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INFINITELY MANY SOLUTIONS FOR $(p(x), q(x))$-LAPLACIAN-LIKE SYSTEMS

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Abstract. Variational method has played an important role in solving problems of uniqueness and existence of the nonlinear works as well as analysis. It will also be extremely useful for researchers in all branches of natural sciences and engineers working with non-linear equations economy, optimization, game theory and medicine. Recently, the existence of infinitely many weak solutions for some non-local problems of Kirchhoff type with Dirichlet boundary condition are studied [14]. Here, a suitable method is presented to treat the elliptic partial derivative equations, especially $(p(x), q(x))$-Laplacian-like systems. This kind of equations are used in the study of fluid flow, diffusive transport akin to diffusion, rheology, probability, electrical networks, etc. Here, the existence of infinitely many weak solutions for some boundary value problems involving the $(p(x), q(x))$-Laplacian-like operators is proved. The method is based on variational methods and critical point theory.

Partial differential equations (PDEs) is used in the study of fluid flow, diffusive transport akin to diffusion, rheology, probability, electrical networks, etc. [3,4,12–15,18–32,34,35]. Also, the existence of solutions for Schrödinger-Hardy systems, p-fractional Hardy-Schrödinger-Kirchhoff systems as well as a class of systems involving fractional (p, q)-Laplacian operators are studied. Recently, the existence of infinitely many weak solutions for $p(x)$-Laplacian-like operators is studied (see [36]).

The purpose of this article is to study the existence of infinitely many weak solutions for $(p(x), q(x))$-Laplacian-like system originated from capillary phenomenon of the following form:

$$(1)\qquad \begin{cases} -div\big((1+\dfrac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}})|\nabla u|^{p(x)-2}\nabla u\big) = \lambda F_u(x,u,v) \quad \text{in} \quad \Omega,\\ -div\big((1+\dfrac{|\nabla v|^{q(x)}}{\sqrt{1+|\nabla v|^{2q(x)}}})|\nabla v|^{q(x)-2}\nabla v\big) = \lambda F_v(x,u,v) \quad \text{in} \quad \Omega,\\ u=v=0 \qquad \text{on} \quad \partial\Omega, \end{cases}$$

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where Ω is an open bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary, $\lambda \in (0, \infty)$, $F : \bar{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ is a function such that $F(\cdot, s, t)$ is measurable in $\bar{\Omega}$ for all $(s, t) \in \mathbb{R}^2$ and $F(x, \cdot, \cdot)$ is C^1 in $\mathbb{R}^2$ for every $x \in \Omega$, and F_u, F_v denote the partial derivatives of F with respect to u, v, respectively. $p(\cdot)$, $q(\cdot) \in C^0(\bar{\Omega})$ with $N < p^- := \inf_{x\in\bar{\Omega}} p(x) \leq p^+ := \sup_{x\in\bar{\Omega}} p(x) < +\infty, N < q^- := \inf_{x\in\bar{\Omega}} q(x) \leq q^+ := \sup_{x\in\bar{\Omega}} q(x) < +\infty$.

Capillary action (sometimes capillarity, capillary motion, capillary effect, or wicking) is the ability of a liquid to flow in narrow spaces without the assistance of, or even in opposition to, external forces like gravity. The effect can be seen in the drawing up of liquids between the hairs of a paint-brush, in a thin tube, in porous materials such as paper and plaster, in some non-porous materials such as sand and liquefied carbon fiber, or in a biological cell. It occurs because of intermolecular forces between the liquid and surrounding solid surfaces. If the diameter of the tube is sufficiently small, then the combination of surface tension (which is caused by cohesion within the liquid) and adhesive forces between the liquid and container wall act to propel the liquid. The study of capillary phenomena has gained some attention recently. This increasing interest is motivated not only by fascination in naturally-occurring phenomena such as motion of drops, bubbles, and waves but also its importance in applied fields ranging from industrial and biomedical and pharmaceutical to microfluidic systems.

The study of differential equations and variational problems with variable exponents have attracted intense research interests in recent years. For some recent work on this subject see [1,9,10,14,17]. In [36] the authors investigate the existence of infinitely many weak solutions for the following $p(x)$-Laplacian-like operators

$$\begin{cases} -div\big((1 + \dfrac{|\nabla u|^{p(x)}}{\sqrt{1 + |\nabla u|^{2p(x)}}})|\nabla u|^{p(x)-2}\nabla u\big) = \lambda f(x, u) \text{ in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Now, we recall some background facts concerning the variable exponent Lebesgue and Sobolev spaces (for more details, see [5,6,8,20] and the references therein). Set $C_+(\Omega) := \{h \in C(\bar{\Omega}) : h(x) > 1 \text{ for all } x \in \bar{\Omega}\}$. For every $p(\cdot) \in C_+(\Omega)$, we define the variable exponent Lebesgue space

$$L^{p(\cdot)}(\Omega) := \{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)} dx < \infty\},$$

which is a Banach space under the Luxemburg norm, $|u|_{p(\cdot)} = \inf\{\mu > 0 : \int_\Omega |\frac{u(x)}{\mu}|^{p(x)} dx \leq 1\}$. The variable exponent Sobolev space is defined by

$$W^{1,p(\cdot)}(\Omega) := \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\}$$

and equipped with the norm $\|u\|_{1,p(\cdot)} := |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)}$. This space is a separable, reflexive uniformly convex Banach space (see [7]). One can define

$W_0^{1,p(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ which can be renormed by the equivalent norm $\|u\|_{p(\cdot)} := |\nabla u|_{p(\cdot)}$. This space is a separable and reflexive Banach space, too.

The following two propositions are from [8].

Proposition 0.1. *Suppose $\frac{1}{p(\cdot)} + \frac{1}{p^*(\cdot)} = 1$. Then $L^{p(\cdot)}(\Omega)$ and $L^{p^*(\cdot)}(\Omega)$ are conjugate spaces. For $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p^*(\cdot)}(\Omega)$, we have*

$$\left|\int_\Omega u(x)v(x)dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{(p^*)^-}\right)|u|_{p(\cdot)}|v|_{p^*(\cdot)} \leq 2|u|_{p(\cdot)}|v|_{p^*(\cdot)},$$

where $p^- := \inf_{x\in\bar\Omega} p(x)$ and $(p^*)^- := \inf_{x\in\bar\Omega} p^*(x)$.

Proposition 0.2. *Set $\rho(u) = \int_\Omega |u|^{p(x)}dx$. For $u, u_n \in L^{p(\cdot)}(\Omega)$, we have*

(1) $|u|_{p(\cdot)} < (=;>)1 \Leftrightarrow \rho(u) < (=;>)1$,
(2) $|u|_{p(\cdot)} > 1 \Rightarrow |u|_{p(\cdot)}^{p^-} \leq \rho(u) \leq |u|_{p(\cdot)}^{p^+}$,
(3) $|u|_{p(\cdot)} < 1 \Rightarrow |u|_{p(\cdot)}^{p^+} \leq \rho(u) \leq |u|_{p(\cdot)}^{p^-}$,
(4) $|u_n|_{p(\cdot)} \to 0 \Leftrightarrow \rho(u_n) \to 0$,
(5) $|u_n|_{p(\cdot)} \to \infty \Leftrightarrow \rho(u_n) \to \infty$.

From Proposition 0.2, for $u \in W_0^{1,p(\cdot)}(\Omega)$, the following inequalities hold:

$$\|u\|_{p(\cdot)}^{p^-} \leq \int_\Omega |\nabla u|^{p(x)}dx \leq \|u\|_{p(\cdot)}^{p^+} \qquad \text{if} \quad \|u\|_{p(\cdot)} \geq 1, \tag{2}$$

$$\|u\|_{p(\cdot)}^{p^+} \leq \int_\Omega |\nabla u|^{p(x)}dx \leq \|u\|_{p(\cdot)}^{p^-} \qquad \text{if} \quad \|u\|_{p(\cdot)} \leq 1. \tag{3}$$

Proposition 0.3 ([11])**.** *If $\Omega \subset \mathbb{R}^N$ is a bounded domain, then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow C^0(\bar\Omega)$ is compact whenever $N < p^-$.*

In the sequel, X denotes the space $W_0^{1,p(\cdot)}(\Omega) \times W_0^{1,q(\cdot)}(\Omega)$, which is a reflexive Banach space respect to the norm

$$\|(u,v)\| = \|u\|_{p(\cdot)} + \|v\|_{q(\cdot)},$$

where

$$\|u\|_{p(\cdot)} = |\nabla u|_{p(\cdot)} \qquad \text{and} \qquad \|v\|_{q(\cdot)} = |\nabla v|_{q(\cdot)}.$$

Since $p^- > N$ and $q^- > N$, so X is compactly embedded in $C^0(\bar\Omega) \times C^0(\bar\Omega)$.

Definition. We say that $(u,v) \in X$ is a weak solution of problem (1) if

$$\int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla z dx - \lambda \int_\Omega F_u(x,u,v) z dx$$
$$+ \int_\Omega \Big(|\nabla v|^{q(x)-2}\nabla v + \frac{|\nabla v|^{2q(x)-2}\nabla v}{\sqrt{1+|\nabla v|^{2q(x)}}}\Big)\nabla w dx - \lambda \int_\Omega F_v(x,u,v) w dx = 0$$

for all $(z,w) \in X$.

Define the functionals $\Phi, \Psi : X \to \mathbb{R}$, by

$$\begin{aligned}\Phi(u,v) = & \int_\Omega \Big(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\sqrt{1+|\nabla u|^{2p(x)}}}{p(x)}\Big)dx \\ & + \int_\Omega \Big(\frac{1}{q(x)}|\nabla v|^{q(x)} + \frac{\sqrt{1+|\nabla v|^{2q(x)}}}{q(x)}\Big)dx \end{aligned} \tag{4}$$

and

$$\Psi(u,v) = \int_\Omega F(x,u,v)dx.$$

Set

$$I_\lambda(u,v) := \Phi(u,v) - \lambda\Psi(u,v) \quad \text{ for all } (u,v) \in X.$$

By a similar argument in [33], Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous whose Gâteaux derivative at (u,v) is the functional $\Phi'(u,v) \in X^*$, given by

$$\begin{aligned}\langle \Phi'(u,v),(z,w)\rangle = & \int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla z dx \\ & + \int_\Omega \Big(|\nabla v|^{q(x)-2}\nabla v + \frac{|\nabla v|^{2q(x)-2}\nabla v}{\sqrt{1+|\nabla v|^{2q(x)}}}\Big)\nabla w dx\end{aligned}$$

for every $(z,w) \in X$.

Proposition 0.4 ([33]). *The functional $\Phi : X \to \mathbb{R}$, given by* (4)*, is convex and mapping $\Phi' : X \to X^*$ is a strictly monotone and bounded homeomorphism.*

Furthermore, Φ is coercive, since

$$\frac{\sqrt{1+|\nabla u|^{2p(x)}}}{p(x)} \geq \frac{1}{p(x)}|\nabla u|^{p(x)},$$

so

$$\Phi(u,v) > \frac{2}{p^+}\int_\Omega |\nabla u|^{p(x)}dx + \frac{2}{q^+}\int_\Omega |\nabla v|^{q(x)}dx.$$

The above inequality and (2), show that for any $(u,v) \in X$ with $\|u\|, \|v\| > 1$ we have

$$\Phi(u,v) > \frac{2}{p^+}\|u\|^{p^-}_{p(\cdot)} + \frac{2}{q^+}\|v\|^{q^-}_{q(\cdot)},$$

which follows $\lim_{\|(u,v)\|\to+\infty} \Phi(u,v) = +\infty$, i.e., Φ is coercive.

Moreover, Ψ is a Gâteaux differentiable functional whose Gâteaux derivative at (u,v) is the functional $\Psi'(u,v) \in X^*$, given by

$$\langle \Psi'(u,v),(z,w)\rangle = \int_\Omega F_u(x,u,v)z dx + \int_\Omega F_v(x,u,v)w dx$$

for every $(z,w) \in X$. Since Ψ has compact derivative, it follows that Ψ is sequentially weakly continuous.

Before proving the result, we recall the following multiple critical points theorem of G. Bonanno [2] which can be regarded as supplements of the variational principle of Ricceri [32] which is our main tools.

Theorem 0.5. *Let X be a reflexive real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is strongly continuous, sequentially weakly lower semi-continuous, coercive, and Ψ is sequentially weakly upper-semi-continuous. For every $r > \inf_X \Phi$, set*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{(\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v)) - \Psi(u)}{r - \Phi(u)},$$

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then

(a) *If $\gamma < +\infty$ then, for each $\lambda \in (0, \frac{1}{\gamma})$, the following alternative holds: Either*

(a1) *$I_\lambda := \Phi - \lambda\Psi$ possesses a global minimum, or*

(a2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.*

(b) *If $\delta < +\infty$ then, for each $\lambda \in (0, \frac{1}{\delta})$, the following alternative holds: Either*

(b1) *there is a global minimum of Φ that is a local minimum of I_λ, or*

(b2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ that weakly converges to a global minimum of Φ with*

$$\lim_{n\to+\infty} \Phi(u_n) = \inf_X \Phi.$$

1. A sequence of unbounded solutions

For fixed $x_0 \in \Omega$, set $R_2 > R_1 > 0$ such that $B(x_0, R_2) \subset \Omega$, where $B(x_0, R_2)$ denotes the ball with center at x_0 and radius R_2. Set

$$C := \max\{\sup_{u\in W_0^{1,p(\cdot)}\setminus\{0\}} \frac{\max_{x\in\Omega} |u(x)|^{p^-}}{\|u\|_{p(\cdot)}^{p^-}}, \sup_{v\in W_0^{1,q(\cdot)}\setminus\{0\}} \frac{\max_{x\in\Omega} |v(x)|^{q^-}}{\|v\|_{q(\cdot)}^{q^-}}\}, \tag{5}$$

(6)

$$\theta_{p^+} := \frac{2\Gamma(1+\frac{N}{2})}{\left((Cp^+)^{\frac{1}{p^-}} + (Cq^+)^{\frac{1}{q^-}}\right)^{\min(p^-,q^-)} \pi^{\frac{N}{2}} (R_2^N - R_1^N)} \left(1 - \frac{2}{(R_2-R_1)^{p^+}+2}\right),$$

$$\theta_{q^+} := \frac{2\Gamma(1+\frac{N}{2})}{\left((Cp^+)^{\frac{1}{p^-}} + (Cq^+)^{\frac{1}{q^-}}\right)^{\min(p^-,q^-)} \pi^{\frac{N}{2}} (R_2^N - R_1^N)} \left(1 - \frac{2}{(R_2-R_1)^{q^+}+2}\right),$$

where Γ denotes the Gamma function. Now we can state the main result.

Theorem 1.1. *Assume that there exist a point $x_0 \in \Omega$ and $R_2 > R_1 > 0$ such that $B(x_0, R_2) \subset \Omega$ and $A < \theta B$, where $\theta := \min\{\theta_{p^+}, \theta_{q^+}\}$,*

$$\begin{aligned} A :=& \quad \liminf_{\xi \to +\infty} \frac{\int_\Omega \sup_{|s|+|t| \le \xi} F(x,s,t)dx}{\xi^{\min(p^-,q^-)}} \quad \textit{and} \\ B :=& \quad \limsup_{s,t \to +\infty} \frac{\int_{B(x^0,R_1)} F(x,s,t)dx}{\frac{s^{p^+}}{p^-} + \frac{t^{q^+}}{q^-}}. \end{aligned}$$

Moreover, suppose $F(x,s,t) \ge 0$ for every $(x,s,t) \in \Omega \times (\mathbb{R}^+)^2$. Then the problem (1) has an unbounded sequence of weak solutions in X, for each

$$\lambda \in \Lambda := \frac{2}{\left((Cp^+)^{\frac{1}{p^-}} + (Cq^+)^{\frac{1}{q^-}}\right)^{\min(p^-,q^-)}} \Big] \frac{1}{\theta B}, \frac{1}{A} \Big[.$$

Proof. We apply part (a) of Theorem 0.5. Certainly, the weak solutions of problem (1) are exactly solutions of the equation $I'_\lambda(u,v) = 0$. The functional Φ and Ψ satisfy the assumptions of Theorem 0.5. We show that $\gamma < +\infty$. Since X is compactly embedded in $C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$ and from (5) one has

$$\sup_{x\in\Omega} |u(x)|^{p^-} \le C\|u\|_{p(\cdot)}^{p^-} \quad \text{and} \quad \sup_{x\in\Omega} |v(x)|^{q^-} \le C\|v\|_{q(\cdot)}^{q^-}$$

for all $(u,v) \in X$. Thus

$$\sup_{x\in\Omega} \Big(\frac{1}{p^+} |u(x)|^{p^-} + \frac{1}{q^+} |v(x)|^{q^-} \Big) < C \Big(\frac{1}{p^+} \|u\|_{p(\cdot)}^{p^-} + \frac{1}{q^+} \|v\|_{q(\cdot)}^{q^-} \Big).$$

So, for each $r > 0$

$$\begin{aligned} \Phi^{-1}(]-\infty, r[) :&= \{(u,v) \in X : \Phi(u,v) < r\} \\ &= \{(u,v) \in X : \frac{2}{p^+} \|u\|_{p(\cdot)}^{p^-} + \frac{2}{q^+} \|v\|_{q(\cdot)}^{q^-} < r\} \qquad (7) \\ &\subseteq \{(u,v) \in X : \frac{1}{p^+} |u(x)|^{p^-} + \frac{1}{q^+} |v(x)|^{q^-} < \frac{Cr}{2} \text{ for all } x \in \Omega\}, \end{aligned}$$

and if we set $\Delta := \{(u,v) \in X : \frac{1}{p^+}|u(x)|^{p^-} + \frac{1}{q^+}|v(x)|^{q^-} < \frac{Cr}{2}$ for all $x \in \Omega\}$, then

$$\sup_{(u,v)\in\Phi^{-1}(]-\infty,r[)} \Psi(u,v) < \sup_{(u,v)\in\Delta} \int_\Omega F(x,u,v)dx.$$

Note that $\Phi(0,0) = 0$ and $\Psi(0,0) \ge 0$. Therefore, for every $r > 0$,

$$\begin{aligned} \varphi(r) :&= \inf_{\Phi(u,v)<r} \frac{\left(\sup_{(u',v')\in\Phi^{-1}(]-\infty,r[)} \Psi(u',v')\right) - \Psi(u,v)}{r - \Phi(u,v)} \\ &\le \frac{\sup_{\Phi^{-1}(]-\infty,r[)} \Psi}{r} \\ &\le \frac{1}{r} \sup_{(u,v)\in\Delta} \int_\Omega F(x,u,v)dx. \end{aligned}$$

Let $\{\xi_n\}$ be a real sequence of positive numbers such that $\lim_{n\to+\infty}\xi_n = +\infty$ and

$$\lim_{n\to+\infty} \frac{\int_\Omega \sup_{|s|+|t|\le\xi_n} F(x,s,t)dx}{\xi_n^{\min(p^-,q^-)}} = A < +\infty. \tag{8}$$

Set $r_n := 2\Big(\frac{\xi_n}{(Cp^+)^{\frac{1}{p^-}}+(Cq^+)^{\frac{1}{q^-}}}\Big)^{\min(p^-,q^-)}$. Let $(u,v) \in \Phi^{-1}(]-\infty, r_n[)$, from (7), one has

$$\frac{1}{p^+}|u(x)|^{p^-} + \frac{1}{q^+}|v(x)|^{q^-} < \frac{Cr_n}{2} \quad \text{for all } x \in \Omega.$$

So,

$$|u(x)| \le (\frac{1}{2}Cr_n p^+)^{\frac{1}{p^-}} \quad \text{and} \quad |v(x)| < (\frac{1}{2}Cr_n q^+)^{\frac{1}{q^-}}.$$

Thus, for each $n \in \mathbb{N}$ large enough ($r_n \ge 2$),

$$\begin{aligned}|u(x)| + |v(x)| &\le (\frac{1}{2}Cr_n p^+)^{\frac{1}{p^-}} + (\frac{1}{2}Cr_n q^+)^{\frac{1}{q^-}}\\ &\le \Big((Cp^+)^{\frac{1}{p^-}} + (Cq^+)^{\frac{1}{q^-}}\Big)(\frac{r_n}{2})^{\frac{1}{\min(p^-,q^-)}} = \xi_n.\end{aligned}$$

Hence

$$\begin{aligned}\varphi(r_n) &\le \frac{\sup_{\{(u,v)\in X:|u(x)|+|v(x)|<\xi_n \text{ for all } x\in\Omega\}}\int_\Omega F(x,u,v)dx}{2\left(\frac{\xi_n}{(Cp^+)^{\frac{1}{p^-}}+(Cq^+)^{\frac{1}{q^-}}}\right)^{\min(p^-,q^-)}}\\ &\le \frac{1}{2}\Big((Cp^+)^{\frac{1}{p^-}} + (Cq^+)^{\frac{1}{q^-}}\Big)^{\min(p^-,q^-)} \frac{\int_\Omega \sup_{|s|+|t|\le\xi_n} F(x,s,t)dx}{\xi_n^{\min(p^-,q^-)}}.\end{aligned} \tag{9}$$

Hence, from (8) and (9), one has

$$\begin{aligned}\gamma &\le \liminf_{n\to+\infty}\varphi(r_n)\\ &\le \frac{1}{2}\Big((Cp^+)^{\frac{1}{p^-}} + (Cq^+)^{\frac{1}{q^-}}\Big)^{\min(p^-,q^-)} \lim_{n\to+\infty}\frac{\int_\Omega \sup_{|s|+|t|\le\xi_n} F(x,s,t)dx}{\xi_n^{\min(p^-,q^-)}}\\ &= \frac{1}{2}\Big((Cp^+)^{\frac{1}{p^-}} + (Cq^+)^{\frac{1}{q^-}}\Big)^{\min(p^-,q^-)} A < +\infty.\end{aligned}$$

This implies

$$\gamma \le \frac{1}{2}\Big((Cp^+)^{\frac{1}{p^-}} + (Cq^+)^{\frac{1}{q^-}}\Big)^{\min(p^-,q^-)} A < \frac{1}{\lambda}.$$

We conclude that $\Lambda \subseteq]0, \frac{1}{\gamma}[$. For $\lambda \in \Lambda$, we show that the functional $I_\lambda = \Phi - \lambda\Psi$ is unbounded from below. Indeed, since

$$\frac{1}{\lambda} < \frac{1}{2}\Big((Cp^+)^{\frac{1}{p^-}} + (Cq^+)^{\frac{1}{q^-}}\Big)^{\min(p^-,q^-)}\theta B,$$

we can consider a sequence d_n of positive numbers and $\eta > 0$ such that $d_n \to +\infty$ as $n \to \infty$ and

$$\frac{1}{\lambda} < \eta < \frac{1}{2}\theta\Big((Cp^+)^{\frac{1}{p^-}} + (Cq^+)^{\frac{1}{q^-}}\Big)^{\min(p^-,q^-)} \frac{\int_{B(x_0,R_1)} F(x,d_n,d_n)dx}{\frac{d_n^{p^+}}{p^-} + \frac{d_n^{q^+}}{q^-}} \tag{10}$$

for n large enough. Suppose $w_n \subseteq X$ is a sequence defined by

$$w_n(x) = \begin{cases} 0 & x \in \bar{\Omega} \setminus B(x_0,R_2), \\ \dfrac{d_n}{R_2 - R_1}\Big(R_2 - \{\Sigma_{i=1}^n (x^i - x_0^i)^2\}^{\frac{1}{2}}\Big) & x \in B(x_0,R_2) \setminus B(x_0,R_1), \\ d_n & x \in B(x_0,R_1). \end{cases}$$

Bearing (6) in mind, we have

$$\begin{aligned} \Phi(w_n,w_n) = & \int_\Omega \frac{1}{p(x)}\Big(|\nabla w_n|^{p(x)} + \sqrt{1 + |\nabla w_n|^{2p(x)}}\Big)dx \\ & + \int_\Omega \frac{1}{q(x)}\Big(|\nabla w_n|^{q(x)} + \sqrt{1 + |\nabla w_n|^{2q(x)}}\Big)dx \\ \leq & \int_{B(x_0,R_2)\setminus B(x_0,R_1)} \frac{1}{p^-}\Big(1 + 2|\nabla w_n|^{p(x)}\Big) + \frac{1}{q^-}\Big(1 + 2|\nabla w_n|^{q(x)}\Big) \\ \leq & \frac{1}{p^-}\frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}(R_2^N - R_1^N)d_n^{p^+}\Big(1 + \frac{2}{(R_2-R_1)^{p^+}}\Big) \\ & + \frac{1}{q^-}\frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}(R_2^N - R_1^N)d_n^{q^+}\Big(1 + \frac{2}{(R_2-R_1)^{q^+}}\Big) \\ = & \frac{2}{\Big((Cp^+)^{\frac{1}{p^-}} + (Cq^+)^{\frac{1}{q^-}}\Big)^{\min(p^-,q^-)}}\Big(\frac{d_n^{p^+}}{p^-\theta_{p^+}} + \frac{d_n^{q^+}}{q^-\theta_{q^+}}\Big). \end{aligned} \tag{11}$$

Moreover, by assumption that $F(x,s,t) \geq 0$, we have

$$\Psi(w_n,w_n) = \int_\Omega F(x,w_n,w_n)dx \geq \int_{B(x_0,R_1)} F(x,d_n,d_n)dx. \tag{12}$$

So, it follows from (10), (11) and (12) that

$$\begin{aligned} I_\lambda(w_n,w_n) &= \Phi(w_n,w_n) - \lambda\Psi(w_n,w_n) \\ &\leq \frac{2}{\Big((Cp^+)^{\frac{1}{p^-}} + (Cq^+)^{\frac{1}{q^-}}\Big)^{\min(p^-,q^-)}}\Big(\frac{d_n^{p^+}}{p^-\theta_{p^+}} + \frac{d_n^{q^+}}{q^-\theta_{q^+}}\Big) \\ &\quad - \lambda\int_{B(x_0,R_1)} F(x,d_n,d_n)dx \\ &\leq \frac{2}{\theta\Big((Cp^+)^{\frac{1}{p^-}} + (Cq^+)^{\frac{1}{q^-}}\Big)^{\min(p^-,q^-)}}\Big(\frac{d_n^{p^+}}{p^-} + \frac{d_n^{q^+}}{q^-}\Big) \end{aligned}$$

$$
\begin{aligned}
&- \lambda \int_{B(x_0,R_1)} F(x, d_n, d_n)dx \\
&< \frac{2(1-\lambda\eta)}{\theta\big((Cp^+)^{\frac{1}{p^-}} + (Cq^+)^{\frac{1}{q^-}}\big)^{\min(p^-,q^-)}} \Big(\frac{d_n^{p^+}}{p^-} + \frac{d_n^{q^+}}{q^-}\Big)
\end{aligned}
$$

for n large enough, so $\lim_{n\to+\infty} I_\lambda(w_n, w_n) = -\infty$ and the claim is done.

The alternative of Theorem 0.5 case (a) assures the existence of unbounded sequence (w_n) of critical points of the functional I_λ. This completes the proof in view of the relation between the critical points of I_λ and the weak solutions of problem (1). $\square$

Now we present an example (see [16,17]) to show the validity of the obtained result.

Example 1.2. Let $\Omega = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 3\}$. $p(x, y)$ and $q(x, y)$ defined on Ω by

$$p(x, y) = x^2 + y^2 + 3 \quad \text{and} \quad q(x, y) = x^2 + y^2 + 4$$

for all $(x, y) \in \Omega$. $\{a_n\}$ is an increasing sequence given by

$$a_1 := 2, \qquad a_{n+1} := n!(a_n)^{\frac{8}{3}} + 2 \ \ (n \geq 1).$$

Define the function $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ by

$$F(x, y, s, t) = \begin{cases} (a_{n+1})^8 e^{1-\frac{1}{1-[(s-a_{n+1})^2+(t-a_{n+1})^2]}+x^2+y^2}, \\ \quad \text{if } (x, y, s, t) \in \Omega \times \cup_{n\geq 1} B\big((a_{n+1}, a_{n+1}), 1\big), \\ 0, \qquad \text{otherwise}, \end{cases}$$

for all $(x, y) \in \Omega$. It is obvious that $p^- = 3, p^+ = 6$ and $q^- = 4, q^+ = 7$. $B\big((a_{n+1}, a_{n+1}), 1\big)$ is an open unit ball of center (a_{n+1}, a_{n+1}).

We see that F is non-negative and $F \in C^1(\Omega \times \mathbb{R}^2)$. For every $n \in \mathbb{N}$, the restriction of F on $B\big((a_{n+1}, a_{n+1}), 1\big)$ attains its maximum in (a_{n+1}, a_{n+1}) and

$$F(x, y, a_{n+1}, a_{n+1}) = e^{x^2+y^2}(a_{n+1})^8,$$

then

$$\limsup_{n\to+\infty} \frac{F(x, y, a_{n+1}, a_{n+1})}{\frac{a_{n+1}^6}{3} + \frac{a_{n+1}^7}{4}} = +\infty.$$

So, we have

$$
\begin{aligned}
B &= \limsup_{s,t\to+\infty} \frac{\int_{B(x_0,R_1)} F(x, y, s, t)dxdy}{\frac{s^6}{3} + \frac{t^7}{4}} \\
&= |B(x_0, R_1)| \limsup_{s,t\to+\infty} \frac{F(x, y, s, t)}{\frac{s^6}{3} + \frac{t^7}{4}} = +\infty.
\end{aligned}
$$

On the other hand, for every $n \in \mathbb{N}$, we have

$$\sup_{|s|+|t|\leq a_{n+1}-1} F(x, y, s, t) = e^{x^2+y^2}(a_n)^8 \text{ for all } n \in \mathbb{N}.$$

Then

$$\lim_{n\to+\infty}\frac{\sup_{|s|+|t|\le a_{n+1}-1}F(x,y,s,t)}{(a_{n+1}-1)^3}=0,$$

thus

$$\lim_{\xi\to+\infty}\frac{\sup_{|s|+|t|\le \xi}F(x,y,s,t)}{\xi^3}=0.$$

Finally,

$$A:=\liminf_{\xi\to+\infty}\frac{\int_\Omega \sup_{|s|+|t|\le \xi}F(x,y,s,t)dxdy}{\xi^3}=0<\theta B.$$

By applying Theorem 1.1, for every $\lambda\in]0,+\infty[$, the system

(13)

$$\begin{cases}-div\big((1+\dfrac{|\nabla u|^{p(x,y)}}{\sqrt{1+|\nabla u|^{2p(x,y)}}})|\nabla u|^{p(x,y)-2}\nabla u\big)=\lambda F_u(x,y,u,v) \text{ in } \Omega,\\ -div\big((1+\dfrac{|\nabla v|^{q(x,y)}}{\sqrt{1+|\nabla v|^{2q(x,y)}}})|\nabla v|^{q(x,y)-2}\nabla v\big)=\lambda F_v(x,y,u,v) \text{ in } \Omega,\\ u=v=0 \text{ on } \partial\Omega,\end{cases}$$

admits infinitely many weak solutions in X.

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