_2(v_2(\varrho),\varrho)))\Big] \\
&\quad + \tau_1(u_1(\varrho) \oplus u_2(\varrho)) \\
&\le (\xi_2^{'} + \xi_1^{'}\gamma_{g_2})(v_1(\varrho) \oplus v_2(\varrho)) + \varphi_2\Big[\gamma_A\gamma_{g_2}(v_1(\varrho) \oplus v_2(\varrho)) \\
&\quad + \frac{\lambda_2}{\omega_2}(\alpha_1^{'}(u_1(\varrho) \oplus u_2(\varrho)) + \alpha_2^{'}\lambda_{f_2}\gamma_{g_2}(v_1(\varrho) \oplus v_2(\varrho)))\Big] \\
&\quad + \tau_1(u_1(\varrho) \oplus u_2(\varrho)) \\
&= \Big[(\xi_2^{'} + \xi_1^{'}\gamma_{g_2}) + \varphi_2\big(\gamma_A\gamma_{g_2} + \frac{\lambda_2\alpha_2^{'}\lambda_{f_2}\gamma_{g_2}}{\omega_2}\big)\Big](v_1(\varrho) \oplus v_2(\varrho)) \\
&\quad + \Big(\frac{\varphi_2\lambda_2\alpha_1^{'}}{\omega_2} + \tau_1\Big)(u_1(\varrho) \oplus u_2(\varrho)).
\end{aligned}
$$

Again, it follows from Definition 2(i) and Lemma 2.2 that

$$
\begin{aligned}
(3.15)\qquad & \|H(u_1(\varrho), v_1(\varrho), \varrho) - H(u_2(\varrho), v_2(\varrho), \varrho)\| \\
&\leq \lambda_{\mathcal{C}} \Big\| \Big[(\xi_2^{'} + \xi_1^{'}\gamma_{g_2}) + \varphi_2\big(\gamma_A\gamma_{g_2} + \frac{\lambda_2\alpha_2^{'}\lambda_{f_2}\gamma_{g_2}}{\omega_2}\big)\Big](v_1(\varrho) \oplus v_2(\varrho)) \\
&\quad + \Big(\frac{\varphi_2\lambda_2\alpha_1^{'}}{\omega_2} + \tau_1\Big) u_1(\varrho) \oplus u_2(\varrho)\Big\| \\
&\leq \lambda_{\mathcal{C}} \Big[(\xi_2^{'} + \xi_1^{'}\gamma_{g_2}) + \varphi_2\big(\gamma_A\gamma_{g_2} + \frac{\lambda_2\alpha_2^{'}\lambda_{f_2}\gamma_{g_2}}{\omega_2}\big)\Big] \|v_1(\varrho) - v_2(\varrho)\| \\
&\quad + \lambda_{\mathcal{C}} \Big(\frac{\varphi_2\lambda_2\alpha_1^{'}}{\omega_2} + \tau_1\Big) \|u_1(\varrho) - u_2(\varrho)\|.
\end{aligned}
$$

Thus from (3.13) and (3.15), we have

$$
\begin{aligned}
(3.16)\qquad & \|G(u_1(\varrho), v_1(\varrho), \varrho) - G(u_2(\varrho), v_2(\varrho), \varrho)\| \\
&\quad + \|H(u_1(\varrho), v_1(\varrho), \varrho) - H(u_2(\varrho), v_2(\varrho), \varrho)\| \\
&\leq \lambda_{\mathcal{C}} \Big[(\xi_2 + \xi_1\gamma_{g_1}) + \varphi_1\big(\gamma_A\gamma_{g_1} + \frac{\lambda_1\alpha_1\lambda_{f_1}\gamma_{g_1}}{\omega_1}\big)\Big] \|u_1(\varrho) - u_2(\varrho)\| \\
&\quad + \lambda_{\mathcal{C}} \Big(\frac{\varphi_1\lambda_1\alpha_2}{\omega_1} + \tau_2\Big) \|v_1(\varrho) - v_2(\varrho)\| \\
&\quad + \lambda_{\mathcal{C}} \Big[(\xi_2^{'} + \xi_1^{'}\gamma_{g_2}) + \varphi_2\big(\gamma_A\gamma_{g_2} + \frac{\lambda_2\alpha_2^{'}\lambda_{f_2}\gamma_{g_2}}{\omega_2}\big)\Big] \|v_1(\varrho) - v_2(\varrho)\| \\
&\quad + \lambda_{\mathcal{C}} \Big(\frac{\varphi_2\lambda_2\alpha_1^{'}}{\omega_2} + \tau_1\Big) \|u_1(\varrho) - u_2(\varrho)\| \\
&= \Theta_1 \|u_1(\varrho) - u_2(\varrho)\| + \Theta_2 \|v_1(\varrho) - v_2(\varrho)\| \\
&\leq \max\{\Theta_1, \Theta_2\} \Big(\|u_1(\varrho) - u_2(\varrho)\| + \|v_1(\varrho) - v_2(\varrho)\| \Big),
\end{aligned}
$$

where, $\Theta_1 = L_1 + \Phi_1;\ \Theta_2 = L_2 + \Phi_2$, and

$$
L_1 = \lambda_{\mathcal{C}}(\xi_2 + \xi_1\gamma_{g_1});\ \Phi_1 = \lambda_{\mathcal{C}}\Big[\varphi_1\big(\gamma_A\gamma_{g_1} + \frac{\lambda_1\alpha_1\lambda_{f_1}\gamma_{g_1}}{\omega_1}\big) + \big(\frac{\varphi_2\lambda_2\alpha_1^{'}}{\omega_2} + \tau_1\big)\Big];
$$

$$
L_2 = \lambda_{\mathcal{C}}(\xi_2^{'} + \xi_1^{'}\gamma_{g_2});\ \Phi_2 = \lambda_{\mathcal{C}}\Big[\varphi_2\big(\gamma_A\gamma_{g_2} + \frac{\lambda_2\alpha_2^{'}\lambda_{f_2}\gamma_{g_2}}{\omega_2}\big) + \big(\frac{\varphi_1\lambda_1\alpha_2}{\omega_1} + \tau_2\big)\Big].
$$

Next, we define a norm $\|\cdot\|_*$ on $\mathcal{B} \times \mathcal{B}$ by

$$
(3.17)\qquad \|(\sigma, \eta)\|_* = \|\sigma\| + \|\eta\|,\ \forall (\sigma, \eta) \in \mathcal{B} \times \mathcal{B}.
$$

One can verify easily that $(\mathcal{B} \times \mathcal{B}, \|\cdot\|_*)$ is a Banach space. Thus from (3.9), (3.16), (3.17), we have

$$
\begin{aligned}
(3.18)\qquad & \|T(u_1(\varrho), v_1(\varrho), \varrho) - T(u_2(\varrho), v_2(\varrho), \varrho)\|_* \\
&\leq \max\{\Theta_1, \Theta_2\} \|u_1(\varrho) - u_2(\varrho)\| + \|v_1(\varrho) - v_2(\varrho)\|.
\end{aligned}
$$

By condition (3.8), $\max\{\Theta_1, \Theta_2\} < 1$, and hence (3.18) yields that T is a contraction mapping. Therefore, there exists a unique point $(u, v) \in \mathcal{B} \times \mathcal{B}, u = u(\varrho), v = v(\varrho), \varrho \in \Omega$ such that

$$T(u, v) = (u, v);$$

which implies that

$$g_1(u(\varrho), \varrho) = R_{\lambda_1, A}^{M_1(\cdot, v, \varrho)}[A - \frac{\lambda_1}{\omega_1} F_1(f_1(\cdot), v, \varrho)] g_1(u(\varrho), \varrho); \tag{3.19}$$

$$g_2(v(\varrho), \varrho) = R_{\lambda_2, A}^{M_2(u, \cdot, \varrho)}[A - \frac{\lambda_2}{\omega_2} F_2(u, f_2(\cdot), \varrho)] g_2(v(\varrho), \varrho). \tag{3.20}$$

Thus, from Lemma 3.2 we conclude that $(u, v) \in \mathcal{B} \times \mathcal{B}, u = u(\varrho), v = v(\varrho), \varrho \in \Omega$ is a unique solution of SGSPOVI (3.1). □

4. Iterative algorithm and convergence analysis

In this section, we suggest an iterative algorithm and approximate the unique solution for SGSPOVI (3.1). Finally, we show that the sequence generated by iterative algorithm converges to the unique solution of SGSPOVI (3.1).

Lemma 4.1 ([17]). *If the nonnegative real sequences $\{k_n\}$ and $\{c_n\}$ satisfying following conditions*:

(i) $0 \leq c_n < 1, \ n = 0, 1, 2, \ldots$ *and* $\limsup_n c_n < 1$;

(ii) $k_{n+1} \leq c_n k_n, \ n = 0, 1, 2, \ldots,$

then $\lim_{n \to \infty} k_n = 0$.

By using Lemma 3.2, we propose the following iterative scheme to find an approximate solution for SGSPOVI (3.1).

Algorithm 4.2. *For given initial value $(u_0, v_0) \in \mathcal{B} \times \mathcal{B}$, $u_0 = u_0(\varrho)$, $v_0 = v_0(\varrho)$, assume that $u_0 \propto u_1$, $v_0 \propto v_1$. We compute the sequence $(u_n, v_n) \in \mathcal{B} \times \mathcal{B}$ such that $u_{n+1} \propto u_n, v_{n+1} \propto v_n$ by the following iterative method*:

$$\begin{aligned} u_{n+1}(\varrho) = {} & \alpha_n [u_n(\varrho) - g_1(u_n(\varrho), \varrho)] \\ & + (1 - \alpha_n) R_{\lambda_1, A}^{M_1(\cdot, v_n, \varrho)} \big[A - \frac{\lambda_1}{\omega_1} F_1(f_1(\cdot), v_n, \varrho)\big] g_1(u_n(\varrho), \varrho); \end{aligned} \tag{4.1}$$

$$\begin{aligned} v_{n+1}(\varrho) = {} & \alpha_n [v_n(\varrho) - g_2(v_n(\varrho), \varrho)] \\ & + (1 - \alpha_n) R_{\lambda_2, A}^{M_2(u_n, \cdot, \varrho)} \big[A - \frac{\lambda_2}{\omega_2} F_2(u_n, f_2(\cdot), \varrho)\big] g_2(v_n(\varrho), \varrho); \end{aligned} \tag{4.2}$$

where $n = 0, 1, 2, \ldots; \alpha_n \in [0, 1)$ with $\limsup_n \alpha_n < 1$ and for each $i = 1, 2$; $\lambda_i, \omega_i > 0$ are constants.

Theorem 4.3. *For each $i = 1, 2$; let M_i, f_i, g_i, F_i and A be same as defined in Theorem 3.3 such that the assumptions of Theorem 3.3 hold. Then the solution $\{(u_n, v_n)\}$ estimated by iterative Algorithm 4.2 converges strongly to the unique solution (u, v) of SGSPOVI* (3.1).

Proof. It is clear from Theorem 3.3 that SGSPOVI (3.1) admits unique solution (u, v), $u = u(\rho), v = v(\rho), \rho \in \Omega$. Thus, Lemma 3.2 gives that

$$\begin{aligned} u(\varrho) &= \alpha_n[u(\varrho) - g_1(u(\varrho), \varrho)] \\ &\quad + (1-\alpha_n) R^{M_1(\cdot,v,\varrho)}_{\lambda_1,A}\big[A - \frac{\lambda_1}{\omega_1} F_1(f_1(\cdot), v, \varrho)\big] g_1(u(\varrho), \varrho); \end{aligned} \tag{4.3}$$

$$\begin{aligned} v(\varrho) &= \alpha_n[v(\varrho) - g_2(v(\varrho), \varrho)] \\ &\quad + (1-\alpha_n) R^{M_2(u,\cdot,\varrho)}_{\lambda_2,A}\big[A - \frac{\lambda_2}{\omega_2} F_2(u, f_2(\cdot), \varrho)\big] g_2(v(\varrho), \varrho). \end{aligned} \tag{4.4}$$

Using (4.1), (4.3) and Lemma 2.4, we have

$$\begin{aligned} \theta &\le u_{n+1}(\varrho) \oplus u(\varrho) \\ &= \alpha_n[u_n(\varrho) - g_1(u_n(\varrho), \varrho)] \\ &\quad + (1-\alpha_n) R^{M_1(\cdot,v_n,\varrho)}_{\lambda_1,A}\big[A - \frac{\lambda_1}{\omega_1} F_1(f_1(\cdot), v_n, \varrho)\big] g_1(u_n(\varrho), \varrho) \\ &\quad \oplus \alpha_n[u(\varrho) - g_1(u(\varrho), \varrho)] \\ &\quad + (1-\alpha_n) R^{M_1(\cdot,v,\varrho)}_{\lambda_1,A}\big[A - \frac{\lambda_1}{\omega_1} F_1(f_1(\cdot), v, \varrho)\big] g_1(u(\varrho), \varrho) \\ &= \alpha_n\big[\big(g_1(u_n(\varrho), \varrho) - u_n(\varrho)\big) \oplus \big(g_1(u(\varrho), \varrho) - u(\varrho)\big)\big] \\ &\quad + (1-\alpha_n)\Big[R^{M_1(\cdot,v_n,\varrho)}_{\lambda_1,A}[A - \frac{\lambda_1}{\omega_1} F_1(f_1(\cdot), v_n, \varrho)] g_1(u_n(\varrho), \varrho) \\ &\quad \oplus R^{M_1(\cdot,v,\varrho)}_{\lambda_1,A}[A - \frac{\lambda_1}{\omega_1} F_1(f_1(\cdot), v, \varrho)] g_1(u(\varrho), \varrho)\Big]. \end{aligned} \tag{4.5}$$

Now, utilizing the same facts as used in (3.12) and (3.13), we have

$$\begin{aligned} &\|u_{n+1}(\varrho) - u(\varrho)\| \\ &\le \lambda_{\mathcal{C}} \Big\| \alpha_n(\xi_2 + \xi_1\gamma_{g_1}) + (1-\alpha_n)\varphi_1\Big(\gamma_A\gamma_{g_1} + \frac{\lambda_1\alpha_1\lambda_{f_1}\gamma_{g_1}}{\omega_1}\Big)(u_n(\varrho) \oplus u(\varrho)) \\ &\quad + (1-\alpha_n)\Big(\frac{\varphi_1\lambda_1\alpha_2}{\omega_1} + \tau_2\Big)(v_n(\varrho) \oplus v(\varrho))\Big\| \\ &\le \lambda_{\mathcal{C}}\Big[\alpha_n(\xi_2 + \xi_1\gamma_{g_1}) + (1-\alpha_n)\varphi_1\Big(\gamma_A\gamma_{g_1} + \frac{\lambda_1\alpha_1\lambda_{f_1}\gamma_{g_1}}{\omega_1}\Big)\Big]\|u_n(\varrho) - u(\varrho)\| \\ &\quad + \lambda_{\mathcal{C}}\Big[(1-\alpha_n)\big(\frac{\varphi_1\lambda_1\alpha_2}{\omega_1} + \tau_2\big)\Big]\|v_n(\varrho) - v(\varrho)\|. \end{aligned} \tag{4.6}$$

Similarly, using (4.2), (4.4) and Lemma 2.4, we have

$$\begin{aligned} \theta &\le v_{n+1}(\varrho) \oplus v(\varrho) \\ &= \alpha_n[v_n(\varrho) - g_2(v_n(\varrho), \varrho)] \end{aligned} \tag{4.7}$$

$$+ (1-\alpha_n)R_{\lambda_2,A}^{M_2(u_n,\cdot,\varrho)}\big[A - \frac{\lambda_2}{\omega_2}F_2(u_n, f_2(\cdot),\varrho)\big]g_2(v_n(\varrho),\varrho)$$
$$\oplus\, \alpha_n[v(\varrho) - g_2(v(\varrho),\varrho)]$$
$$+ (1-\alpha_n)R_{\lambda_2,A}^{M_2(u,\cdot,\varrho)}\big[A - \frac{\lambda_2}{\omega_2}F_2(u, f_2(\cdot),\varrho)\big]g_2(v(\varrho),\varrho)$$
$$= \alpha_n\big[(g_2(v_n(\varrho),\varrho) - u_n(\varrho)) \oplus (g_2(v(\varrho),\varrho) - v(\varrho))\big]$$
$$+ (1-\alpha_n)\Big[R_{\lambda_2,A}^{M_2(u_n,\cdot,\varrho)}[A - \frac{\lambda_2}{\omega_2}F_2(u_n, f_2(\cdot),\varrho)]g_2(v_n(\varrho),\varrho)$$
$$\oplus\, R_{\lambda_2,A}^{M_2(u,\cdot,\varrho)}[A - \frac{\lambda_2}{\omega_2}F_2(u, f_2(\cdot),\varrho)]g_2(v(\varrho),\varrho)\Big].$$

Again, utilizing the same facts as used in (3.14) and (3.15), we have

$$\|v_{n+1}(\varrho) - v(\varrho)\| \tag{4.8}$$
$$\leq \lambda_{\mathcal{C}}\Big\|\alpha_n(\xi_2^{'} + \xi_1^{'}\gamma_{g_2}) + (1-\alpha_n)\varphi_2\Big(\gamma_A\gamma_{g_2} + \frac{\lambda_2\alpha_2^{'}\lambda_{f_2}\gamma_{g_2}}{\omega_2}\Big)(v_n(\varrho)\oplus v(\varrho))$$
$$+ (1-\alpha_n)\Big(\frac{\varphi_2\lambda_2\alpha_1^{'}}{\omega_2} + \tau_1\Big)(u_n(\varrho)\oplus u(\varrho))\Big\|$$
$$\leq \lambda_{\mathcal{C}}\Big[\alpha_n(\xi_2^{'} + \xi_1^{'}\gamma_{g_2}) + (1-\alpha_n)\varphi_2\Big(\gamma_A\gamma_{g_2} + \frac{\lambda_2\alpha_2^{'}\lambda_{f_2}\gamma_{g_2}}{\omega_2}\Big)\Big]\|v_n(\varrho) - v(\varrho)\|$$
$$+ \lambda_{\mathcal{C}}(1-\alpha_n)\Big(\frac{\varphi_2\lambda_2\alpha_1^{'}}{\omega_2} + \tau_1\Big)\|u_n(\varrho) - u(\varrho)\|.$$

Thus, it follows from (4.6) and (4.8) that

$$\|u_{n+1}(\varrho) - u(\varrho)\| + \|v_{n+1}(\varrho) - v(\varrho)\| \tag{4.9}$$
$$\leq \lambda_{\mathcal{C}}\Big[\alpha_n(\xi_2 + \xi_1\gamma_{g_1}) + (1-\alpha_n)\varphi_1\Big(\gamma_A\gamma_{g_1} + \frac{\lambda_1\alpha_1\lambda_{f_1}\gamma_{g_1}}{\omega_1}\Big)\Big]\|u_n(\varrho) - u(\varrho)\|$$
$$+ \lambda_{\mathcal{C}}\Big[(1-\alpha_n)\Big(\frac{\varphi_1\lambda_1\alpha_2}{\omega_1} + \tau_2\Big)\Big]\|v_n(\varrho) - v(\varrho)\|$$
$$+ \lambda_{\mathcal{C}}\Big[\alpha_n(\xi_2^{'} + \xi_1^{'}\gamma_{g_2}) + (1-\alpha_n)\varphi_2\Big(\gamma_A\gamma_{g_2} + \frac{\lambda_2\alpha_2^{'}\lambda_{f_2}\gamma_{g_2}}{\omega_2}\Big)\Big]\|v_n(\varrho) - v(\varrho)\|$$
$$+ \lambda_{\mathcal{C}}\Big[(1-\alpha_n)\Big(\frac{\varphi_2\lambda_2\alpha_1^{'}}{\omega_2} + \tau_1\Big)\Big]\|u_n(\varrho) - u(\varrho)\|$$
$$= \alpha_n(L_1\|u_n(\varrho) - u(\varrho)\| + L_2\|v_n(\varrho) - v(\varrho)\|)$$
$$+ (1-\alpha_n)(\Phi_1\|u_n(\varrho) - u(\varrho)\| + \Phi_2\|v_n(\varrho) - v(\varrho)\|),$$

which implies that

$$\begin{aligned} &\|u_{n+1}(\varrho) - u(\varrho)\| + \|v_{n+1}(\varrho) - v(\varrho)\| \\ &\leq [\alpha_n L + (1-\alpha_n)\Phi](\|u_n(\varrho) - u(\varrho)\| + \|v_n(\varrho) - v(\varrho)\|), \end{aligned} \tag{4.10}$$

where $L = \max\{L_1, L_2\}$, $\Phi = \max\{\Phi_1, \Phi_2\}$. Assume that $k_n = \|u_n(\varrho) - u(\varrho)\| + \|v_n(\varrho) - v(\varrho)\|$ and $c_n = \Phi + (L-\Phi)\alpha_n$, then (4.10) can be written as

$$k_{n+1} \leq c_n k_n, \; n = 0, 1, 2, \ldots.$$

Choosing c_n, we have $\limsup_n c_n < 1$. Then from Lemma 4.1, we get $0 \leq c_n < 1$. Hence, the sequence $\{(u_n, v_n)\}$ converges strongly to the unique solution $(u, v) \in \mathcal{B} \times \mathcal{B}$, $u = u(\varrho), v = v(\varrho), \varrho \in \Omega$ of SGSPOVI (3.1). This completes the proof. □

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