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ON AN INTERESTING EXTENSION OF KUMMER'S SECOND THEOREM WITH APPLICATIONS

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Abstract. In this research paper, an attempt has been made to provide an interesting extension of the well-known and useful Kummer's second theorem. Several applications have also been given.

1. Introduction

In 1812, C. F. Gauss [14] defined his famous infinite series as follows:

$$1+\frac{ab}{c}\frac{z}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+\cdots . \tag{1.1}$$

The series (1.1) is denoted by the symbol

$${}_2F_1\left[\begin{matrix} a, \quad b \\ \\ c \end{matrix} \; ; \quad z\right] \text{ or } {}_2F_1\left[\begin{matrix} a, \quad b; \\ \\ c \end{matrix} \; ; \quad z\right] \text{ or } {}_2F_1\left[a,b;c;z\right]$$

or simply F and is popularly known as the Gauss's function or the Gauss's hypergeometric function. Here a, b and c are known as parameters of the series and z is called the variable of the series. All a, b, c and z may be real or complex with one assumption that c should not be zero nor a negative integer. The series (1.1) is known as "hypergeometric series" because either $a = 1$ and $b = c$ or $b = 1$ and $a = c$, it reduces to the well known Geometric series.

The Pochhammer's symbol $(a)_n$ is defined by

$$(a)_n = \begin{cases} a(a+1)\cdots(a+n-1), & n \in \mathbb{N}, \\ 1, & n = 0. \end{cases} \tag{1.2}$$

Also, in terms of Gamma function, $(a)_n$ can be represented as

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{1.3}$$

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Thus, in terms of Pochhammer's symbol, the hypergeometric series (1.1) is represented by

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & & ; & z \\ c & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}. \tag{1.4} $$

For a detailed discussion about it's convergence (including absolute) conditions and properties, we refer [2, 4, 30, 31, 45, 49, 50].

The limiting case of (1.4) is worth meaning. For this, if we replace z by $\frac{z}{b}$ in (1.4) and take the limit as $b \to \infty$, then since

$$ \frac{(b)_n}{b^n} z^n \to z^n $$

we arrive at the following series which is in the literature known as the Kummer's series or the confluent hypergeometric series [24, 25]

$$ {}_1F_1\left[\begin{matrix} a \\ & ; & z \\ c \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}. \tag{1.5} $$

It is not out of place to mention here that almost all the elementary functions of mathematics and mathematical physics are special cases or limiting cases of ${}_2F_1$ or ${}_1F_1$.

The generalized hypergeometric function ${}_pF_q$ with p numerator and q denominator parameters is defined [2, 4, 30, 45, 49, 50] by

$$ \begin{aligned} {}_pF_q\left[\begin{matrix} \alpha_1, & \ldots, & \alpha_p \\ & & & ; & z \\ \beta_1, & \ldots, & \beta_q \end{matrix}\right] &= {}_pF_q[\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z] \\ &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{z^n}{n!}. \end{aligned} \tag{1.6} $$

For convergence conditions and other details, we refer [2, 4, 30, 31, 45, 49, 50]. It is interesting to mention here that whenever a hypergeometric function ${}_2F_1$ or the generalized hypergeometric function ${}_pF_q$ reduce to the ratio of the products of gamma functions, the results are important from the point of view of applications. Thus the classical summation theorems such as those of Gauss, Gauss second, Kummer and Bailey for the series ${}_2F_1$; Watson, Dixon, Whipple and Saalschütz for the series ${}_3F_2$ and others play an important role. In a very popular, interesting and useful research paper, Bailey [5] had obtained a large number of very interesting results involving products of generalized hypergeometric functions.

The well-known Kummer's first theorem or Kummer's first transformation for the series ${}_1F_1$ is given by

$$e^{-x}\ {}_1F_1\left[\begin{matrix} a \\ b \end{matrix}\ ;\ x\right] = {}_1F_1\left[\begin{matrix} b-a \\ b \end{matrix}\ ;\ -x\right]. \tag{1.7}$$

The result (1.7) was obtained by Kummer [24, 25] from the theory of differential equations. Bailey [5] re-derived this result by employing classical Gauss's summation theorem. Paris [27] generalized (1.7) in the form

$$e^{-x}\ {}_2F_2\left[\begin{matrix} a, & 1+d \\ 1+b, & d \end{matrix}\ ;\ x\right] = {}_2F_2\left[\begin{matrix} b-a, & f+1 \\ 1+b, & f \end{matrix}\ ;\ -x\right], \tag{1.8}$$

where $f = \frac{d(a-b)}{(a-d)}$.

For $d = b$, we recover (1.7) and for $d = \frac{a}{2}$, we recover a result obtained earlier by Exton [13]. Also, from the theory of differential equations, Kummer [24, 25] established the following result which is in the literature known as the Kummer's second theorem or Kummer's second transformation viz.

$$e^{-\frac{x}{2}}\ {}_1F_1\left[\begin{matrix} a \\ 2a \end{matrix}\ ;\ x\right] = {}_0F_1\left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix}\ ;\ \frac{x^2}{16}\right]. \tag{1.9}$$

Bailey [5] re-derived (1.9) by employing Gauss's second summation theorem and Rathie and Choi [39] re-derived (1.9) by employing Gauss's summation theorem. From (1.9), it is not difficult to obtain the following two results for the terminating ${}_2F_1(2)$ recorded in Rainville [31] for $n \in \mathbb{N}_0$ viz.

$${}_2F_1\left[\begin{matrix} -2n, & a \\ 2a & \end{matrix}\ ;\ 2\right] = \frac{\left(\frac{1}{2}\right)_n}{\left(a+\frac{1}{2}\right)_n} \tag{1.10}$$

and

$${}_2F_1\left[\begin{matrix} -2n-1, & a \\ 2a & \end{matrix}\ ;\ 2\right] = 0. \tag{1.11}$$

In 1995, Rathie and Nagar [42] established two results closely related to (1.9) out of which one result is given below:

$$\begin{aligned} e^{-\frac{x}{2}}\ {}_1F_1\left[\begin{matrix} a \\ 2a+1 \end{matrix}\ ;\ x\right] &= {}_0F_1\left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix}\ ;\ \frac{x^2}{16}\right] \\ &\quad - \frac{x}{2(2a+1)}\ {}_0F_1\left[\begin{matrix} - \\ a+\frac{3}{2} \end{matrix}\ ;\ \frac{x^2}{16}\right]. \end{aligned} \tag{1.12}$$

From (1.12), the following result is recorded by Kim et al. [19, 20, 22] viz.

$$ {}_2F_1\left[\begin{matrix} -2n, \quad a \\ 2a+1 \end{matrix} ; \; 2\right] = \frac{\left(\frac{1}{2}\right)_n}{\left(a+\frac{1}{2}\right)_n} \tag{1.13} $$

and

$$ {}_2F_1\left[\begin{matrix} -2n-1, \quad a \\ 2a+1 \end{matrix} ; \; 2\right] = \frac{\left(\frac{3}{2}\right)_n}{(2a+1)\left(a+\frac{3}{2}\right)_n}. \tag{1.14} $$

In 2010, Kim et al. [19] generalized Kummer's second theorem (1.9) and obtained explicit expression of

$$ e^{-\frac{x}{2}} \; {}_1F_1\left[\begin{matrix} a \\ 2a+i \end{matrix} ; \; x\right] \tag{1.15} $$

for $i = 0, \pm1, \pm2, \ldots, \pm5$. and discussed some applications. One of such results is given below.

$$ \begin{aligned} e^{-\frac{x}{2}} \; {}_1F_1\left[\begin{matrix} a \\ 2a+2 \end{matrix} ; \; x\right] &= {}_0F_1\left[\begin{matrix} - \\ a+\frac{3}{2} \end{matrix} ; \; \frac{x^2}{16}\right] \\ &\quad - \frac{x}{2(a+1)} \; {}_0F_1\left[\begin{matrix} - \\ a+\frac{3}{2} \end{matrix} ; \; \frac{x^2}{16}\right] \\ &\quad + \frac{x^2}{4(a+1)(2a+3)} \; {}_0F_1\left[\begin{matrix} - \\ a+\frac{5}{2} \end{matrix} ; \; \frac{x^2}{16}\right]. \end{aligned} \tag{1.16} $$

In the same paper, among other results for the terminating ${}_2F_1(2)$, they have obtained the following two results viz.

$$ {}_2F_1\left[\begin{matrix} -2n, \quad a \\ 2a+2 \end{matrix} ; \; 2\right] = \frac{\left(\frac{1}{2}\right)_n \left(\frac{a+3}{2}\right)_n}{\left(a+\frac{3}{2}\right)_n \left(\frac{a+1}{2}\right)_n} \tag{1.17} $$

and

$$ {}_2F_1\left[\begin{matrix} -2n-1, \quad a \\ 2a+2 \end{matrix} ; \; 2\right] = \frac{\left(\frac{3}{2}\right)_n}{\left(a+\frac{3}{2}\right)_n (a+1)}. \tag{1.18} $$

Motivated by extension of Kummer's first theorem established by Paris [27], Rathie and Pagany [43] established the extension of the Kummer's second theorem (1.9) in the following form:

$$e^{-\frac{x}{2}}\,{}_2F_2\left[\begin{matrix} a, & d+1 \\ 2a+1, & d \end{matrix}\;;\; x\right] = {}_0F_1\left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix}\;;\; \frac{x^2}{16}\right] + \frac{x(2a-d)}{2d(2a+1)}\,{}_0F_1\left[\begin{matrix} - \\ a+\frac{3}{2} \end{matrix}\;;\; \frac{x^2}{16}\right]. \tag{1.19}$$

It is obvious that the case $d = 2a$ of (1.19) reduces at once to (1.9). For another extensions of (1.9), we refer Rakha et al. [32, 36]. Also, from the theory of differential equations, Kummer [24,25] established the following transformation formula viz.

$$(1-x)^{-b}\,{}_2F_1\left[\begin{matrix} a, & b \\ 2a & \end{matrix}\;;\; -\frac{2x}{1-x}\right] = {}_2F_1\left[\begin{matrix} \frac{b}{2}, & \frac{b+1}{2} \\ a+\frac{1}{2} & \end{matrix}\;;\; x^2\right]. \tag{1.20}$$

Kim et al. [19] obtained several results closely related to (1.20) out of which, we would like to mention two of them below:

$$(1-x)^{-b}\,{}_2F_1\left[\begin{matrix} a, & b \\ 2a+1 & \end{matrix}\;;\; -\frac{2x}{1-x}\right] = {}_2F_1\left[\begin{matrix} \frac{b}{2}, & \frac{b+1}{2} \\ a+\frac{1}{2} & \end{matrix}\;;\; x^2\right] + \frac{bx}{(2a+1)}\,{}_2F_1\left[\begin{matrix} \frac{b+1}{2}, & \frac{b}{2}+1 \\ a+\frac{3}{2} & \end{matrix}\;;\; x^2\right] \tag{1.21}$$

and

$$(1-x)^{-b}\,{}_2F_1\left[\begin{matrix} a, & b \\ 2a+2 & \end{matrix}\;;\; -\frac{2x}{1-x}\right] = {}_2F_1\left[\begin{matrix} \frac{b}{2}, & \frac{b+1}{2} \\ a+\frac{3}{2} & \end{matrix}\;;\; x^2\right] + \frac{bx}{(a+1)}\,{}_2F_1\left[\begin{matrix} \frac{b+1}{2}, & \frac{b}{2}+1 \\ a+\frac{3}{2} & \end{matrix}\;;\; x^2\right] + \frac{b(b+1)x^2}{(a+1)(2a+3)}\,{}_2F_1\left[\begin{matrix} \frac{b}{2}+1, & \frac{b+3}{2} \\ a+\frac{5}{2} & \end{matrix}\;;\; x^2\right]. \tag{1.22}$$

Next, using theory of differential equations, Preece [29] established the following interesting identity involving product of generalized hypergeometric functions viz.

$${}_1F_1\left[\begin{matrix} a \\ 2a \end{matrix}\;;\; x\right] \times {}_1F_1\left[\begin{matrix} a \\ 2a \end{matrix}\;;\; -x\right] = {}_1F_2\left[\begin{matrix} a \\ a+\frac{1}{2}, \; 2a \end{matrix}\;;\; \frac{x^2}{4}\right]. \tag{1.23}$$

Using (1.7), we can express (1.23) in the form

$$\left\{ {}_1F_1\left[\begin{array}{c} a \\ 2a \end{array} ; x\right]\right\}^2 = e^x \, {}_1F_2\left[\begin{array}{c} a \\ a+\frac{1}{2}, \quad 2a \end{array} ; \frac{x^2}{4}\right]. \tag{1.24}$$

Rathie [37] proved Preece's identity (1.21) by a very short method and obtained a few results closely related to it, and see also Choi and Rathie [38] for more contiguous results. Bailey [5] generalized Preece's identity (1.20) by employing the classical Watson's theorem for the series ${}_3F_2$ with unit argument in the following form:

$$\begin{aligned} {}_1F_1\left[\begin{array}{c} a \\ 2a \end{array} ; x\right] \times {}_1F_1\left[\begin{array}{c} b \\ 2b \end{array} ; -x\right] \\ = {}_2F_3\left[\begin{array}{c} \frac{1}{2}(a+b), \quad \frac{1}{2}(a+b+1) \\ a+\frac{1}{2}, \quad b+\frac{1}{2}, \quad a+b \end{array} ; \frac{x^2}{4}\right]. \end{aligned} \tag{1.25}$$

Using (1.7), we can express (1.22) in the form

$$\begin{aligned} {}_1F_1\left[\begin{array}{c} a \\ 2a \end{array} ; x\right] \times {}_1F_1\left[\begin{array}{c} b \\ 2b \end{array} ; x\right] \\ = e^x \, {}_2F_3\left[\begin{array}{c} \frac{1}{2}(a+b), \quad \frac{1}{2}(a+b+1) \\ a+\frac{1}{2}, \quad b+\frac{1}{2}, \quad a+b \end{array} ; \frac{x^2}{4}\right]. \end{aligned} \tag{1.26}$$

Obviously, when $b = a$, we get at once the Preece's identity (1.23). Rathie and Choi [40, 41] established the Preece's identity (1.23) by a short method and obtained four results closely related to it. Very recently, Kim et al. [15,16] established the extension of Bailey's identity (1.25) in the following form:

(1.27)

$$\begin{aligned} &{}_2F_2\left[\begin{array}{c} a, \quad d+1 \\ 2a+1, \quad d \end{array} ; x\right] \times {}_2F_2\left[\begin{array}{c} b, \quad e+1 \\ 2b+1, \quad e \end{array} ; x\right] \\ &= e^x \left\{ {}_2F_3\left[\begin{array}{c} \frac{1}{2}(a+b), \quad \frac{1}{2}(a+b+1) \\ a+\frac{1}{2}, \quad b+\frac{1}{2}, \quad a+b \end{array} ; \frac{x^2}{4}\right] \right. \\ &\quad + \frac{x(2a-d)}{2d(2a+1)} \, {}_2F_3\left[\begin{array}{c} \frac{1}{2}(a+b+1), \quad \frac{1}{2}(a+b+2) \\ a+\frac{3}{2}, \quad b+\frac{1}{2}, \quad a+b+1 \end{array} ; \frac{x^2}{4}\right] \\ &\quad + \frac{x(2b-e)}{2e(2b+1)} \, {}_2F_3\left[\begin{array}{c} \frac{1}{2}(a+b+1), \quad \frac{1}{2}(a+b+2) \\ a+\frac{1}{2}, \quad b+\frac{3}{2}, \quad a+b+1 \end{array} ; \frac{x^2}{4}\right] \end{aligned}$$

$$+\frac{x^2(2a-d)(2b-e)}{4de(2a+1)(2b+1)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+2), & \frac{1}{2}(a+b+3) & \\ a+\frac{3}{2}, & b+\frac{3}{2}, & a+b+2\end{matrix}\ ;\ \frac{x^2}{4}\right]\Bigg\}.$$

For $d = 2a$ and $e = 2b$, we immediate recover Bailey's identity (1.23). In establishing Preece identity (1.21), Bailey's identity (1.26). In establishing Preece identity (1.21) and its generalization (1.26), they have used the following well known result due to Bailey [5].

$$(1.28)\qquad {}_0F_1\left[\begin{matrix}-\\ \rho\end{matrix}\ ;\ x\right]\times {}_0F_1\left[\begin{matrix}-\\ \sigma\end{matrix}\ ;\ x\right]$$

$$= {}_2F_3\left[\begin{matrix}\frac{1}{2}(\rho+\sigma-1), & \frac{1}{2}(\rho+\sigma) & \\ \rho, & \sigma, & \rho+\sigma-1\end{matrix}\ ;\ 4x\right].$$

For another extension of (1.26), we refer Rakha et al. [33]. Next, by employing results (1.10) and (1.11), Berndt [6] established the following very interesting result of Great Indian Mathematician S. Ramanujan given in the following theorem [6]:

Theorem 1.1. *Let $\phi(t)$ be analytic for $|t-1| < \mathbb{R}$ where $\mathbb{R} > 1$. Suppose that $a, d+1$ and $\phi(t)$ are such that the order of summation in*

$$\sum_{k=0}^{\infty}\frac{2^k(a)_k}{(2a)_k k!}\sum_{n=k}^{\infty}\frac{(-1)^n(-n)_k}{n!}\phi^{(n)}(1)$$

may be inverted. Then

$$(1.29)\qquad \sum_{k=0}^{\infty}\frac{2^k(a)_k\phi^{(k)}(0)}{(2a)_k k!} = \sum_{k=0}^{\infty}\frac{\phi^{(2k)}(1)}{2^{2k}\left(a+\frac{1}{2}\right)_k k!}.$$

Kim et al. [21] generalized Ramanujan's result (1.29) in the form

$$\sum_{k=0}^{\infty}\frac{2^k(a)_k\phi^{(k)}(0)}{(2a+j)_k k!}$$

for $j \in \mathbb{Z}$.

For another extension of (1.29), we refer Rakha et al. [34] and for its generalization, we refer Kim et al. [21]. In addition to this, the Beta integral $B(a,b)$ is defined by the first integral and known to be evaluated as the second one as follows:

$$(1.30)\qquad B(a,b) = \begin{cases}\int_0^1 x^{a-1}(1-x)^{b-1}dx, & (Re(a) > 0; Re(b) > 0),\\ \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, & (a, b \in \mathbb{C}\backslash\mathbb{Z}_0^-).\end{cases}$$

In a paper, Krattenthaler and Rao [23] made a systematic use of so-called Beta integral method, a method of deriving new hypergeometric identities from old ones by mainly using the Beta integral in (1.30) based on the MATHEMATICA package HYP, to illustrate several interesting identities for the hypergeometric function and the Kampé de Fériet function. The same procedure can be done for Gamma integral.

The vast popularity and immense usefulness of the hypergeometric function ${}_2F_1$ and the generalized hypergeometric function ${}_pF_q$ in one variable have inspired and simulated a large number of researchers in mathematics to study hypergeometric functions of two and more variables. In this regard, serious and very significant study of the functions of two variables initiated by Appell [3], who introduced the so-called functions F_1, F_2, F_3 and F_4 named in the literature, the Appell functions which are the natural generalizations of hypergeometric function ${}_2F_1$ and the generalized hypergeometric functions ${}_pF_q$, Also their confluent forms were studied by Humbert [49] and a complete list of these functions can be seen in the standard texts [30, 49].

Later on, the Appell functions F_1, F_2, F_3 and F_4 and their confluent forms were further generalized by Kampé de Fériet [3] who introduced a more general function in two variables. The notation for this function was subsequently abbreviated by Burchnall and Chaundy [7, 8]. However, in our present investigation , we recall here the definition of a more general function in two variables (than the one defined by Kampé de Fériet in a slightly modified notation which is due to Srivastava and Panda [51] defined as follows:

$$F^{H:A;B}_{G:C;D}\left[\begin{array}{cccc} (h_H): & (a_A); & (b_B) & \\ & & & ;x,\ y \\ (g_G): & (c_C); & (d_D) & \end{array}\right] \tag{1.31}$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((h_H))_{m+n}((a_A))_m((b_B))_n x^m y^n}{((g_G))_{m+n}((c_C))_m((d_D))_n m!n!},$$

where (h_H) denotes the sequence of parameters $(h_1, h_2, \ldots, h_H)$ and for $n \in \mathbb{N}_0$, define the pochhammer symbol

$$((h_H))_n := (h_1)_n \cdots (h_H)_n,$$

where, when $n = 0$, the product is understood to reduce to unity. The symbol (h) is a convenient contraction for the sequence of the parameters $(h_1, h_2, \ldots, h_H)$ and Pochhammer symbol $(h)_n$ is the same as defined in (1.2), For more details about the convergence for the function, we refer [49].

The Srivastava-Daoust generalized Kampé de Fériet function of two variables initially introduced in [46–48] will be defined and represented in the following manner:

$$S^{A:B;B'}_{C:D;D'}\left[\begin{array}{c} x \\ y \end{array}\right] = S^{A:B;B'}_{C:D;D'}\left[\begin{array}{cccc} [(a):\theta,\phi]: & [(b):\psi]; & [(b'):\psi'] & \\ & & & ;x,\ y \\ [(c):\delta,\epsilon]: & [(d):\eta]; & [(d)':\eta'] & \end{array}\right]$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Pi_{j=1}^{A}\Gamma(a_j+m\theta_j+n\phi_j)\ \Pi_{j=1}^{B}\Gamma(b_j+m\psi_j)\ \Pi_{j=1}^{B'}\Gamma(b'_j+n\psi'_j)x^m y^n}{\Pi_{j=1}^{C}\Gamma(c_j+m\delta_j+n\epsilon_j)\ \Pi_{j=1}^{D}\Gamma(d_j+m\eta_j)\ \Pi_{j=1}^{D'}\Gamma(d'_j+m\eta'_j)m!n!}, \tag{1.32}$$

where, for convergence

$$\begin{cases} 1+\sum_{j=1}^{C}\delta_j+\sum_{j=1}^{D}\eta_j-\sum_{j=1}^{A}\theta_j-\sum_{j=1}^{B}\psi_j>0, \\ 1+\sum_{j=1}^{C}\epsilon_j+\sum_{j=1}^{D^{'}}\eta_j^{'}-\sum_{j=1}^{A}\phi_j-\sum_{j=1}^{B^{'}}\psi_j^{'}>0. \end{cases}$$

A detailed account of the above function can be founded in the research papers [46–48] and in the standard text [49].

Later on, a unification of Lauricella fourteen triple hypergeometric series $F_1,\ldots,F_{14}$ and the additional three triple hypergeometric series H_A, H_B and H_C was introduced by Srivastava [49] who defined the following general triple hypergeometric series $F^{(3)}[x,y,z]$in the following form:

$$F^{(3)}[x,y,z]=F^{(3)}\left[\begin{array}{c} (a)::(b);\quad (b^{'});\quad (b^{''}):(c);\quad (c^{'});\quad (c^{''}) \\ \\ (e)::(g);\quad (g^{'});\quad (g^{''}):(h);\quad (h^{'});\quad (h^{''}) \end{array};x,\ y,\ z\right]$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\triangle\,(m,n,p)\frac{x^m}{m!}\frac{x^n}{n!}\frac{x^p}{p!}, \tag{1.33}$$

where, for convenience

$$\triangle(m,n,p)=\frac{\Pi_{j=1}^{A}(a_j)_{m+n+p}\ \Pi_{j=1}^{B}(b_j)_{m+n}\ \Pi_{j=1}^{B^{'}}(b_j^{'})_{n+p}\ \Pi_{j=1}^{B^{''}}(b_j^{''})_{m+p}}{\Pi_{j=1}^{E}(e_j)_{m+n+p}\ \Pi_{j=1}^{G}(g_j)_{m+n}\ \Pi_{j=1}^{G^{'}}(b_j^{'})_{n+p}\ \Pi_{j=1}^{G^{''}}(g_j^{''})_{m+p}}$$

$$\times\frac{\Pi_{j=1}^{C}(c_j)_m\ \Pi_{j=1}^{C^{'}}(c_j^{'})_n\ \Pi_{j=1}^{C^{''}}(c_j^{''})_p}{\Pi_{j=1}^{H}(h_j)_m\ \Pi_{j=1}^{H^{'}}(h_j^{'})_n\ \Pi_{j=1}^{H^{''}}(h_j^{''})_p} \tag{1.34}$$

and (a) abbreviates the array of A parameters $a_1,\ldots,a_A$, with similar interpretations for (b), $(b^{'})$, $(b^{''})$, and so on. The detailed about the triple series for its convergence conditions, we refer [49]. By employing the Beta-integral method to Bailey's identity (1.26), Pogany and Rathie [28] established the following interesting formula for the Srivastava-Daoust double hypergeometric function in terms of the Kampé de Fériet viz.

(1.35)

$$S^{1:0;2}_{1:0;3}\left[\begin{array}{llll} [d:1,2]:-; & \left[\frac{1}{2}(\alpha+\beta);1\right]; & \left[\frac{1}{2}(\alpha+\beta+1);1\right] & ;x \\ [e:1,2]:-; & \left[\alpha+\frac{1}{2};1\right]; & \left[\beta+\frac{1}{2};1\right]; \quad \left[\frac{1}{2}(\alpha+\beta);1\right] & ;\frac{x^2}{4} \end{array}\right]$$

$$=\frac{\sqrt{\pi}\Gamma(d)2^{1-\alpha-\beta}}{\Gamma(e)\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\beta+\frac{1}{2}\right)}\ F^{1:1;1}_{1:1;1}\left[\begin{array}{c} d:\alpha;\quad \beta \\ \\ e:2\alpha;\quad 2\beta \end{array};x,\ x\right]$$

provided that $Re(d)>0$, $Re(e)>0$ for all $x\in\mathbb{C}$.

Very recently, Kim et al. [18] have established the following reduction formulas for the Kampé de Fériet function, Srivastava-Daoust double hypergeometric function and Srivastava triple series, viz.

$$F^{1:1;1}_{1:1;1}\left[\begin{array}{l} d:\alpha;\quad \beta \\ \\ e:2\alpha;\quad 2\beta \end{array} ;x,\ -x\right] \tag{1.36}$$

$$= {}_4F_5\left[\begin{array}{cccccc} \frac{\alpha+\beta}{2}, & \frac{\alpha+\beta+1}{2}, & \frac{d}{2}, & \frac{d+1}{2} & & \\ & & & & & ;\frac{x^2}{4} \\ \alpha+\frac{1}{2}, & \beta+\frac{1}{2}, & \alpha+\beta, & \frac{e}{2}, & \frac{e+1}{2} & \end{array}\right],$$

$$F^{(3)}\left[\begin{array}{llllll} d::-; & -; & -; & -; & \alpha; & \beta \\ & & & & & \\ e::-; & -; & -; & -; & 2\alpha; & 2\beta \end{array} ;-x,\ x,\ x\right] \tag{1.37}$$

$$= {}_4F_5\left[\begin{array}{cccccc} \frac{\alpha+\beta}{2}, & \frac{\alpha+\beta+1}{2}, & \frac{d}{2}, & \frac{d+1}{2} & & \\ & & & & & ;\frac{x^2}{4} \\ \alpha+\frac{1}{2}, & \beta+\frac{1}{2}, & \alpha+\beta, & \frac{e}{2}, & \frac{e+1}{2} & \end{array}\right].$$

From (1.36) and (1.37), it follows that

$$F^{(3)}\left[\begin{array}{llllll} d::-; & -; & -; & -; & \alpha; & \beta \\ & & & & & \\ e::-; & -; & -; & -; & 2\alpha; & 2\beta \end{array} ;-x,\ x,\ x\right] \tag{1.38}$$

$$= F^{1:1;1}_{1:1;1}\left[\begin{array}{l} d:\alpha;\quad \beta \\ \\ e:2\alpha;\quad 2\beta \end{array} ;x,\ -x\right].$$

In recent years, several researchers contributed a lot in finding reduction formulas for the above mentioned functions. For this, we mention research paper by Kim et al. [18], Rakha et al. [35], Choi and Rathie [9–11], Choi et al. [12].

The paper is organized as follows: In Section 2, we provide an extension of the well known and useful Kummer's second theorem (1.9). As special cases, we recover Kummer's second theorem and two results closely related to it obtained earlier by Rathie and Nagar [42] and Kim et al. [17].

In Section 3, two interesting results for the terminating ${}_3F_2(2)$ series have been established with the help of the result given in Section 2. As special cases, we recover two results for the terminating ${}_2F_1(2)$ recorded in Rainville [31] and four results obtained earlier by Kim et al. [20].

In Section 4, we provide an extension of the well known Kummer's transformation formula (1.20). The result is derived with the help of the results given in Section 3, As special cases, we recover Kummer's transformation formula (1.20) and two results obtained earlier by Kim et al. [19].

In Section 5, we establish an extension of the well known Bailey's identity (1.26) involving products of two generalized hypergeometric functions. The result is obtained with the help of the result given in Section 2. As special cases, we recover Bailey's identity (1.26), Preece's identity (1.24) and a few results obtained earlier by Rathie and Choi et al. [40, 41].

In Section 6, an extension of well known and useful result due to Ramanujan is given. The result is obtained with the help of the results given in Section 3, A few known special cases obtained earlier by Kim et al. [19] have been obtained.

In Section 7, we establish two interesting reduction formulas for the Kampé de Fériet double hypergeometric function with the help of the result given in Section 2 and considered some interesting special cases.

In Section 8, we provide two formulas of the Kampé de Fériet function expressed in terms of the Srivastava-Daoust S-function of two variables with the help of the result given in Section 5. We also consider some special cases in this section.

In Section 9, we provide two formulas of the generalized hypergeometric function expressed in terms of S-function of two variables with the help of the result given in Section 2 together with some special cases.

In the last Section 10, we aim to provide two formulas for Srivastava's triple hypergeometric series expressed in terms of generalized hypergeometric function with the help of the result given in Section 5. We also consider some special cases of our main findings.

2. Extension of Kummer's second theorem

The extension of the well-known and useful Kummer's second theorem to be proved in this section is the following.

$$(2.1)\qquad e^{-\frac{x}{2}}\ {}_2F_2\left[\begin{matrix} a, & d+1 \\ & & ; & x \\ 2a+2 & d \end{matrix}\right]$$

$$= {}_0F_1\left[\begin{matrix} - \\ & ; & \frac{x^2}{16} \\ a+\frac{3}{2} \end{matrix}\right] + \frac{x\left(\frac{a}{d}-1\right)}{2(a+1)}\ {}_0F_1\left[\begin{matrix} - \\ & ; & \frac{x^2}{16} \\ a+\frac{3}{2} \end{matrix}\right]$$

$$+ \frac{x^2\left(1-\frac{a}{d}\right)}{4(a+1)(2a+3)}\ {}_0F_1\left[\begin{matrix} - \\ & ; & \frac{x^2}{16} \\ a+\frac{5}{2} \end{matrix}\right],$$

provided $2a+2, d \notin \mathbb{Z}_0^-$.

Proof. In order to establish the identity (2.1), we shall first establish the following result.

$$(2.2)\qquad {}_2F_2\left[\begin{matrix} a, & d+1 \\ & & ; & x \\ 2a+2 & d \end{matrix}\right]$$

$$= {}_1F_1\left[\begin{matrix} a \\ & ; & x \\ 2a+2 \end{matrix}\right] + \frac{ax}{2d(a+1)}\ {}_1F_1\left[\begin{matrix} a+1 \\ & ; & x \\ 2a+3 \end{matrix}\right].$$

For this, starting with the left-hand side of (2.2), we have, upon using its definition

$$S = {}_2F_2\left[\begin{matrix} a, & d+1 \\ 2a+2 & d \end{matrix} ; x\right] = \sum_{n=0}^{\infty} \frac{(a)_n (d+1)_n}{(2a+2)_n (d)_n} \frac{x^n}{n!}.$$

Using the result $\frac{(d+1)_n}{(d)_n} = 1 + \frac{n}{d}$, we have

$$S = \sum_{n=0}^{\infty} \frac{(a)_n}{(2a+2)_n} \frac{x^n}{n!} \left(1 + \frac{n}{d}\right).$$

Separating into two series and summing up the first series, we have

$$S = {}_1F_1\left[\begin{matrix} a \\ 2a+2 \end{matrix} ; x\right] + \frac{1}{d} \sum_{n=1}^{\infty} \frac{(a)_n}{(2a+2)_n} \frac{x^n}{(n-1)!}.$$

Now, putting $n-1 = N$ and using the result $(a)_{N+1} = a(a+1)_N$, we have after some algebra

$$S = {}_1F_1\left[\begin{matrix} a \\ 2a+2 \end{matrix} ; x\right] + \frac{ax}{2d(a+1)} \sum_{N=0}^{\infty} \frac{(a+1)_N}{(2a+3)_{N+1}} \frac{x^N}{N!}.$$

Finally, summing up the series, we easily arrive at the right-hand side of (2.2).

Now, we are ready to establish our identity (2.1). For this, multiply both sides of (2.2) by $e^{-\frac{x}{2}}$, we have

$$(2.3) \quad e^{-\frac{x}{2}} {}_2F_2\left[\begin{matrix} a, & d+1 \\ 2a+2 & d \end{matrix} ; x\right]$$
$$= e^{-\frac{x}{2}} {}_1F_1\left[\begin{matrix} a \\ 2a+2 \end{matrix} ; x\right] + \frac{ax}{2d(a+1)} e^{-\frac{x}{2}} {}_1F_1\left[\begin{matrix} a+1 \\ 2a+3 \end{matrix} ; x\right].$$

Now, observe here that for the first and second expressions appearing on the right-hand side of (2.3), we can apply the known result (1.16) and (1.12) and after some simplification, we can easily arrive at the result (2.1). This completes the proof of (2.1). □

2.1. Special cases

In this section, we shall mention two known special cases of (2.1).

1. In (2.1), if we set $d = a$, we get at once the result

$$e^{-\frac{x}{2}} {}_1F_1\left[\begin{matrix} a+1 \\ 2a+2 \end{matrix} ; x\right] = {}_0F_1\left[\begin{matrix} - \\ a+\frac{3}{2} \end{matrix} ; \frac{x^2}{16}\right]$$

which is nothing but Kummer's second theorem (1.9) (by replacing a by $a-1$).

2. In (2.1), if we set $d = 2a+1$, we get

$$e^{-\frac{x}{2}}\,{}_1F_1\left[\begin{matrix} a \\ 2a+1 \end{matrix}\;;\; x\right] \tag{2.4}$$

$$= {}_0F_1\left[\begin{matrix} - \\ a+\frac{3}{2} \end{matrix}\;;\; \tfrac{x^2}{16}\right] - \frac{x}{2(2a+1)}\,{}_0F_1\left[\begin{matrix} - \\ a+\frac{3}{2} \end{matrix}\;;\; \tfrac{x^2}{16}\right]$$

$$+ \frac{x^2}{4(2a+1)(2a+3)}\,{}_0F_1\left[\begin{matrix} - \\ a+\frac{5}{2} \end{matrix}\;;\; \tfrac{x^2}{16}\right].$$

But it is not difficult to see that

$${}_0F_1\left[\begin{matrix} - \\ a+\frac{3}{2} \end{matrix}\;;\; \tfrac{x^2}{16}\right] + \frac{x^2}{4(2a+1)(2a+3)}\,{}_0F_1\left[\begin{matrix} - \\ a+\frac{5}{2} \end{matrix}\;;\; \tfrac{x^2}{16}\right] \tag{2.5}$$

$$= {}_0F_1\left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix}\;;\; \tfrac{x^2}{16}\right].$$

Thus, using (2.5) in (2.4), we immediately get the known result (1.12) due to Rathie and Nagar [42] which is also recorded in Kim et al. [19].

3. In (2.1), if we let $d \to \infty$, we get (1.16). Similarly, other results can be obtained.

3. Two new results for terminating ${}_3F_2\,(2)$

In this section, we shall establish the following two new and interesting results for terminating ${}_3F_2\,(2)$.

$${}_3F_2\left[\begin{matrix} -2n, & a, & d+1 \\ 2a+2, & d & \end{matrix}\;;\; 2\right] = \frac{\left(\frac{1}{2}\right)_n}{\left(a+\frac{3}{2}\right)_n}\left\{1 + \frac{2\left(1-\frac{a}{d}\right)n}{a+1}\right\} \tag{3.1}$$

and

$${}_3F_2\left[\begin{matrix} -2n-1, & a, & d+1 \\ 2a+2, & d & \end{matrix}\;;\; 2\right] = -\left(\frac{a}{d}-1\right)\frac{\left(\frac{3}{2}\right)_n}{(a+1)\left(a+\frac{3}{2}\right)_n} \tag{3.2}$$

each for $n \in \mathbb{N}_0$.

Proof. It is a simple exercise to show that the even term of the expression appearing on the right-hand side of (2.1) can be written as

$$
{}_0F_1\left[\begin{matrix} - \\ a+\frac{3}{2}\end{matrix} ; \frac{x^2}{16}\right] + \frac{x^2\left(1-\frac{a}{d}\right)}{4(a+1)(2a+3)}\,{}_0F_1\left[\begin{matrix} - \\ a+\frac{5}{2}\end{matrix} ; \frac{x^2}{16}\right]
$$

$$
= \sum_{n=0}^{\infty}\left\{1+\frac{2n\left(1-\frac{a}{d}\right)}{a+1}\right\}\frac{x^{2n}}{\left(a+\frac{3}{2}\right)_n 2^{4n} n!}
$$

and odd term is equal to

$$
\frac{x\left(\frac{a}{d}-1\right)}{2(a+1)}\,{}_0F_1\left[\begin{matrix} - \\ a+\frac{3}{2}\end{matrix} ; \frac{x^2}{16}\right] = \frac{\left(\frac{a}{d}-1\right)}{2(a+1)}\sum_{n=0}^{\infty}\frac{x^{2n+1}}{\left(a+\frac{3}{2}\right)_n 2^{4n} n!}.
$$

Hence, right-hand side of (2.1) can be written as

$$
\text{R.H.S} = \sum_{n=0}^{\infty}\left\{1+\frac{2n\left(1-\frac{a}{d}\right)}{a+1}\right\}\frac{x^{2n}}{\left(a+\frac{3}{2}\right)_n 2^{4n} n!} + \frac{\left(\frac{a}{d}-1\right)}{2(a+1)}\sum_{n=0}^{\infty}\frac{x^{2n+1}}{\left(a+\frac{3}{2}\right)_n 2^{4n} n!}. \tag{3.3}
$$

Also, start with the left-hand side of (2.1)

$$
\text{L.H.S} = e^{-\frac{x}{2}}\,{}_2F_2\left[\begin{matrix} a, & d+1 \\ 2a+2 & d\end{matrix} ; x\right].
$$

Expressing both functions as series

$$
= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n}{2^n n!}\frac{(a)_m(d+1)_m}{(2a+2)_m(d)_m m!}x^{n+m}.
$$

Changing n to $n-m$ and using the result [31, Lemma-10, p. 56],

$$
\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k) \tag{3.4}
$$

we have

$$
\text{L.H.S} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^{n-m}}{2^{n-m}(n-m)!}\frac{(a)_m(d+1)_m}{(2a+2)_m(d)_m m!}x^n.
$$

Using the identity $(n-m)! = \frac{(-1)^m n!}{(-n)_m}$ and after some calculation

$$
\text{L.H.S} = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^n n!}\sum_{m=0}^{n}\frac{(-n)_m(a)_m(d+1)_m}{(2a+2)_m(d)_m m!}2^m.
$$

Summing up the inner series

$$\text{L.H.S} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n n!} \, {}_3F_2 \left[\begin{matrix} -n, & a, & d+1 \\ 2a+2, & d & \end{matrix} ; \; 2 \right].$$

Separating into even and odd powers of x and using the identities $(2n)! = 2^{2n} \left(\frac{1}{2}\right)_n n!$ and $(2n+1)! = 2^{2n} \left(\frac{3}{2}\right)_n n!$, we have

$$\text{L.H.S} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^{4n} \left(\frac{1}{2}\right)_n n!} \, {}_3F_2 \left[\begin{matrix} -2n, & a, & d+1 \\ 2a+2, & d & \end{matrix} ; \; 2 \right]$$

$$- \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^{4n+1} \left(\frac{3}{2}\right)_n n!} \, {}_3F_2 \left[\begin{matrix} -2n-1, & a, & d+1 \\ 2a+2, & d & \end{matrix} ; \; 2 \right]. \tag{3.5}$$

Hence, from (3.3) and (3.5), equating coefficient of x^{2n} and x^{2n+1}, we arrive at the desired results (3.1) and (3.2). □

3.1. Special cases

In this section, we shall mention four known special cases of (3.1) and (3.2).

1. In (3.1) and (3.2), if we set $d = a$, we get the known results (1.10) and (1.11) respectively.
2. In (3.1) and (3.2), if we set $d = 2a+1$, we get the known results (1.13) and (1.14) respectively.
3. In (3.1) and (3.2), if we let $d \to \infty$, we get the known results (1.17) and (1.18).

4. Extension of a transformation formula due to kummer

In this section, we shall establish the extension of the well-known and useful transformation formula (1.20) due to Kummer. The result to be proved in this section is the following:

$$(1-x)^{-b} \, {}_3F_2 \left[\begin{matrix} a, & b, & d+1 \\ 2a+2, & d & \end{matrix} ; \; \frac{-2x}{1-x} \right] \tag{4.1}$$

$$= {}_3F_2 \left[\begin{matrix} \frac{b}{2}, & \frac{b+1}{2}, & c+1 \\ a+\frac{3}{2}, & c & \end{matrix} ; \; x^2 \right]$$

$$+ \frac{b\left(1-\frac{a}{d}\right)x}{(a+1)} \, {}_2F_1 \left[\begin{matrix} \frac{b+1}{2}, & \frac{b}{2}+1 \\ a+\frac{3}{2} & \end{matrix} ; x^2 \right],$$

where $c = \frac{(a+1)}{2\left(1-\frac{a}{d}\right)}$.

Proof. In order to establish the result (4.1), we proceed as follows. Denoting the left-hand side of (4.1) by S, expressing ${}_3F_2$ as a series, we have

$$S=\sum_{k=0}^{\infty}\frac{(-2)^k(a)_k(b)_k(d+1)_k}{(2a+2)_k(d)_k k!}x^k(1-x)^{-(b+k)}.$$

Now using the Binomial theorem,

$$(1-x)^{-(b+k)}=\sum_{n=0}^{\infty}\frac{(b+k)_n}{n!}x^n$$

we have

$$S=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-2)^k(a)_k(b)_k(d+1)_k(b+k)_n}{(2a+2)_k(d)_k k!n!}x^{k+n}.$$

Using the identity

$$(b)_k(b+k)_n=(b)_{n+k}$$

this becomes

$$S=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-2)^k(a)_k(d+1)_k(b)_{n+k}}{(2a+2)_k(d)_k k!n!}x^{k+n}.$$

Now, replacing n by $n-k$ and using (3.4), we have

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-2)^k(a)_k(d+1)_k(b)_n}{(2a+2)_k(d)_k(n-k)!k!}x^n.$$

Using the identity,

$$(n-m)!=\frac{(-1)^m n!}{(-n)_m}$$

we have, after some simplification

$$S=\sum_{n=0}^{\infty}\frac{(b)_n}{n!}x^n\sum_{k=0}^{n}\frac{(2)^k(a)_k(-n)_k(d+1)_k}{(2a+2)_k(d)_k k!}x^n.$$

Summing up the inner series, we have

$$S=\sum_{n=0}^{\infty}\frac{(b)_n}{n!}x^n\ {}_3F_2\left[\begin{matrix}-n, & a, & d+1 & \\ & & ; & 2\\ 2a+2, & d & & \end{matrix}\right].$$

Separating into even and odd power of x, we have

$$S=\sum_{n=0}^{\infty}\frac{(b)_{2n}}{2n!}x^{2n}\ {}_3F_2\left[\begin{matrix}-2n, & a, & d+1 & \\ & & ; & 2\\ 2a+2, & d & & \end{matrix}\right]$$
$$+\sum_{n=0}^{\infty}\frac{(b)_{2n+1}}{(2n+1)!}x^{2n+1}\ {}_3F_2\left[\begin{matrix}-2n-1, & a, & d+1 & \\ & & ; & 2\\ 2a+2, & d & & \end{matrix}\right].$$

Finally, using the result (3.1) and (3.2) and making use of the following identities

$$(b)_{2n} = 2^{2n}\left(\frac{b}{2}\right)_n\left(\frac{b+1}{2}\right)_n,$$

$$(b)_{2n+1} = b2^{2n}\left(\frac{b+1}{2}\right)_n\left(\frac{b}{2}+1\right)_n$$

and

$$(2n)! = 2^{2n}\left(\frac{1}{2}\right)_n n!,$$

$$(2n+1)! = 2^{2n}\left(\frac{3}{2}\right)_n n!$$

and after some simplifications, we get

$$\sum_{n=0}^{\infty}\frac{\left(\frac{b}{2}\right)_n\left(\frac{b+1}{2}\right)_n(c+1)_n}{\left(a+\frac{3}{2}\right)_n(c)_n n!}x^{2n} + \frac{\left(1-\frac{a}{d}\right)bx}{(a+1)}\sum_{n=0}^{\infty}\frac{\left(\frac{b+1}{2}\right)_n\left(\frac{b}{2}+1\right)_n}{\left(a+\frac{3}{2}\right)_n n!}x^{2n}.$$

Finally, summing up both the series, we easily arrive at the right-hand side of the result (4.1). This completes the proof of (4.1). □

4.1. Special cases

In this section, we shall consider some special cases of (4.1).

(1) In (4.1), if we take $d = a$, we get

$$(1-x)^{-b}\ {}_2F_1\left[\begin{matrix} a+1, & b \\ & & ;\frac{-2x}{1-x} \\ 2a+2 & \end{matrix}\right] = {}_2F_1\left[\begin{matrix} \frac{b+1}{2}, & \frac{b}{2}+1 \\ & & ;x^2 \\ a+\frac{3}{2} & \end{matrix}\right] \tag{4.2}$$

which is Kummer's transformation (1.20) (by replacing a by $a-1$).

(2) In (4.1), if we take $d = 2a+1$, we get the known result (1.21) due to Kim et al. [19].

(3) In (4.1), if we take $d \to \infty$, we get another known result (1.22) due to Kim et al. [19].

Similarly, other results can be obtained.

5. Extension of Bailey's identity

The extension of the well-known and useful Bailey's identity (1.26) to be proved in this section is the following

(5.1)

$$ {}_2F_2\left[\begin{matrix} a, & d+1 \\ & & ;\ x \\ 2a+2 & d \end{matrix}\right] \times {}_2F_2\left[\begin{matrix} b, & f+1 \\ & & ;\ x \\ 2b+2 & f \end{matrix}\right]$$

$$= e^x\left\{ {}_2F_3\left[\begin{matrix} \frac{1}{2}(a+b+3), & \frac{1}{2}(a+b+2) & \\ & & & ;\ \frac{x^2}{4} \\ a+\frac{3}{2} & b+\frac{3}{2} & a+b+2 \end{matrix}\right]\right.$$

$$+ \frac{\left(\frac{a}{d}-1\right)x}{2(a+1)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+3), & \frac{1}{2}(a+b+2) & \\ a+\frac{3}{2} & b+\frac{3}{2} & a+b+2\end{matrix};\ \frac{x^2}{4}\right]$$
$$+ \frac{\left(\frac{b}{f}-1\right)x}{2(b+1)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+3), & \frac{1}{2}(a+b+2) & \\ a+\frac{3}{2}, & b+\frac{3}{2} & a+b+2\end{matrix};\ \frac{x^2}{4}\right]$$
$$- \frac{\left(\frac{a}{d}-1\right)x^2}{4(a+1)(2a+3)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+4), & \frac{1}{2}(a+b+3) & \\ a+\frac{5}{2}, & b+\frac{3}{2} & a+b+3\end{matrix};\ \frac{x^2}{4}\right]$$
$$- \frac{\left(\frac{b}{f}-1\right)x^2}{4(b+1)(2b+3)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+4), & \frac{1}{2}(a+b+3) & \\ a+\frac{3}{2}, & b+\frac{5}{2} & a+b+3\end{matrix};\ \frac{x^2}{4}\right]$$
$$+ \frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{f}-1\right)x^2}{4(a+1)(b+1)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+3), & \frac{1}{2}(a+b+2) & \\ a+\frac{3}{2} & b+\frac{3}{2} & a+b+2\end{matrix};\ \frac{x^2}{4}\right]$$
$$- \frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{f}-1\right)x^3}{8(a+1)(b+1)(2a+3)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+4), & \frac{1}{2}(a+b+3) & \\ a+\frac{5}{2}, & b+\frac{3}{2} & a+b+3\end{matrix};\ \frac{x^2}{4}\right]$$
$$- \frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{f}-1\right)x^3}{8(a+1)(b+1)(2b+3)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+4), & \frac{1}{2}(a+b+3) & \\ a+\frac{3}{2}, & b+\frac{5}{2} & a+b+3\end{matrix};\ \frac{x^2}{4}\right]$$
$$+ \frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{f}-1\right)x^4}{16(a+1)(b+1)(2a+3)(2b+3)} \times\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+5), & \frac{1}{2}(a+b+4) & \\ a+\frac{5}{2}, & b+\frac{5}{2} & a+b+4\end{matrix};\ \frac{x^2}{4}\right]\Bigg\}$$

provided $2a+2, 2b+2, d, f \notin \mathbb{Z}_0^-$.

Proof. In order to establish the result (5.1), it is sufficient to show that

(5.2)

$$e^{-x}\ {}_2F_2\left[\begin{matrix}a, & d+1\\ 2a+2 & d\end{matrix};\ x\right] \times\ {}_2F_2\left[\begin{matrix}b, & f+1\\ 2b+2 & f\end{matrix};\ x\right]$$
$$= {}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+3), & \frac{1}{2}(a+b+2) & \\ a+\frac{3}{2} & b+\frac{3}{2} & a+b+2\end{matrix};\ \frac{x^2}{4}\right]$$
$$+ \frac{\left(\frac{a}{d}-1\right)x}{2(a+1)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+3), & \frac{1}{2}(a+b+2) & \\ a+\frac{3}{2} & b+\frac{3}{2} & a+b+2\end{matrix};\ \frac{x^2}{4}\right]$$
$$+ \frac{\left(\frac{b}{f}-1\right)x}{2(b+1)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+3), & \frac{1}{2}(a+b+2) & \\ a+\frac{3}{2}, & b+\frac{3}{2} & a+b+2\end{matrix};\ \frac{x^2}{4}\right]$$
$$- \frac{\left(\frac{a}{d}-1\right)x^2}{4(a+1)(2a+3)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+4), & \frac{1}{2}(a+b+3) & \\ a+\frac{5}{2}, & b+\frac{3}{2} & a+b+3\end{matrix};\ \frac{x^2}{4}\right]$$
$$- \frac{\left(\frac{b}{f}-1\right)x^2}{4(b+1)(2b+3)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+4), & \frac{1}{2}(a+b+3) & \\ a+\frac{3}{2}, & b+\frac{5}{2} & a+b+3\end{matrix};\ \frac{x^2}{4}\right]$$

$$+\frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{f}-1\right)x^2}{4(a+1)(b+1)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+3), & \frac{1}{2}(a+b+2) & \\ & & ;\ \frac{x^2}{4}\\ a+\frac{3}{2} & b+\frac{3}{2} & a+b+2\end{matrix}\right]$$
$$-\frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{f}-1\right)x^3}{8(a+1)(b+1)(2a+3)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+4), & \frac{1}{2}(a+b+3) & \\ & & ;\ \frac{x^2}{4}\\ a+\frac{5}{2}, & b+\frac{3}{2} & a+b+3\end{matrix}\right]$$
$$-\frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{f}-1\right)x^3}{8(a+1)(b+1)(2b+3)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+4), & \frac{1}{2}(a+b+3) & \\ & & ;\ \frac{x^2}{4}\\ a+\frac{3}{2}, & b+\frac{5}{2} & a+b+3\end{matrix}\right]$$
$$+\frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{f}-1\right)x^4}{16(a+1)(b+1)(2a+3)(2b+3)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+5), & \frac{1}{2}(a+b+4) & \\ & & ;\ \frac{x^2}{4}\\ a+\frac{5}{2}, & b+\frac{5}{2} & a+b+4\end{matrix}\right].$$

Now, start with the left-hand side of (5.2).

$$L.H.S = e^{-x}\ {}_2F_2\left[\begin{matrix}a, & d+1 & \\ & & ;\ x\\ 2a+2 & d & \end{matrix}\right]\times\ {}_2F_2\left[\begin{matrix}b, & f+1 & \\ & & ;\ x\\ 2b+2 & f & \end{matrix}\right]$$
$$=\left\{e^{-\frac{x}{2}}\ {}_2F_2\left[\begin{matrix}a, & d+1 & \\ & & ;\ x\\ 2a+2 & d & \end{matrix}\right]\right\}\left\{e^{-\frac{x}{2}}\ {}_2F_2\left[\begin{matrix}b, & f+1 & \\ & & ;\ x\\ 2b+2 & f & \end{matrix}\right]\right\}.$$

Using (2.1) in both the expressions, we have

$$=\left\{{}_0F_1\left[\begin{matrix}- & \\ & ;\ \frac{x^2}{16}\\ a+\frac{3}{2} & \end{matrix}\right]+\frac{x\left(\frac{a}{d}-1\right)}{2(a+1)}\ {}_0F_1\left[\begin{matrix}- & \\ & ;\ \frac{x^2}{16}\\ a+\frac{3}{2} & \end{matrix}\right]\right.$$
$$\left.-\frac{x^2\left(\frac{a}{d}-1\right)}{4(a+1)(2a+3)}\ {}_0F_1\left[\begin{matrix}- & \\ & ;\ \frac{x^2}{16}\\ a+\frac{5}{2} & \end{matrix}\right]\right\}$$
$$\times\left\{{}_0F_1\left[\begin{matrix}- & \\ & ;\ \frac{x^2}{16}\\ b+\frac{3}{2} & \end{matrix}\right]+\frac{x\left(\frac{b}{f}-1\right)}{2(b+1)}\ {}_0F_1\left[\begin{matrix}- & \\ & ;\ \frac{x^2}{16}\\ b+\frac{3}{2} & \end{matrix}\right]\right.$$
$$\left.-\frac{x^2\left(\frac{b}{f}-1\right)}{4(b+1)(2b+3)}\ {}_0F_1\left[\begin{matrix}- & \\ & ;\ \frac{x^2}{16}\\ b+\frac{5}{2} & \end{matrix}\right]\right\}.$$

After simplification, we have

$$L.H.S = {}_0F_1\left[\begin{matrix}- & \\ & ;\ \frac{x^2}{16}\\ a+\frac{3}{2} & \end{matrix}\right]\times\ {}_0F_1\left[\begin{matrix}- & \\ & ;\ \frac{x^2}{16}\\ b+\frac{3}{2} & \end{matrix}\right]$$
$$+\frac{x\left(\frac{a}{d}-1\right)}{2(a+1)}\ {}_0F_1\left[\begin{matrix}- & \\ & ;\ \frac{x^2}{16}\\ a+\frac{3}{2} & \end{matrix}\right]\times\ {}_0F_1\left[\begin{matrix}- & \\ & ;\ \frac{x^2}{16}\\ b+\frac{3}{2} & \end{matrix}\right]$$

$$+\frac{x\left(\frac{b}{f}-1\right)}{2(b+1)}\,{}_0F_1\left[\begin{matrix}-\\ a+\frac{3}{2}\end{matrix};\frac{x^2}{16}\right]\times{}_0F_1\left[\begin{matrix}-\\ b+\frac{3}{2}\end{matrix};\frac{x^2}{16}\right]$$

$$-\frac{x^2\left(\frac{a}{d}-1\right)}{4(a+1)(2a+3)}\,{}_0F_1\left[\begin{matrix}-\\ a+\frac{5}{2}\end{matrix};\frac{x^2}{16}\right]\times{}_0F_1\left[\begin{matrix}-\\ b+\frac{3}{2}\end{matrix};\frac{x^2}{16}\right]$$

$$-\frac{x^2\left(\frac{b}{f}-1\right)}{4(b+1)(2b+3)}\,{}_0F_1\left[\begin{matrix}-\\ a+\frac{3}{2}\end{matrix};\frac{x^2}{16}\right]\times{}_0F_1\left[\begin{matrix}-\\ b+\frac{5}{2}\end{matrix};\frac{x^2}{16}\right]$$

$$+\frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{f}-1\right)x^2}{4(a+1)(b+1)}\,{}_0F_1\left[\begin{matrix}-\\ a+\frac{3}{2}\end{matrix};\frac{x^2}{16}\right]\times{}_0F_1\left[\begin{matrix}-\\ b+\frac{3}{2}\end{matrix};\frac{x^2}{16}\right]$$

$$-\frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{f}-1\right)x^3}{8(a+1)(b+1)(2a+3)}\,{}_0F_1\left[\begin{matrix}-\\ a+\frac{5}{2}\end{matrix};\frac{x^2}{16}\right]\times{}_0F_1\left[\begin{matrix}-\\ b+\frac{3}{2}\end{matrix};\frac{x^2}{16}\right]$$

$$-\frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{f}-1\right)x^3}{8(a+1)(b+1)(2b+3)}\,{}_0F_1\left[\begin{matrix}-\\ a+\frac{3}{2}\end{matrix};\frac{x^2}{16}\right]\times{}_0F_1\left[\begin{matrix}-\\ b+\frac{5}{2}\end{matrix};\frac{x^2}{16}\right]$$

$$+\frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{f}-1\right)x^4}{16(a+1)(b+1)(2a+3)(2b+3)}\,{}_0F_1\left[\begin{matrix}-\\ a+\frac{5}{2}\end{matrix};\frac{x^2}{16}\right]\times{}_0F_1\left[\begin{matrix}-\\ b+\frac{5}{2}\end{matrix};\frac{x^2}{16}\right].$$

Finally, applying the known result (1.28) due to Bailey in each of the nine expressions, we easily arrive at the right- hand side of (5.1). This completes the proof of (5.1). □

5.1. Special case

In this section, we shall mention some of the interesting known as well as new special cases of (5.1).

1. In (5.1), if we take $d = a$ and $f = b$, then we get at once Bailey's identity (1.26), by replacing $a \to a-1$, $b \to b-1$.
2. In (5.1), if we set $d = 2a+1$ and $f = 2b+1$, we get a known result recorded in Kim et al. [19].
3. In (5.1), if we set $d = a$ and $f = 2b+1$ or $d = 2a+1$ and $f = b$, we get a known result recorded in Kim et al. [19].
4. In (5.1), if we take $b = a$ and $f = d$, we get the following interesting result

$$\left\{e^{-\frac{x}{2}}\,{}_2F_2\left[\begin{matrix}a, & d+1\\ 2a+2, & d\end{matrix};x\right]\right\}^2 \tag{5.3}$$

$$={}_1F_2\left[\begin{matrix}a+1\\ a+\frac{3}{2},\quad 2a+2\end{matrix};\frac{x^2}{4}\right]$$

$$+\frac{\left(\frac{a}{d}-1\right)x}{(a+1)}\ {}_1F_2\left[\begin{matrix}a+2\\ \\ a+\frac{3}{2},\quad 2a+2\end{matrix}\ ;\frac{x^2}{4}\right]$$

$$-\frac{\left(\frac{a}{d}-1\right)x^2}{2(a+1)(2a+3)}\ {}_1F_2\left[\begin{matrix}a+2\\ \\ a+\frac{5}{2},\quad 2a+3\end{matrix}\ ;\frac{x^2}{4}\right]$$

$$+\frac{\left(\frac{a}{d}-1\right)^2x^2}{4(a+1)^2}\ {}_1F_2\left[\begin{matrix}a+1\\ \\ a+\frac{3}{2},\quad 2a+2\end{matrix}\ ;\frac{x^2}{4}\right]$$

$$-\frac{\left(\frac{a}{d}-1\right)^2x^3}{4(a+1)^2(2a+3)}\ {}_1F_2\left[\begin{matrix}a+2\\ \\ a+\frac{5}{2},\quad 2a+3\end{matrix}\ ;\frac{x^2}{4}\right]$$

$$+\frac{\left(\frac{a}{d}-1\right)^2x^4}{16(a+1)^2(2a+3)^2}\ {}_1F_2\left[\begin{matrix}a+2\\ \\ a+\frac{5}{2},\quad 2a+4\end{matrix}\ ;\frac{x^2}{4}\right].$$

5. In (5.3), if we take $d = a$, we recover Preece's identity (1.24), by replacing $a \to a-1$.
6. In (5.3), if we take $d = 2a+1$, we get a known result recorded due to Choi and Rathie [38]. Similarly, other results can be obtained.

6. Extension of Ramanujan's result

In this section, we shall establish the following interesting extension of the well known and useful Ramanujan's result (1.29).

6.1. Main result

Let $\phi(t)$ be analytic for $|t-1| < \mathbb{R}$ where $\mathbb{R} > 1$. Suppose that $a, d+1$ and $\phi(t)$ are such that the order of summation in

$$\sum_{k=0}^{\infty}\frac{2^k(a)_k(d+1)_k}{(2a+2)_k(d)_k k!}\sum_{n=k}^{\infty}\frac{(-1)^n(-n)_k}{n!}\phi^{(n)}(1)$$

may be inverted. Then

$$\sum_{k=0}^{\infty}\frac{2^k(a)_k(d+1)_k}{(2a+2)_k(d)_k k!}\phi^{(k)}(0)=\sum_{k=0}^{\infty}\frac{\phi^{(2k)}(1)}{2^{2k}\left(a+\frac{3}{2}\right)_k k!}\left\{1+\frac{2k\left(1-\frac{a}{d}\right)}{a+1}\right\}$$
$$+\frac{\left(\frac{a}{d}-1\right)}{a+1}\sum_{k=0}^{\infty}\frac{\phi^{(2k+1)}(1)}{2^{2k}\left(a+\frac{3}{2}\right)_k k!}. \tag{6.1}$$

Proof. Since $\phi(t)$ is analytic for $|t-1| < \mathbb{R}$ where $\mathbb{R} > 1$, we have

$$\phi^{(k)}(0)=\sum_{n=k}^{\infty}\frac{(-1)^n(-n)_k}{n!}\phi^{(n)}(1). \tag{6.2}$$

Now, denoting the left-hand side of (6.1) by S, we have

$$S=\sum_{k=0}^{\infty}\frac{2^k(a)_k(d+1)_k}{(2a+2)_k(d)_k k!}\phi^{(k)}(0).$$

Using (6.2), we have

$$S=\sum_{k=0}^{\infty}\frac{2^k(a)_k(d+1)_k}{(2a+2)_k(d)_k k!}\sum_{n=k}^{\infty}\frac{(-1)^n(-n)_k}{n!}\phi^{(n)}(1).$$

Upon inversion of summation, this becomes

$$S=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\phi^{(n)}(1)\sum_{k=0}^{n}\frac{2^k(-n)_k(a)_k(d+1)_k}{(2a+2)_k(d)_k k!}.$$

Summing up the inner series

$$S=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\phi^{(n)}(1)\ {}_3F_2\left[\begin{matrix}-n, & a, & d+1 \\ 2a+2, & d & \end{matrix}\ ;\ 2\right].$$

Separating into even and odd terms and using the identities

$$(2n)!=2^{2n}\left(\frac{1}{2}\right)_n n!$$

and

$$(2n+1)!=2^{2n}\left(\frac{3}{2}\right)_n n!$$

we have

$$\begin{aligned}S=&\sum_{n=0}^{\infty}\frac{\phi^{(2n)}(1)}{2^{2n}\left(\frac{1}{2}\right)_n n!}\ {}_3F_2\left[\begin{matrix}-2n, & a, & d+1 \\ 2a+2, & d & \end{matrix}\ ;\ 2\right]\\ &-\sum_{n=0}^{\infty}\frac{\phi^{(2n+1)}(1)}{2^{2n}\left(\frac{3}{2}\right)_n n!}\ {}_3F_2\left[\begin{matrix}-2n-1, & a, & d+1 \\ 2a+2, & d & \end{matrix}\ ;\ 2\right].\end{aligned}$$

Finally using the result (3.1) and (3.2), we easily arrive at the right-hand side of (6.1). This completes the proof of (6.1). □

6.1.1. *Special cases.* In this section, we shall consider some known special cases of our main result (6.1).

(1) In (6.1), if we take $d=a$, we get

$$\sum_{k=0}^{\infty}\frac{2^k(a+1)_k}{(2a+2)_k k!}\phi^{(k)}(0)=\sum_{k=0}^{\infty}\frac{\phi^{(2k)}(1)}{2^{2k}\left(a+\frac{3}{2}\right)_k k!}\tag{6.3}$$

which is Ramanujan's result (1.29) (by changing a to $a-1$).

(2) In (6.1), if we take $d = 2a + 1$, we get

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{2^k (a)_k}{(2a+1)_k k!} \phi^{(k)}(0) = \sum_{k=0}^{\infty} \frac{\phi^{(2k)}(1)}{2^{2k} \left(a + \frac{1}{2}\right)_k k!} \\ - \frac{1}{(2a+1)} \sum_{k=0}^{\infty} \frac{\phi^{(2k+1)}(1)}{2^{2k} \left(a + \frac{3}{2}\right)_k k!}\end{aligned} \tag{6.4}$$

a result obtained earlier by Kim et al. [21].

(3) In (6.1), if we take $d \to \infty$ and noting that $\frac{(d+1)_k}{(d)_k} \to 1$ as $d \to \infty$, we get

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{2^k (a)_k}{(2a+2)_k k!} \phi^{(k)}(0) = \sum_{k=0}^{\infty} \frac{\phi^{(2k)}(1) \left(\frac{a+3}{2}\right)_k}{2^{2k} \left(a + \frac{3}{2}\right)_k \left(\frac{a+1}{2}\right)_k k!} \\ - \frac{1}{(a+1)} \sum_{k=0}^{\infty} \frac{\phi^{(2k+1)}(1)}{2^{2k} \left(a + \frac{3}{2}\right)_k k!}\end{aligned} \tag{6.5}$$

a result obtained earlier by Kim et al. [21].

Similarly, other results can be obtained.

7. Reduction formulas of Kampé de Fériet functions

In this section, we shall establish the following two reduction formulae for Kampé de Fériet functions.

$$F_{1:0;2}^{1:0;2}\left[\begin{matrix} b: & -; & a, & d+1 \\ c: & -; & 2a+2, & d \end{matrix} ; -\tfrac{1}{2}x,\ x\right] \tag{7.1}$$

$$\begin{aligned}= {}_2F_3\left[\begin{matrix} \frac{b}{2}, & \frac{b+1}{2} & \\ a+\frac{3}{2}, & \frac{c}{2} & \frac{c+1}{2} \end{matrix} ; \frac{x^2}{16}\right] \\ + \frac{b\left(\frac{a}{d}-1\right)x}{2c(a+1)}\, {}_2F_3\left[\begin{matrix} \frac{b+1}{2}, & \frac{b}{2}+1 & \\ a+\frac{3}{2}, & \frac{c+1}{2} & \frac{c}{2}+1 \end{matrix} ; \frac{x^2}{16}\right] \\ + \frac{b(b+1)\left(1-\frac{a}{d}\right)x^2}{4c(c+1)(a+1)(2a+3)}\, {}_2F_3\left[\begin{matrix} \frac{b}{2}+1, & \frac{b+3}{2} & \\ a+\frac{5}{2}, & \frac{c}{2}+1 & \frac{c+3}{2} \end{matrix} ; \frac{x^2}{16}\right],\end{aligned}$$

$$F_{0:0;2}^{1:0;2}\left[\begin{matrix} b: & -; & a, & d+1 \\ -: & -; & 2a+2, & d \end{matrix} ; -\tfrac{1}{2}x,\ x\right] \tag{7.2}$$

$$= {}_2F_1\left[\begin{matrix} \frac{b}{2}, & \frac{b+1}{2} \\ a+\frac{3}{2} & \end{matrix} ; \frac{x^2}{4}\right]$$

$$+\frac{b\left(\frac{a}{d}-1\right)x}{2(a+1)}\ {}_2F_1\left[\begin{matrix}\frac{b+1}{2}, & \frac{b}{2}+1\\ & \\ a+\frac{3}{2} & \end{matrix};\ \frac{x^2}{4}\right]$$

$$+\frac{b(b+1)\left(1-\frac{a}{d}\right)x^2}{4(a+1)(2a+3)}\ {}_2F_1\left[\begin{matrix}\frac{b}{2}+1, & \frac{b+3}{2}\\ & \\ a+\frac{5}{2} & \end{matrix};\ \frac{x^2}{4}\right].$$

Proof. **Derivation of (7.1):** In order to establish the reduction formula (7.1), we proceed as follows. Replacing x by xt in (2.1), we have

$$e^{-\frac{1}{2}xt}\ {}_2F_2\left[\begin{matrix}a, & d+1\\ & \\ 2a+2 & d\end{matrix};\ xt\right] \tag{7.3}$$

$$={}_0F_1\left[\begin{matrix}-\\ \\ a+\frac{3}{2}\end{matrix};\ \frac{x^2t^t}{16}\right]+\frac{xt\left(\frac{a}{d}-1\right)}{2(a+1)}\ {}_0F_1\left[\begin{matrix}-\\ \\ a+\frac{3}{2}\end{matrix};\ \frac{x^2t^2}{16}\right]$$

$$+\ \frac{x^2t^2\left(1-\frac{a}{d}\right)}{4(a+1)(2a+3)}\ {}_0F_1\left[\begin{matrix}-\\ \\ a+\frac{5}{2}\end{matrix};\ \frac{x^2t^2}{16}\right].$$

Next, multiply both sides of (7.3) by $t^{b-1}(1-t)^{c-b-1}$ and integrating with respect to t in the interval (0,1), then for $Re(c) > Re(b) > 0$, we have

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}\ e^{-\frac{1}{2}xt}\ {}_2F_2\left[\begin{matrix}a, & d+1\\ & \\ 2a+2 & d\end{matrix};\ xt\right]dt \tag{7.4}$$

$$=\int_0^1 t^{b-1}(1-t)^{c-b-1}\ {}_0F_1\left[\begin{matrix}-\\ \\ a+\frac{3}{2}\end{matrix};\ \frac{x^2t^2}{16}\right]dt$$

$$+\frac{x\left(\frac{a}{d}-1\right)}{2(a+1)}\int_0^1 t^{b}(1-t)^{c-b-1}\ {}_0F_1\left[\begin{matrix}-\\ \\ a+\frac{3}{2}\end{matrix};\ \frac{x^2t^2}{16}\right]dt$$

$$+\frac{x^2\left(1-\frac{a}{d}\right)}{4(a+1)(2a+3)}\int_0^1 t^{b+1}(1-t)^{c-b-1}\ {}_0F_1\left[\begin{matrix}-\\ \\ a+\frac{5}{2}\end{matrix};\ \frac{x^2t^2}{16}\right]dt.$$

Now, L.H.S. of (7.4)

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}\ {}_0F_0\left[\begin{matrix}-\\ \\ -\end{matrix};\ -\tfrac{1}{2}xt\right]\ {}_2F_2\left[\begin{matrix}a, & d+1\\ & \\ 2a+2 & d\end{matrix};\ xt\right]dt.$$

Now expressing both the generalized hypergeometric functions as series, change the order of integration and series, which is easily seen to be justified due to the uniform convergence of the series involved in the process, we have L.H.S. of (7.4)

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)^m x^m}{m!}\frac{(a)_n(d+1)_n}{(2a+2)_n(d)_n}\frac{x^n}{n!}\int_0^1 t^{b+m+n-1}(1-t)^{c-b-1}dt.$$

Evaluating the beta-integral and using the identity $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, we have L.H.S. of (7.4)

$$= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)^m x^m x^n}{m!n!}\frac{(a)_n(d+1)_n(b)_{m+n}}{(2a+2)_n(d)_n(c)_{m+n}}.$$

Using the definition of Kampé de Fériet function (1.31), summing up the series, we finally have L.H.S. of (7.4)

$$(7.5)\qquad = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\, F^{1:0;2}_{1:0;2}\left[\begin{array}{lllll} b: & -; & a, & d+1 & \\ & & & & ;-\frac{1}{2}x,\ x \\ c: & -; & 2a+2, & d & \end{array}\right].$$

Similarly, for the right-hand side of (7.4), doing the same process and using the definition of generalized hypergeometric function, we finally have R.H.S of (7.4)

$$(7.6)\quad = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\Bigg\{ {}_2F_3\left[\begin{array}{llll} \frac{b}{2}, & \frac{b+1}{2} & & \\ & & & ;\ \frac{x^2}{16} \\ a+\frac{3}{2}, & \frac{c}{2}, & \frac{c+1}{2} & \end{array}\right]$$

$$+\frac{b\left(\frac{a}{d}-1\right)x}{2c(a+1)}\,{}_2F_3\left[\begin{array}{llll} \frac{b+1}{2}, & \frac{b}{2}+1 & & \\ & & & ;\ \frac{x^2}{16} \\ a+\frac{3}{2}, & \frac{c+1}{2}, & \frac{c}{2}+1 & \end{array}\right]$$

$$+\frac{b(b+1)\left(1-\frac{a}{d}\right)x^2}{4c(c+1)(a+1)(2a+3)}\,{}_2F_3\left[\begin{array}{llll} \frac{b}{2}, & \frac{b+3}{2} & & \\ & & & ;\ \frac{x^2}{16} \\ a+\frac{5}{2}, & \frac{c}{2}+1, & \frac{c+3}{2} & \end{array}\right]\Bigg\}.$$

Therefore, equating (7.5) and (7.6), we easily arrive at the desired result (7.1). This completes the proof of (7.1).

Derivation of (7.2): Multiply both sides of (7.3) by $t^{b-1}e^{-t}$ and integrating with respect to t in the interval $(0,\infty)$, then for $Re(b) > 0$, we have

$$(7.7)\qquad \int_0^{\infty} t^{b-1}e^{-t}\ e^{-\frac{1}{2}xt}\ {}_2F_2\left[\begin{array}{lll} a, & d+1 & \\ & & ;\ xt \\ 2a+2 & d & \end{array}\right]dt$$

$$= \int_0^{\infty} t^{b-1}e^{-t}\ {}_0F_1\left[\begin{array}{ll} - & \\ & ;\ \frac{x^2t^2}{16} \\ a+\frac{3}{2} & \end{array}\right]dt$$

$$+\frac{x\left(\frac{a}{d}-1\right)}{2(a+1)}\int_0^\infty t^{b-1}e^{-t}\ {}_0F_1\left[\begin{matrix}-\\ a+\frac{3}{2}\end{matrix};\frac{x^2t^2}{16}\right]dt$$

$$+\frac{x^2\left(1-\frac{a}{d}\right)}{4(a+1)(2a+3)}\int_0^\infty t^{b-1}e^{-t}\ {}_0F_1\left[\begin{matrix}-\\ a+\frac{5}{2}\end{matrix};\frac{x^2t^2}{16}\right]dt.$$

Now, L.H.S of (7.7)

$$=\int_0^\infty e^{-t}\ t^{b-1}\ {}_0F_0\left[\begin{matrix}-\\ -\end{matrix};-\tfrac{1}{2}xt\right]\ {}_2F_2\left[\begin{matrix}a, & d+1\\ 2a+2 & d\end{matrix};xt\right]dt.$$

Now expressing both the generalized hypergeometric functions as series, change the order of integration and summation, which is easily seen to be justified due to the uniform convergence of the series involved in the process, we have L.H.S. of (7.7)

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)^m x^m}{m!}\frac{(a)_n(d+1)_n}{(2a+2)_n(d)_n}\frac{x^n}{n!}\int_0^\infty t^{b+n+m-1}e^{-t}dt.$$

Evaluating the gamma-integral and making use the identity $(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}$, we have L.H.S. of (7.7)

$$=\Gamma(b)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)^m x^m x^n}{m!n!}\frac{(a)_n(d+1)_n\cdot(b)_{m+n}}{(2a+2)_n(d)_n}.$$

Using the definition of Kampé de Fériet function (1.31), summing up the series, we finally have L.H.S. of (7.7)

$$=\Gamma(b)\ F^{1:0;2}_{0:0;2}\left[\begin{matrix}b: & -; & a, & d+1\\ c: & -; & 2a+2, & d\end{matrix};-\tfrac{1}{2}x,\ x\right]. \tag{7.8}$$

Similarly, for the right-hand side of (7.7), doing the same procedure and using the definition of generalized hypergeometric functions, we finally have R.H.S. of (7.7)

$$=\Gamma(b)\left\{{}_2F_1\left[\begin{matrix}\frac{b}{2}, & \frac{b+1}{2}\\ a+\frac{3}{2}\end{matrix};\frac{x^2}{4}\right]\right. \tag{7.9}$$

$$+\frac{b\left(\frac{a}{d}-1\right)x}{2(a+1)}\ {}_2F_1\left[\begin{matrix}\frac{b+1}{2}, & \frac{b}{2}+1\\ a+\frac{3}{2}\end{matrix};\frac{x^2}{4}\right]$$

$$\left.+\frac{b(b+1)\left(1-\frac{a}{d}\right)x^2}{4(a+1)(2a+3)}\ {}_2F_1\left[\begin{matrix}\frac{b}{2}+1, & \frac{b+3}{2}\\ a+\frac{5}{2}\end{matrix};\frac{x^2}{4}\right]\right\}.$$

Therefore, equating (7.8) and (7.9), we easily arrive at the desired result (7.2). This completes the proof of (7.2). □

7.1. Special cases

Here we shall consider some special cases of our main results (7.1) and (7.2).

(1) In (7.1), if we set $d = a$, we get

$$F^{1:0;1}_{1:0;1}\left[\begin{array}{lll} b: & -; & a+1 \\ & & ;-\frac{1}{2}x,\ x \\ c: & -; & 2a+2, \end{array}\right] \tag{7.10}$$
$$= {}_2F_3\left[\begin{array}{lll} \frac{b}{2}, & \frac{b+1}{2} & \\ & & ;\ \frac{x^2}{16} \\ a+\frac{3}{2}, & \frac{c}{2} & \frac{c+1}{2} \end{array}\right].$$

(2) In (7.1), if we set $d = 2a + 1$, we get

$$F^{1:0;1}_{1:0;1}\left[\begin{array}{ll} b: & a \\ & ;-\frac{1}{2}x,\ x \\ c: & 2a+1, \end{array}\right] \tag{7.11}$$
$$= {}_2F_3\left[\begin{array}{lll} \frac{b}{2}, & \frac{b+1}{2} & \\ & & ;\ \frac{x^2}{16} \\ a+\frac{3}{2}, & \frac{c}{2} & \frac{c+1}{2} \end{array}\right]$$
$$- \frac{bx}{2c(2a+1)}\ {}_2F_3\left[\begin{array}{lll} \frac{b+1}{2}, & \frac{b}{2}+1 & \\ & & ;\ \frac{x^2}{16} \\ a+\frac{3}{2}, & \frac{c}{2}+\frac{1}{2} & \frac{c}{2}+1 \end{array}\right]$$
$$+ \frac{b(b+1)x^2}{4c(c+1)(2a+1)(2a+3)}\ {}_2F_3\left[\begin{array}{lll} \frac{b}{2}+1, & \frac{b+3}{2} & \\ & & ;\ \frac{x^2}{16} \\ a+\frac{5}{2}, & \frac{c}{2}+1 & \frac{c+3}{2} \end{array}\right].$$

(3) In (7.2), if we set $d = a$, we get

(7.12)
$$F^{1:0;1}_{0:0;1}\left[\begin{array}{lll} b: & -; & a+1 \\ & & ;-\frac{1}{2}x,\ x \\ -: & -; & 2a+2 \end{array}\right] = {}_2F_1\left[\begin{array}{ll} \frac{b}{2}, & \frac{b+1}{2} \\ & ;\ \frac{x^2}{4} \\ a+\frac{3}{2} & \end{array}\right].$$

(3) In (7.2), if we take $d = 2a + 1$, we get

$$F^{1:0;1}_{0:0;1}\left[\begin{array}{lll} b: & -; & a \\ & & ;-\frac{1}{2}x,\ x \\ -: & -; & 2a+1 \end{array}\right] \tag{7.13}$$
$$= {}_2F_1\left[\begin{array}{ll} \frac{b}{2}, & \frac{b+1}{2} \\ & ;\ \frac{x^2}{4} \\ a+\frac{3}{2} & \end{array}\right]$$

$$- \frac{bx}{2(2a+1)}\, {}_2F_1\left[\begin{array}{c} \frac{b+1}{2}, \quad \frac{b}{2}+1 \\ a+\frac{3}{2} \end{array}; \frac{x^2}{4}\right]$$
$$+ \frac{b(b+1)x^2}{4(2a+1)(2a+3)}\, {}_2F_1\left[\begin{array}{c} \frac{b}{2}+1, \quad \frac{b+3}{2} \\ a+\frac{5}{2} \end{array}; \frac{x^2}{4}\right].$$

Similarly, other result can be obtained.

We conclude this section by remarking that the derivations of the results presented in Sections 7, 8 and 9 together with their special cases are of similar type that given in this section. So, they are given in this paper without proof and are left as an exercise to the interested readers.

8. Formulas for the Kampé de Fériet functions expressed in terms of Srivastava-Daoust S-function of two variables

In this section, we shall establish the following two formulas for the Kampé de Fériet functions expressed in terms of Srivastava-Daoust S-function of two variables.

(8.1)
$$F^{1:2;2}_{1:2;2}\left[\begin{array}{ccccc} f: & a, & d+1; & b, & e+1 \\ g: & 2a+2, & d; & 2b+2, & e \end{array}; x,\ x\right]$$
$$= S^{1:0;2}_{1:0;3}\left[\begin{array}{llll} [f:1,2]: & -; & \left[\frac{1}{2}(a+b+3),1\right], & \left[\frac{1}{2}(a+b+2),1\right] \\ [g:1,2]: & -; & \left[a+\frac{3}{2},1\right], & \left[b+\frac{3}{2},1\right], \quad [a+b+2,1] \end{array}; \begin{array}{l} x \\ \frac{x^2}{4} \end{array}\right]$$
$$+ \frac{\left(\frac{a}{d}-1\right) fx}{2g(a+1)}$$
$$\times S^{1:0;2}_{1:0;3}\left[\begin{array}{llll} [f+1:1,2]: & -; & \left[\frac{1}{2}(a+b+3),1\right], & \left[\frac{1}{2}(a+b+2),1\right] \\ [g+1:1,2]: & -; & \left[a+\frac{3}{2},1\right], & \left[b+\frac{3}{2},1\right], \quad [a+b+2,1] \end{array}; \begin{array}{l} x \\ \frac{x^2}{4} \end{array}\right]$$
$$+ \frac{\left(\frac{b}{e}-1\right) fx}{2g(b+1)}$$
$$\times S^{1:0;2}_{1:0;3}\left[\begin{array}{llll} [f+1:1,2]: & -; & \left[\frac{1}{2}(a+b+3),1\right], & \left[\frac{1}{2}(a+b+2),1\right] \\ [g+1:1,2]: & -; & \left[a+\frac{3}{2},1\right], & \left[b+\frac{3}{2},1\right], \quad [a+b+2,1] \end{array}; \begin{array}{l} x \\ \frac{x^2}{4} \end{array}\right]$$
$$- \frac{\left(\frac{a}{d}-1\right) f(f+1)x^2}{4g(g+1)(a+1)(2a+3)}$$
$$\times S^{1:0;2}_{1:0;3}\left[\begin{array}{llll} [f+2:1,2]: & -; & \left[\frac{1}{2}(a+b+4),1\right], & \left[\frac{1}{2}(a+b+3),1\right] \\ [g+2:1,2]: & -; & \left[a+\frac{5}{2},1\right], & \left[b+\frac{3}{2},1\right], \quad [a+b+3,1] \end{array}; \begin{array}{l} x \\ \frac{x^2}{4} \end{array}\right]$$
$$- \frac{\left(\frac{b}{e}-1\right) f(f+1)x^2}{4g(g+1)(b+1)(2b+3)}$$
$$\times S^{1:0;2}_{1:0;3}\left[\begin{array}{llll} [f+2:1,2]: & -; & \left[\frac{1}{2}(a+b+4),1\right], & \left[\frac{1}{2}(a+b+3),1\right] \\ [g+2:1,2]: & -; & \left[a+\frac{3}{2},1\right], & \left[b+\frac{5}{2},1\right], \quad [a+b+3,1] \end{array}; \begin{array}{l} x \\ \frac{x^2}{4} \end{array}\right]$$

$$+\frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{e}-1\right)f(f+1)x^2}{4g(g+1)(a+1)(b+1)}$$
$$\times\ S^{1:0;2}_{1:0;3}\left[\begin{array}{llllll}[f+2:1,2]: & -; & \left[\frac{1}{2}(a+b+3),1\right], & \left[\frac{1}{2}(a+b+2),1\right] & & ;x \\ [g+2:1,2]: & -; & \left[a+\frac{3}{2},1\right], & \left[b+\frac{3}{2},1\right], & [a+b+2,1] & ;\frac{x^2}{4}\end{array}\right]$$
$$-\frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{e}-1\right)f(f+1)(f+2)x^3}{8g(g+1)(a+1)(b+1)(2a+3)}$$
$$\times\ S^{1:0;2}_{1:0;3}\left[\begin{array}{llllll}[f+3:1,2]: & -; & \left[\frac{1}{2}(a+b+4),1\right], & \left[\frac{1}{2}(a+b+3),1\right] & & ;x \\ [g+3:1,2]: & -; & \left[a+\frac{5}{2},1\right], & \left[b+\frac{3}{2},1\right], & [a+b+3,1] & ;\frac{x^2}{4}\end{array}\right]$$
$$-\frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{e}-1\right)f(f+1)(f+2)x^3}{8g(g+1)(a+1)(b+1)(2b+3)}$$
$$\times\ S^{1:0;2}_{1:0;3}\left[\begin{array}{llllll}[f+3:1,2]: & -; & \left[\frac{1}{2}(a+b+4),1\right], & \left[\frac{1}{2}(a+b+3),1\right] & & ;x \\ [g+3:1,2]: & -; & \left[a+\frac{3}{2},1\right], & \left[b+\frac{5}{2},1\right], & [a+b+3,1] & ;\frac{x^2}{4}\end{array}\right]$$
$$+\frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{e}-1\right)f(f+1)(f+2)(f+3)x^4}{16g(g+1)(g+2)(g+3)(a+1)(b+1)(2a+3)(2b+3)}$$
$$\times\ S^{1:0;2}_{1:0;3}\left[\begin{array}{llllll}[f+4:1,2]: & -; & \left[\frac{1}{2}(a+b+5),1\right], & \left[\frac{1}{2}(a+b+4),1\right] & & ;x \\ [g+4:1,2]: & -; & \left[a+\frac{5}{2},1\right], & \left[b+\frac{5}{2},1\right], & [a+b+4,1] & ;\frac{x^2}{4}\end{array}\right],$$

(8.2)
$$F^{1:2;2}_{0:2;2}\left[\begin{array}{llllll} c: & a, & d+1; & b, & e+1 & \\ & & & & & ;x,\ x \\ -: & 2a+2, & d; & 2b+2, & e & \end{array}\right]$$
$$=S^{1:0;2}_{0:0;3}\left[\begin{array}{llllll}[c:1,2]: & -; & \left[\frac{1}{2}(a+b+3),1\right], & \left[\frac{1}{2}(a+b+2),1\right] & & ;x \\ -: & -; & \left[a+\frac{3}{2},1\right], & \left[b+\frac{3}{2},1\right], & [a+b+2,1] & ;\frac{x^2}{4}\end{array}\right]$$
$$+\frac{\left(\frac{a}{d}-1\right)cx}{2(a+1)}$$
$$S^{1:0;2}_{0:0;3}\left[\begin{array}{llllll}[c+1:1,2]: & -; & \left[\frac{1}{2}(a+b+3),1\right], & \left[\frac{1}{2}(a+b+2),1\right] & & ;x \\ -: & -; & \left[a+\frac{3}{2},1\right], & \left[b+\frac{3}{2},1\right], & [a+b+2,1] & ;\frac{x^2}{4}\end{array}\right]$$
$$+\frac{\left(\frac{b}{e}-1\right)cx}{2(b+1)}$$
$$S^{1:0;2}_{0:0;3}\left[\begin{array}{llllll}[c+1:1,2]: & -; & \left[\frac{1}{2}(a+b+3),1\right], & \left[\frac{1}{2}(a+b+2),1\right] & & ;x \\ -: & -; & \left[a+\frac{3}{2},1\right], & \left[b+\frac{3}{2},1\right], & [a+b+2,1] & ;\frac{x^2}{4}\end{array}\right]$$
$$-\frac{\left(\frac{a}{d}-1\right)c(c+1)x^2}{4(a+1)(2a+3)}$$
$$\times\ S^{1:0;2}_{0:0;3}\left[\begin{array}{llllll}[c+2:1,2]: & -; & \left[\frac{1}{2}(a+b+4),1\right], & \left[\frac{1}{2}(a+b+3),1\right] & & ;x \\ -: & -; & \left[a+\frac{5}{2},1\right], & \left[b+\frac{3}{2},1\right], & [a+b+3,1] & ;\frac{x^2}{4}\end{array}\right]$$
$$-\frac{\left(\frac{b}{e}-1\right)c(c+1)x^2}{4(b+1)(2b+3)}$$
$$\times\ S^{1:0;2}_{0:0;3}\left[\begin{array}{llllll}[c+2:1,2]: & -; & \left[\frac{1}{2}(a+b+4),1\right], & \left[\frac{1}{2}(a+b+3),1\right] & & ;x \\ -: & -; & \left[a+\frac{3}{2},1\right], & \left[b+\frac{5}{2},1\right], & [a+b+3,1] & ;\frac{x^2}{4}\end{array}\right]$$

$$+\frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{e}-1\right)c(c+1)x^2}{4(a+1)(b+1)}$$
$$\times\ S^{1:0;2}_{0:0;3}\left[\begin{array}{lllll} [c+2:1,2]: & -; & \left[\frac{1}{2}(a+b+4),1\right], & \left[\frac{1}{2}(a+b+3),1\right] & ;x \\ -: & -; & \left[a+\frac{5}{2},1\right], & \left[b+\frac{3}{2},1\right], \quad [a+b+2,1] & ;\frac{x^2}{4}\end{array}\right]$$
$$-\frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{e}-1\right)c(c+1)(c+2)x^3}{8(a+1)(b+1)(2a+3)}$$
$$\times\ S^{1:0;2}_{0:0;3}\left[\begin{array}{lllll} [c+3:1,2]: & -; & \left[\frac{1}{2}(a+b+4),1\right], & \left[\frac{1}{2}(a+b+3),1\right] & ;x \\ -: & -; & \left[a+\frac{5}{2},1\right], & \left[b+\frac{3}{2},1\right], \quad [a+b+3,1] & ;\frac{x^2}{4}\end{array}\right]$$
$$-\frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{e}-1\right)c(c+1)(c+2)x^3}{8(a+1)(b+1)(2b+3)}$$
$$\times\ S^{1:0;2}_{0:0;3}\left[\begin{array}{lllll} [c+3:1,2]: & -; & \left[\frac{1}{2}(a+b+4),1\right], & \left[\frac{1}{2}(a+b+3),1\right] & ;x \\ -: & -; & \left[a+\frac{3}{2},1\right], & \left[b+\frac{5}{2},1\right], \quad [a+b+3,1] & ;\frac{x^2}{4}\end{array}\right]$$
$$+\frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{e}-1\right)c(c+1)(c+2)(c+3)x^4}{16(a+1)(b+1)(2a+3)(2b+3)}$$
$$\times\ S^{1:0;2}_{0:0;3}\left[\begin{array}{lllll} [c+4:1,2]: & -; & \left[\frac{1}{2}(a+b+5),1\right], & \left[\frac{1}{2}(a+b+4),1\right] & ;x \\ -: & -; & \left[a+\frac{5}{2},1\right], & \left[b+\frac{5}{2},1\right], \quad [a+b+4,1] & ;\frac{x^2}{4}\end{array}\right].$$

8.1. Special cases

In this section, we shall mention special cases of our results (8.1) and (8.2).

(1) In (8.1), if we take $d=a$ and $e=b$, we get the following result

$$F^{1:1;1}_{1:1;1}\left[\begin{array}{llll} f: & a+1; & b+1, & \\ & & & ;x,\ x \\ g: & 2a+2; & 2b+2, & \end{array}\right] \tag{8.3}$$
$$=S^{1:0;2}_{1:0;3}\left[\begin{array}{lllll} [f:1,2]: & -; & \left[\frac{1}{2}(a+b+3),1\right], & \left[\frac{1}{2}(a+b+2),1\right] & ;x \\ [g:1,2]: & -; & \left[a+\frac{3}{2},1\right], & \left[b+\frac{3}{2},1\right], \quad [a+b+2,1] & \frac{x^2}{4}\end{array}\right].$$

(2) In (8.1), if we take $d=a$, we get the following result

$$F^{1:1;2}_{1:1;2}\left[\begin{array}{llllll} f: & a+1; & b, & e+1 & & \\ & & & & & ;x,\ x \\ g: & 2a+2, & d; & 2b+2, & e & \end{array}\right] \tag{8.4}$$
$$=S^{1:0;2}_{1:0;3}\left[\begin{array}{lllll} [f:1,2]: & -; & \left[\frac{1}{2}(a+b+3),1\right], & \left[\frac{1}{2}(a+b+2),1\right] & ;x \\ [g:1,2]: & -; & \left[a+\frac{3}{2},1\right], & \left[b+\frac{3}{2},1\right], \quad [a+b+2,1] & ;\frac{x^2}{4}\end{array}\right]$$
$$+\frac{\left(\frac{b}{e}-1\right)fx}{2g(b+1)}$$
$$\times\ S^{1:0;2}_{1:0;3}\left[\begin{array}{lllll} [f+1:1,2]: & -; & \left[\frac{1}{2}(a+b+3),1\right], & \left[\frac{1}{2}(a+b+2),1\right] & ;x \\ [g+1:1,2]: & -; & \left[a+\frac{3}{2},1\right], & \left[b+\frac{3}{2},1\right], \quad [a+b+2,1] & ;\frac{x^2}{4}\end{array}\right]$$
$$-\frac{\left(\frac{b}{e}-1\right)f(f+1)x^2}{4g(g+1)(b+1)(2b+3)}$$
$$\times\ S^{1:0;2}_{1:0;3}\left[\begin{array}{lllll} [f+2:1,2]: & -; & \left[\frac{1}{2}(a+b+4),1\right], & \left[\frac{1}{2}(a+b+3),1\right] & ;x \\ [g+2:1,2]: & -; & \left[a+\frac{3}{2},1\right], & \left[b+\frac{5}{2},1\right], \quad [a+b+3,1] & ;\frac{x^2}{4}\end{array}\right].$$

(2) In (8.2), if we take $d = a$, we get the following result

(8.5)
$$F^{1:1;1}_{0:1;1}\left[\begin{array}{lll} c: & a+1; & b+1 \\ & & ;x,\ x \\ -: & 2a+2; & 2b+2 \end{array}\right]$$
$$= S^{1:0;2}_{0:0;3}\left[\begin{array}{llllll} [c:1,2]: & -; & \left[\frac{1}{2}(a+b+3),1\right], & \left[\frac{1}{2}(a+b+2),1\right] & & ;x \\ -: & -; & \left[a+\frac{3}{2},1\right], & \left[b+\frac{3}{2},1\right], & [a+b+2,1] & ;\frac{x^2}{4} \end{array}\right].$$

(3) In (8.2), if we take $d = a$, we get the following result

(8.6)
$$F^{1:1;2}_{0:1;2}\left[\begin{array}{llll} c: & a+1; & b, & e+1 \\ & & & ;x,\ x \\ -: & 2a+2; & 2b+2, & e \end{array}\right]$$
$$= S^{1:0;2}_{0:0;3}\left[\begin{array}{llllll} [c:1,2]: & -; & \left[\frac{1}{2}(a+b+3),1\right], & \left[\frac{1}{2}(a+b+2),1\right] & & ;x \\ -: & -; & \left[a+\frac{3}{2},1\right], & \left[b+\frac{3}{2},1\right], & [a+b+2,1] & ;\frac{x^2}{4} \end{array}\right]$$
$$+ \frac{\left(\frac{b}{e}-1\right)cx}{2(b+1)}$$
$$\times\ S^{1:0;2}_{0:0;3}\left[\begin{array}{llllll} [c+1:1,2]: & -; & \left[\frac{1}{2}(a+b+3),1\right], & \left[\frac{1}{2}(a+b+2),1\right] & & ;x \\ -: & -; & \left[a+\frac{3}{2},1\right], & \left[b+\frac{3}{2},1\right], & [a+b+2,1] & ;\frac{x^2}{4} \end{array}\right]$$
$$- \frac{\left(\frac{b}{e}-1\right)c(c+1)x^2}{4(b+1)(2b+3)}$$
$$\times\ S^{1:0;2}_{0:0;3}\left[\begin{array}{llllll} [c+2:1,2]: & -; & \left[\frac{a+b+4}{2},1\right], & \left[\frac{a+b+3}{2},1\right] & & ;x \\ -: & -; & \left[a+\frac{3}{2},1\right], & \left[b+\frac{5}{2},1\right], & [a+b+3,1] & ;\frac{x^2}{4} \end{array}\right].$$

Similarly, other results can be obtained.

9. Formulas for the generalized hypergeometric functions expressed in terms of Srivastava-Daoust S-function of two variables

In this section, we shall establish the following two formulas for the generalized hypergeometric functions expressed in terms of Srivastava-Daoust S-function of two variables.

(9.1)
$${}_3F_3\left[\begin{array}{llll} a, & b, & d+1 & \\ & & & ;x \\ c & 2a+2, & d & \end{array}\right]$$
$$= S^{1:0;0}_{1:0;1}\left[\begin{array}{llll} [b:1,2]: & -; & - & ;\frac{1}{2}x \\ [c:1,2]: & -; & \left[a+\frac{3}{2},1\right] & ;\frac{x^2}{16} \end{array}\right]$$
$$+ \frac{\left(\frac{a}{d}-1\right)bx}{2c(a+1)}\ S^{1:0;0}_{1:0;1}\left[\begin{array}{llll} [b+1:1,2]: & -; & - & ;\frac{1}{2}x \\ [c+1:1,2]: & -; & \left[a+\frac{3}{2},1\right] & ;\frac{x^2}{16} \end{array}\right]$$
$$+ \frac{\left(1-\frac{a}{d}\right)b(b+1)x^2}{4c(c+1)(a+1)(2a+3)}\ S^{1:0;0}_{1:0;1}\left[\begin{array}{llll} [b+2:1,2]: & -; & - & ;\frac{1}{2}x \\ [c+2:1,2]: & -; & \left[a+\frac{5}{2},1\right] & ;\frac{x^2}{16} \end{array}\right],$$

(9.2)

$$
{}_3F_2\left[\begin{matrix} a, & b, & d+1 \\ & & & ;x \\ 2a+2, & d & \end{matrix}\right]
$$

$$
= S^{1:0;0}_{0:0;1}\left[\begin{matrix} [b:1,2]: & -; & - & ;\frac{1}{2}x \\ -: & -; & \left[a+\frac{3}{2},1\right] & ;\frac{x^2}{16} \end{matrix}\right]
$$

$$
+\frac{\left(\frac{a}{d}-1\right)bx}{2(a+1)}\,S^{1:0;0}_{0:0;1}\left[\begin{matrix} [b+1:1,2]: & -; & - & ;\frac{1}{2}x \\ -: & -; & \left[a+\frac{3}{2},1\right] & ;\frac{x^2}{16} \end{matrix}\right]
$$

$$
+\frac{\left(1-\frac{a}{d}\right)b(b+1)x^2}{4(a+1)(2a+3)}\,S^{1:0;0}_{0:0;1}\left[\begin{matrix} [b+2:1,2]: & -; & - & ;\frac{1}{2}x \\ -: & -; & \left[a+\frac{5}{2},1\right] & ;\frac{x^2}{16} \end{matrix}\right].
$$

9.1. Special cases

In this section, we shall mention some interesting special cases of our results (9.1) and (9.2).

(1) In (9.1), if we take $d = a$, we get

(9.3)

$$
{}_2F_2\left[\begin{matrix} b, & a+1 \\ & & ;x \\ c, & 2a+2 \end{matrix}\right] = S^{1:0;0}_{1:0;1}\left[\begin{matrix} [b:1,2]: & -; & - & ;\frac{1}{2}x \\ [c:1,2]: & -; & \left[a+\frac{3}{2},1\right] & ;\frac{x^2}{16} \end{matrix}\right].
$$

(2) In (9.1), if we take $d = 2a+1$, we get

(9.4)

$$
{}_2F_2\left[\begin{matrix} b, & a \\ & & ;x \\ c, & 2a+1 \end{matrix}\right]
$$

$$
= S^{1:0;0}_{1:0;1}\left[\begin{matrix} [b:1,2]: & -; & - & ;\frac{1}{2}x \\ [c:1,2]: & -; & \left[a+\frac{3}{2},1\right] & ;\frac{x^2}{16} \end{matrix}\right]
$$

$$
-\frac{bx}{2c(2a+1)}\,S^{1:0;0}_{1:0;1}\left[\begin{matrix} [b+1:1,2]: & -; & - & ;\frac{1}{2}x \\ [c+1:1,2]: & -; & \left[a+\frac{5}{2},1\right] & ;\frac{x^2}{16} \end{matrix}\right]
$$

$$
+\frac{b(b+1)x^2}{4c(c+1)(2a+1)(2a+3)}\,S^{1:0;0}_{1:0;1}\left[\begin{matrix} [b+2:1,2]: & -; & - & ;\frac{1}{2}x \\ [c+2:1,2]: & -; & \left[a+\frac{5}{2},1\right] & ;\frac{x^2}{16} \end{matrix}\right].
$$

(3) In (9.2), if we take $d = a$, we get

(9.5)

$$
{}_2F_1\left[\begin{matrix} b, & a+1 \\ & & ;x \\ 2a+2 & \end{matrix}\right] = S^{1:0;0}_{0:0;1}\left[\begin{matrix} [b:1,2]: & -; & - & ;\frac{1}{2}x \\ -: & -; & \left[a+\frac{3}{2},1\right] & ;\frac{x^2}{16} \end{matrix}\right].
$$

(4) In (9.2), if we take $d = 2a + 1$, we get

(9.6)

$$
{}_2F_1\left[\begin{matrix} b, & a+1 \\ 2a+2 & \end{matrix} ; x\right]
= S^{1:0;0}_{0:0;1}\left[\begin{matrix} [b:1,2]: & -; & - & ; \frac{1}{2}x \\ -: & -; & [a+\frac{3}{2},1] & ; \frac{x^2}{16} \end{matrix}\right]
$$
$$
- \frac{bx}{2(2a+1)} S^{1:0;0}_{0:0;1}\left[\begin{matrix} [b+1:1,2]: & -; & - & ; \frac{1}{2}x \\ -: & -; & [a+\frac{3}{2},1] & ; \frac{x^2}{16} \end{matrix}\right]
$$
$$
+ \frac{b(b+1)x^2}{4(2a+1)(2a+3)} S^{1:0;0}_{0:0;1}\left[\begin{matrix} [b+2:1,2]: & -; & - & ; \frac{1}{2}x \\ -: & -; & [a+\frac{5}{2},1] & ; \frac{x^2}{16} \end{matrix}\right].
$$

Similarly, other results can be obtained.

10. Formulas for the Srivastava's triple series expressed in terms of generalized hypergeometric functions

In this section, we shall establish the following two formulas for the Srivastava's triple series expressed in terms of hypergeometric function.

(10.1)

$$
F^{(3)}\left[\begin{matrix} f:: & -; & -; & -; & -; & a, & d+1; & b, & e+1 \\ g:: & -; & -; & -; & -; & 2a+2, & d; & 2b+2, & e \end{matrix} ; -x,\ x,\ x\right]
$$
$$
= {}_4F_5\left[\begin{matrix} \frac{a+b+3}{2}, & \frac{a+b+2}{2}, & \frac{f}{2}, & \frac{f+1}{2} & \\ a+\frac{3}{2}, & b+\frac{3}{2}, & a+b+2, & \frac{g}{2}, & \frac{g+1}{2} \end{matrix} ; \frac{x^2}{4}\right]
$$
$$
+ \frac{\left(\frac{a}{d}-1\right) fx}{2g(a+1)}\, {}_4F_5\left[\begin{matrix} \frac{a+b+3}{2}, & \frac{a+b+2}{2}, & \frac{f}{2}, & \frac{f+1}{2} & \\ a+\frac{3}{2}, & b+\frac{3}{2}, & a+b+2, & \frac{g}{2}, & \frac{g+1}{2} \end{matrix} ; \frac{x^2}{4}\right]
$$
$$
+ \frac{\left(\frac{b}{e}-1\right) fx}{2g(b+1)}\, {}_4F_5\left[\begin{matrix} \frac{a+b+3}{2}, & \frac{a+b+2}{2}, & \frac{f+1}{2}, & \frac{f}{2}+1 & \\ a+\frac{3}{2}, & b+\frac{3}{2}, & a+b+2, & \frac{g+1}{2}, & \frac{g}{2}+1 \end{matrix} ; \frac{x^2}{4}\right]
$$
$$
- \frac{\left(\frac{a}{d}-1\right) f(f+1)x^2}{4g(g+1)(a+1)(2a+3)}\, {}_4F_5\left[\begin{matrix} \frac{a+b+4}{2}, & \frac{a+b+3}{2}, & \frac{f}{2}+1, & \frac{f+3}{2} & \\ a+\frac{5}{2}, & b+\frac{3}{2}, & a+b+3, & \frac{g}{2}+1, & \frac{g+3}{2} \end{matrix} ; \frac{x^2}{4}\right]
$$
$$
- \frac{\left(\frac{b}{e}-1\right) f(f+1)x^2}{4g(g+1)(b+1)(2b+3)}\, {}_4F_5\left[\begin{matrix} \frac{a+b+4}{2}, & \frac{a+b+3}{2}, & \frac{f}{2}+1, & \frac{f+3}{2} & \\ a+\frac{3}{2}, & b+\frac{5}{2}, & a+b+3, & \frac{g}{2}+1, & \frac{g+3}{2} \end{matrix} ; \frac{x^2}{4}\right]
$$
$$
+ \frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{e}-1\right) f(f+1)x^2}{4g(g+1)(a+1)(b+1)}
$$
$$
\times\ {}_4F_5\left[\begin{matrix} \frac{a+b+3}{2}, & \frac{a+b+2}{2}, & \frac{f}{2}+1, & \frac{f+3}{2} & \\ a+\frac{3}{2}, & b+\frac{3}{2}, & a+b+2, & \frac{g}{2}+1, & \frac{g+3}{2} \end{matrix} ; \frac{x^2}{4}\right]
$$

$$
\begin{aligned}
&-\frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{e}-1\right)f(f+1)(f+2)x^3}{8g(g+1)(g+2)(a+1)(b+1)(2a+3)} \\
&\times\ {}_4F_5\left[\begin{matrix} \frac{a+b+4}{2}, & \frac{a+b+3}{2}, & \frac{f+3}{2}, & \frac{f}{2}+2 & \\ a+\frac{5}{2}, & b+\frac{3}{2}, & a+b+3, & \frac{g+3}{2}, & \frac{g}{2}+2 \end{matrix};\frac{x^2}{4}\right] \\
&-\frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{e}-1\right)f(f+1)(f+2)x^3}{8g(g+1)(g+2)(a+1)(b+1)(2b+3)} \\
&\times\ {}_4F_5\left[\begin{matrix} \frac{a+b+4}{2}, & \frac{a+b+3}{2}, & \frac{f+3}{2}, & \frac{f}{2}+2 & \\ a+\frac{3}{2}, & b+\frac{5}{2}, & a+b+3, & \frac{g+3}{2}, & \frac{g}{2}+2 \end{matrix};\frac{x^2}{4}\right] \\
&+\frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{e}-1\right)f(f+1)(f+2)(f+3)x^4}{16g(g+1)(g+2)(g+3)(a+1)(b+1)(2a+3)(2b+3)} \\
&\times\ {}_4F_5\left[\begin{matrix} \frac{a+b+5}{2}, & \frac{a+b+4}{2}, & \frac{f}{2}+2, & \frac{f+5}{2} & \\ a+\frac{5}{2}, & b+\frac{5}{2}, & a+b+4, & \frac{g}{2}+2, & \frac{g+5}{2} \end{matrix};\frac{x^2}{4}\right],
\end{aligned}
$$

$$
\begin{aligned}
(10.2)\quad & F^{(3)}\left[\begin{matrix} c:: & -; & -; & -; & -; & a, & d+1; & b, & e+1 \\ -:: & -; & -; & -; & -; & 2a+2, & d; & 2b+2, & e \end{matrix};-x,\ x,\ x\right] \\
&= {}_4F_3\left[\begin{matrix} \frac{a+b+3}{2}, & \frac{a+b+2}{2}, & \frac{c}{2}, & \frac{c+1}{2} \\ a+\frac{3}{2}, & b+\frac{3}{2}, & a+b+2 & \end{matrix};x^2\right] \\
&\quad+\frac{\left(\frac{a}{d}-1\right)cx}{2(a+1)}\ {}_4F_3\left[\begin{matrix} \frac{a+b+3}{2}, & \frac{a+b+2}{2}, & \frac{c+1}{2}, & \frac{c}{2}+1 \\ a+\frac{3}{2}, & b+\frac{3}{2}, & a+b+2 & \end{matrix};x^2\right] \\
&\quad+\frac{\left(\frac{b}{e}-1\right)cx}{2(b+1)}\ {}_4F_3\left[\begin{matrix} \frac{a+b+3}{2}, & \frac{a+b+2}{2}, & \frac{c+1}{2}, & \frac{c}{2}+1 \\ a+\frac{3}{2}, & b+\frac{3}{2}, & a+b+2 & \end{matrix};x^2\right] \\
&\quad-\frac{\left(\frac{a}{d}-1\right)c(c+1)x^2}{4(a+1)(2a+3)}\ {}_4F_3\left[\begin{matrix} \frac{a+b+4}{2}, & \frac{a+b+3}{2}, & \frac{c}{2}+1, & \frac{c+3}{2} \\ a+\frac{5}{2}, & b+\frac{3}{2}, & a+b+3 & \end{matrix};x^2\right] \\
&\quad-\frac{\left(\frac{b}{e}-1\right)c(c+1)x^2}{4(b+1)(2b+3)}\ {}_4F_3\left[\begin{matrix} \frac{a+b+4}{2}, & \frac{a+b+3}{2}, & \frac{c}{2}+1, & \frac{c+3}{2} \\ a+\frac{3}{2}, & b+\frac{5}{2}, & a+b+3 & \end{matrix};x^2\right] \\
&\quad+\frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{e}-1\right)c(c+1)x^2}{4(a+1)(b+1)}\ {}_4F_3\left[\begin{matrix} \frac{a+b+3}{2}, & \frac{a+b+2}{2}, & \frac{c}{2}+1, & \frac{c+3}{2} \\ a+\frac{3}{2}, & b+\frac{3}{2}, & a+b+2 & \end{matrix};x^2\right] \\
&\quad-\frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{e}-1\right)c(c+1)(c+2)x^3}{8(a+1)(b+1)(2a+3)} \\
&\quad\times\ {}_4F_3\left[\begin{matrix} \frac{a+b+4}{2}, & \frac{a+b+3}{2}, & \frac{c+3}{2}, & \frac{c}{2}+2 \\ a+\frac{5}{2}, & b+\frac{3}{2}, & a+b+3 & \end{matrix};x^2\right] \\
&\quad-\frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{e}-1\right)c(c+1)(c+2)x^3}{8(a+1)(b+1)(2b+3)}
\end{aligned}
$$

$$\times\ {}_4F_3\left[\begin{matrix}\frac{a+b+4}{2}, & \frac{a+b+3}{2}, & \frac{c+3}{2}, & \frac{c}{2}+2 \\ a+\frac{3}{2}, & b+\frac{5}{2}, & a+b+3 & \end{matrix};x^2\right]$$

$$+\frac{\left(\frac{a}{d}-1\right)\left(\frac{b}{e}-1\right)c(c+1)(c+2)(c+3)x^4}{16(a+1)(b+1)(2a+3)(2b+3)}$$

$$\times\ {}_4F_3\left[\begin{matrix}\frac{a+b+5}{2}, & \frac{a+b+4}{2}, & \frac{c}{2}+2, & \frac{c+5}{2} \\ a+\frac{5}{2}, & b+\frac{5}{2}, & a+b+4 & \end{matrix};x^2\right].$$

10.1. Special cases

In this section, we shall mention special cases of our results (10.1) and (10.2).

(1) In (10.1), if we take $d = a$ and $e = b$, we get

$$F^{(3)}\left[\begin{matrix}f:: & -; & -; & -; & -; & a+1; & b+1 \\ g:: & -; & -; & -; & -; & 2a+2; & 2b+2\end{matrix};-x,\ x,\ x\right] \tag{10.3}$$

$$= {}_4F_5\left[\begin{matrix}\frac{a+b+3}{2}, & \frac{a+b+2}{2}, & \frac{f}{2}, & \frac{f+1}{2} & \\ a+\frac{3}{2}, & b+\frac{3}{2}, & a+b+2, & \frac{g}{2}, & \frac{g+1}{2}\end{matrix};\frac{x^2}{4}\right].$$

(2) In (10.1), if we take $d = 2a+1$, we get

(10.4)

$$F^{(3)}\left[\begin{matrix}f:: & -; & -; & -; & -; & a; & b, & e+1 \\ g:: & -; & -; & -; & -; & 2a+1; & 2b+2, & e\end{matrix};-x,\ x,\ x\right]$$

$$= {}_4F_5\left[\begin{matrix}\frac{a+b+3}{2}, & \frac{a+b+2}{2}, & \frac{f}{2}, & \frac{f+1}{2} & \\ a+\frac{3}{2}, & b+\frac{3}{2}, & a+b+2, & \frac{g}{2}, & \frac{g+1}{2}\end{matrix};\frac{x^2}{4}\right]$$

$$+\frac{\left(\frac{b}{e}-1\right)fx}{2g(b+1)}\ {}_4F_5\left[\begin{matrix}\frac{a+b+3}{2}, & \frac{a+b+2}{2}, & \frac{f+1}{2}, & \frac{f}{2}+1 & \\ a+\frac{3}{2}, & b+\frac{3}{2}, & a+b+2, & \frac{g+1}{2}, & \frac{g}{2}+1\end{matrix};\frac{x^2}{4}\right]$$

$$-\frac{\left(\frac{b}{e}-1\right)f(f+1)x^2}{4g(g+1)(b+1)(2b+3)}$$

$$\times\ {}_4F_5\left[\begin{matrix}\frac{a+b+4}{2}, & \frac{a+b+3}{2}, & \frac{f}{2}+1, & \frac{f+3}{2} & \\ a+\frac{3}{2}, & b+\frac{5}{2}, & a+b+3, & \frac{g}{2}+1, & \frac{g+3}{2}\end{matrix};\frac{x^2}{4}\right].$$

(3) In (10.2), if we take $d = a$ and $e = b$, we get

$$F^{(3)}\left[\begin{matrix}c:: & -; & -; & -; & -; & a+1; & b+1 \\ -:: & -; & -; & -; & -; & 2a+2; & 2b+2\end{matrix};-x,\ x,\ x\right] \tag{10.5}$$

$$= {}_4F_3\left[\begin{array}{cccc} \frac{a+b+3}{2}, & \frac{a+b+2}{2}, & \frac{c}{2}, & \frac{c+1}{2} \\ & & & ;x^2 \\ a+\frac{3}{2}, & b+\frac{3}{2}, & a+b+2 & \end{array}\right].$$

(4) In (10.2), if we take $d = 2a + 1$, we get

(10.6)

$$F^{(3)}\left[\begin{array}{cccccccc} c::& -; & -; & -; & -; & a; & b, & e+1 \\ & & & & & & & ;-x,\ x,\ x \\ -:: & -; & -; & -; & -; & 2a+2; & 2b+2, & e \end{array}\right]$$

$$= {}_4F_3\left[\begin{array}{cccc} \frac{a+b+3}{2}, & \frac{a+b+2}{2}, & \frac{c}{2}, & \frac{c+1}{2} \\ & & & ;x^2 \\ a+\frac{3}{2}, & b+\frac{3}{2}, & a+b+2 & \end{array}\right]$$

$$+ \frac{\left(\frac{b}{e}-1\right)cx}{2(b+1)}\ {}_4F_3\left[\begin{array}{cccc} \frac{a+b+3}{2}, & \frac{a+b+2}{2}, & \frac{c+1}{2}, & \frac{c}{2}+1 \\ & & & ;x^2 \\ a+\frac{3}{2}, & b+\frac{3}{2}, & a+b+2 & \end{array}\right]$$

$$- \frac{\left(\frac{b}{e}-1\right)c(c+1)x^2}{4(b+1)(2b+3)}\ {}_4F_3\left[\begin{array}{cccc} \frac{a+b+4}{2}, & \frac{a+b+3}{2}, & \frac{c}{2}+1, & \frac{c+3}{2} \\ & & & ;x^2 \\ a+\frac{3}{2}, & b+\frac{5}{2}, & a+b+3 & \end{array}\right].$$

Similarly, other results can be obtained.

Concluding remark

In the theory of hypergeometric and generalized hypergeometric functions, Kummer's well-known first and second theorems play a key role. Both the theorems contain the function ${}_1F_1$ which is in the literature popularly known as the Confluent hypergeometric function or the Kummer function. The Kummer confluent hypergeometric function belongs to an important class of special functions of mathematical physics with a large number of applications in different branches of Pure and Applied Mathematics, Mathematical Statistics, Theory of Probability, Distribution Theory, Quantum (Wave) Mechanics, Atomic Physics, Quantum Theory, Nuclear Physics, Quantum Electronics, Elastic Theory, Acoustics, Theory of Oscillating Strings, Hydrodynamics, Random Walk Theory, Optics, Wave Theory, Fibre Optics, Electromagnetic Field Theory and Plasma Physics. For details, we refer the standard texts of N. N. Lebedev [26], M. Abramowitz and I. Stegun [1], M. I. Zhurina and L. N. Osipova [53], F. G. Tricomi [52] and L. J. Slater [44].

In this paper we have provided an interesting extension of the Kummer's second theorem and as an applications, we have given a large number of very interesting results involving functions in two and three variables which generalize and unify the results already available in the literature. The applications of the extension of Kummer's second theorem for solving real practical problems mentioned in the above second paragraph are under investigations and the same will form a part of the subsequent paper in this direction.

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