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SOME PROPERTIES OF GENERALIZED BESSEL FUNCTION ASSOCIATED WITH GENERALIZED FRACTIONAL CALCULUS OPERATORS

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Abstract. This paper devoted to obtain some fractional integral properties of generalized Bessel function using pathway fractional integral operator. We also find the pathway transform of the generalized Bessel function in terms of Fox H-function.

1. Introduction and preliminaries

Bessel functions are directly associated with problems having circular and cylindrical symmetry. They arise in the study of free vibrations of a circular membrane and in determining the temperature distribution in a circular cylinder. They also occur in electromagnetic theory and several other areas of science and engineering. In 1824, F. W. Bessel gave the systematic study of Bessel function. The Bessel function of first kind $J_\nu(z)$ [1] is represented as

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1+\nu+k)k!} \left(\frac{z}{2}\right)^{2k+\nu}, \tag{1}$$

where $|z| < \infty, |\arg z| < \pi$.

Wright [16] generalized Bessel function which is defined by

$$J_\nu^h(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{\Gamma(1+\nu+hk)k!}, \tag{2}$$

where $h > 0, |z| < \infty, |\arg z| < \pi$.

Galue [1] generalized Bessel function which is represented by

$$_hJ_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1+\nu+hk)k!} \left(\frac{z}{2}\right)^{2k+\nu}, \tag{3}$$

where $h > 0, |z| < \infty, |\arg z| < \pi$.

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Wright [15] extended the generalized hypergeometric function in the form of Fox-Wright function which is defined as

$$ {}_p\Psi_q(z) = {}_p\Psi_q\left[\begin{matrix}(u_i, U_i)_{1,p}\\(v_j, V_j)_{1,q}\end{matrix}\middle| z\right] = \sum_{k=0}^{\infty}\frac{\Gamma(u_1+U_1k)\cdots\Gamma(u_p+U_pk)}{\Gamma(v_1+V_1k)\cdots\Gamma(v_q+V_qk)}\frac{z^k}{k!}, \tag{4}$$

where $i = 1,2,\dots,p$; $j = 1,2,\dots,q$ and $u_i, v_j \in \mathbf{C}$, and the coefficients $U_1,\dots,U_p \in \mathbf{R}^+$ and $V_1,\dots,V_q \in \mathbf{R}^+$ satisfying the condition

$$\sum_{j=1}^{q} V_j - \sum_{i=1}^{p} U_i > -1. \tag{5}$$

In particular, when $U_i = V_j = 1\,(i = 1,2,\dots,p;\ j = 1,2,\dots,q)$, the equation (4) reduces to

$$ {}_p\Psi_q\left[\begin{matrix}(u_1,1),\dots,(u_p,1)\\(v_1,1),\dots,(v_q,1)\end{matrix}\middle| z\right] = \frac{\prod_{i=1}^{p}\Gamma(u_i)}{\prod_{j=1}^{q}\Gamma(v_j)}\,{}_pF_q\left[\begin{matrix}u_1,\dots,u_p\\v_1,\dots,v_q\end{matrix}\middle| z\right], \tag{6}$$

where ${}_pF_q(\cdot)$ is the generalized hypergeometric function [14].

The notion of the Hadamard product [3, 13] of two holomorphic functions is very useful in our present investigation. If one of the power series is an entire function, then the Hadamard product of two power series is also an entire function. This can help us to degrade a newly emerged function into two known functions. Let two power series with radius of convergence R_1 and R_2,

$$f(z) := \sum_{n=0}^{\infty} u_n z^n, \quad (|z| < R_1)$$
$$g(z) := \sum_{n=0}^{\infty} v_n z^n, \quad (|z| < R_2).$$

Then, the Hadamard product of two given power series defined by

$$(f * g)(z) := \sum_{n=0}^{\infty} u_n v_n z^n = (g * f)(z), \quad (|z| < R), \tag{7}$$

where

$$R = \lim_{n\to\infty}\left|\frac{u_n v_n}{u_{n+1}v_{n+1}}\right| = \left(\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right|\right)\cdot\left(\lim_{n\to\infty}\left|\frac{v_n}{v_{n+1}}\right|\right) = R_1 \cdot R_2.$$

Therefore, in general, $R \geq R_1 \cdot R_2$.

The H-function [11] is defined by means of a Mellin-Barnes type integral as follows:

$$H^{m,n}_{p,q}(z) = H^{m,n}_{p,q}\left[z\middle|\begin{matrix}(u_1,U_1),\dots,(u_p,U_p)\\(v_1,V_1),\dots,(v_q,V_q)\end{matrix}\right] = \frac{1}{2\pi i}\int_L h(s) z^{-s} ds, \tag{8}$$

where

$$h(s) = \frac{\left\{\prod_{j=1}^{m}\Gamma(v_j+V_js)\right\}\left\{\prod_{j=1}^{n}\Gamma(1-u_j-U_js)\right\}}{\left\{\prod_{j=m+1}^{q}\Gamma(1-v_j-V_js)\right\}\left\{\prod_{j=n+1}^{p}\Gamma(u_j+U_js)\right\}}. \tag{9}$$

Here $z^{-s} = \exp[-s\{|z| + i \arg z\}]$, $z \neq 0$, $i = \sqrt{-1}$, where $\ln|z|$ represents the natural logarithm of $|z|$ and $\arg z$ is not necessarily the principal value and m, n, p, q are integers such that $1 \leq m \leq q$, $0 \leq n \leq p$, $U_i, V_j \in \mathbf{R}^+$, $u_i, v_j \in \mathbf{C}$, $(i = 1, 2, \ldots, p; j = 1, 2, \ldots, q)$. An empty product in (9) is always interpreted as unity. The contour L in (8) separates the poles of the gamma functions $\Gamma(v_j + V_js)$, $j = 1, \ldots, m$ from those of the Gamma functions $\Gamma(1 - u_j - U_js)$, $j = 1, \ldots, n$.

The specific and vast theory of H-function have well explained in the books of Mathai [6], Mathai and Saxena ([10], Ch. 2) and Kilbass and Saigo ([2], Ch. 1 and Ch. 2).

2. Pathway integral representation of ${}_hJ_\nu(z)$

In this section, we establish fractional integration formula of pathway type of generalized Bessel function (3) in terms of generalized Wright hypergeometric function.

In 2009, Nair [12] introduced the pathway fractional integral operator by using the pathway idea of Mathai [7] which was further developed by Mathai and Haubold [8, 9]. This operator is a generalization of the classical Riemann-Liouville fractional integral operator which is defined below:

Let $f(x) \in L(p,q)$, $\xi \in \mathbf{C}$, $\Re(\xi) > 0$, $p \in \mathbf{R}^+$ and $\omega < 1$ be the pathway operator. Then

$$\left(P_{0+}^{\xi,\omega} f\right)(x) = x^{\xi} \int_0^{\left[\frac{x}{p(1-w)}\right]} \left[1 - \frac{p(1-\omega)t}{x}\right]^{\frac{\xi}{1-\omega}} f(t)dt, \tag{10}$$

where $L(p,q)$ is the set of Lebesgue measurable function. For a real scalar ω, the pathway model for scalar random variables is represented by the following probability density function (p.d.f.):

$$f(x) = c|x|^{v-1}[1 - p(1-\omega)|x|^{\tau}]^{\frac{\lambda}{1-\omega}}, \tag{11}$$

provided that $x \in \mathbf{R}$, $v, \tau \in \mathbf{R}^+$, $\lambda \geq 0$, $1 - p(1-\omega)|x|^{\tau} > 0$, where c is normalizing constant and ω is pathway parameter.

The normalizing constants c for pathway parameter ω is as follows:

$$c = \begin{cases} \frac{1}{2}\frac{\tau[p(1-\omega)]^{\frac{v}{\tau}}\Gamma(\frac{v}{\tau}+\frac{\xi}{1-\omega}+1)}{\Gamma(\frac{v}{\tau})\Gamma(\frac{\xi}{1-\omega}+1)}, & (\omega < 1) \\ \frac{1}{2}\frac{\tau[p(\omega-1)]^{\frac{v}{\tau}}\Gamma(\frac{\xi}{\omega-1})}{\Gamma(\frac{v}{\tau})\Gamma(\frac{\xi}{\omega-1}-\frac{v}{\tau})}, & \left(\frac{1}{\omega-1} - \frac{v}{\tau} > 0,\ \omega > 1\right) \\ \frac{1}{2}\frac{(p\lambda)^{\frac{v}{\tau}}}{\Gamma(\frac{v}{\tau})}. & (\omega \to 1) \end{cases}$$

For $\omega < 1$, it is a finite range density with $1 - p(1-\omega)|x|^\tau > 0$ which is known as extended generalized type-1 beta family of densities for real x. It also includes the type-1 beta density function, triangular density function, the uniform density function and many other probability density function.

If $\omega > 1$, then (10) can be represented as follows:

$$\left(P_{0+}^{\xi,\omega} f\right)(x) = x^\xi \int_0^{\left[\frac{x}{-p(\omega-1)}\right]} \left[1 + \frac{p(\omega-1)t}{x}\right]^{\frac{\xi}{-(\omega-1)}} f(t)dt, \tag{12}$$

and

$$f(x) = c|x|^{v-1}[1 + p(\omega-1)|x|^\tau]^{\frac{\lambda}{-(\omega-1)}}, \tag{13}$$

provided that $x \in \mathbf{R}$, $v, \tau \in \mathbf{R}^+, \lambda \geq 0$, $\omega > 1$ which is known as extended generalized type-2 beta family of densities for real x. It also includes the type-2 beta density function, the F density function, the Student-t density function and many different density function.

For $\omega \to 1$, (11) and (13) will give the exponential form as

$$\begin{aligned}\lim_{\omega\to 1} c|x|^{v-1}[1 - p(1-\omega)|x|^\tau]^{\frac{\lambda}{1-\omega}} &= \lim_{\omega\to 1} c|x|^{v-1}[1 + p(\omega-1)|x|^\tau]^{\frac{\lambda}{-(\omega-1)}} \\ &= c|x|^{v-1}e^{-p\lambda}|x|^\tau. \end{aligned} \tag{14}$$

When $p = 1$, $\omega = 0$ and replacing ξ by $\xi - 1$, then (10) gives

$$\left(P_{0+}^{\xi-1,0} f\right)(x) = \int_0^x (x-t)^{\xi-1} f(t)dt = \Gamma(\xi)\left(I_{0+}^{\xi} f\right)(x), \tag{15}$$

where $\left(I_{0+}^{\xi} f\right)(x)$ denotes the left-sided Riemann-Liouville fractional integral operator [12].

Now, $\left[1 - \frac{p(1-\omega)t}{x}\right]^{\frac{\xi}{1-\omega}} \to e^{-\frac{p\xi}{x}t}$ as $\omega \to 1_-$, then the operator (10) reduces to the Laplace transform of f with parameter $\frac{p\xi}{x}$, which is represented as

$$\lim_{\omega\to 1_-} \left(P_{0+}^{\xi,\omega} f\right)(x) = x^\xi \int_0^\infty e^{-\frac{p\xi}{x}t} f(t)dt = x^\xi L_f\left(\frac{p\xi}{x}\right). \tag{16}$$

Now we prove Theorem 2.2 using the following Lemma 2.1 given in [12].

Lemma 2.1. *Let $\xi \in \mathbf{C}$, $\Re(\xi) > 0$, $\mu \in \mathbf{C}$, $p \in \mathbf{R}^+$ and $\omega < 1$. If $\Re(\mu) > 0$ and $\Re\left(\frac{\xi}{1-\omega}\right) > -1$, then*

$$\left(P_{0+}^{\xi,\omega} t^{\mu-1}\right)(x) = \frac{x^{\xi+\mu}}{[p(1-\omega)]^\mu} \frac{\Gamma\left(1 + \frac{\xi}{1-\omega}\right)\Gamma(\mu)}{\Gamma\left(1 + \mu + \frac{\xi}{1-\omega}\right)}. \tag{17}$$

Theorem 2.2. *Let $\xi, \mu, \nu \in \mathbf{C}$, $h > 0$, $p \in \mathbf{R}^+$ and $\omega < 1$, such that $\Re(\xi) > 0$, $\Re(\mu) > 0$, $\Re(\nu) > -1$, $\Re(\mu+\nu) > 0$, $\Re\left(\frac{\xi}{1-\omega}\right) > -1$. Then*

$$\begin{aligned}\left(P_{0+}^{\xi,\omega}\, t^{\mu-1}{}_hJ_\nu(t)\right)(x) &= \frac{x^{\xi+\mu}}{[p(1-\omega)]^{\mu+\nu}}\left(\frac{x}{2}\right)^\nu \Gamma\left(1+\frac{\xi}{1-\omega}\right)\\ &\times\ {}_1\Psi_2\left[\left.\begin{matrix}(\mu+\nu,2)\\ (1+\nu,h),(1+\mu+\nu+\frac{\xi}{1-\omega},2)\end{matrix}\right| -\frac{x^2}{4[p(1-\omega)]^2}\right].\end{aligned}\tag{18}$$

Proof. Let $\mathcal{P}_1$ be the left-hand side of (18). Then, using (3), we have

$$\mathcal{P}_1 = \left(P_{0+}^{\xi,\omega}\, t^{\mu-1}\sum_{m=0}^\infty \frac{(-1)^m}{\Gamma(1+\nu+hm)m!}\left(\frac{t}{2}\right)^{\nu+2m}\right)(x).$$

On reversing the order of summation and integration, we obtain

$$\mathcal{P}_1 = \sum_{m=0}^\infty \frac{(-1)^m}{\Gamma(1+\nu+hm)m!}\left(\frac{1}{2}\right)^{\nu+2m}\left(P_{0+}^{\xi,\omega}\, t^{\mu+\nu+2m-1}\right)(x).$$

On using Lemma 2.1, we obtain

$$\begin{aligned}\mathcal{P}_1 &= \sum_{m=0}^\infty \frac{(-1)^m}{\Gamma(1+\nu+hm)m!}\left(\frac{1}{2}\right)^{\nu+2m}\\ &\times \frac{x^{\xi+\mu+\nu+2m}}{[p(1-\omega)]^{\mu+\nu+2m}}\frac{\Gamma\left(\frac{\xi}{1-\omega}+1\right)\Gamma(\mu+\nu+2m)}{\Gamma\left(1+\mu+\nu+2m+\frac{\xi}{1-\omega}\right)},\end{aligned}\tag{19}$$

with the help of (4), this leads to the right-hand side of (18). □

When $\omega \to 1_-$, $\frac{\xi}{1-\omega} \to \infty$, then in (19), we can expand gamma function by using Stirling formula:

$$\begin{aligned}&\frac{1}{(1-\omega)^{\mu+\nu+2m}}\frac{\Gamma\left(\frac{\xi}{1-\omega}+1\right)}{\Gamma\left(1+\mu+\nu+2m+\frac{\xi}{1-\omega}\right)}\\ &\to \frac{\sqrt{2\pi}\left(\frac{\xi}{1-\omega}\right)^{\frac{\xi}{1-\omega}+\frac{1}{2}} e^{-\frac{\xi}{1-\omega}}}{(1-\omega)^{\mu+\nu+2m}\sqrt{2\pi}\left(\frac{\xi}{1-\omega}\right)^{\mu+\nu+2m+\frac{\xi}{1-\omega}+\frac{1}{2}} e^{-\frac{\xi}{1-\omega}}} = \frac{1}{\xi^{\mu+\nu+2m}}.\end{aligned}$$

Hence,

$$\lim_{\omega\to 1_-}\left[P_{0+}^{\xi,\omega}\, t^{\mu-1}{}_hJ_\nu(t)\right](x) = \frac{x^{\xi+\mu+\nu}}{2^\nu(p\xi)^{\mu+\nu}}\ {}_1\Psi_1\left[\left.\begin{matrix}(\mu+\nu,2)\\(1+\nu,h)\end{matrix}\right| -\frac{x^2}{4p^2\xi^2}\right],$$

which corresponds to the Laplace transform of the function $t^{\mu-1}{}_hJ_\nu(t)$.

Now, we prove the following result by assuming the case $\omega > 1$ and using the equation (12).

Theorem 2.3. *Let $\xi, \mu, \nu \in \mathbf{C}$, $h > 0$, $p \in \mathbf{R}^+$ and $\omega > 1$, such that $\Re(\xi) > 0$, $\Re(\mu) > 0$, $\Re(\nu) > -1$, $\Re(\mu+\nu) > 0$, $\Re\left(1 - \frac{\xi}{\omega-1}\right) > 0$. Then*

$$\begin{aligned}\left(P_{0+}^{\xi,\omega}\ t^{\mu-1}{}_hJ_\nu(t)\right)(x) &= \frac{x^{\xi+\mu}}{[-p(\omega-1)]^{\mu+\nu}}\left(\frac{x}{2}\right)^{\nu}\Gamma\left(1-\frac{\xi}{\omega-1}\right)\\ &\times\ {}_1\Psi_2\left[\left.\begin{matrix}(\mu+\nu,2)\\(1+\nu,h),(1+\mu+\nu-\frac{\xi}{\omega-1},2)\end{matrix}\right| -\frac{x^2}{4[-p(\omega-1)]^2}\right].\end{aligned} \tag{20}$$

Proof. Proof of Theorem 2.3 is similar as given in Theorem 2.2. □

2.1. Representations in terms of product of two functions

In this section, we establish some results in terms of product of two functions using Hadamard product of power series defined in (7).

Theorem 2.4. *Let $\xi, \mu, \nu \in \mathbf{C}$, $h > 0$, $p \in \mathbf{R}^+$ and $\omega < 1$, such that $\Re(\xi) > 0$, $\Re(\mu) > 0$, $\Re(\nu) > -1$, $\Re(\mu+\nu) > 0$, $\Re\left(\frac{\xi}{1-\omega}\right) > -1$. Then*

$$\begin{aligned}\left(P_{0+}^{\xi,\omega}\ t^{\mu-1}{}_hJ_\nu(t)\right)(x) &= \frac{x^{\xi+\mu}}{[p(1-\omega)]^{\mu+\nu}}\left(\frac{x}{2}\right)^{\nu}\Gamma\left(1+\frac{\xi}{1-\omega}\right)\\ &\times J_\nu^h\left(\frac{x^2}{4[p(1-\omega)]^2}\right)\cdot{}_2\Psi_1\left[\left.\begin{matrix}(1,1),(\mu+\nu,2)\\(1+\mu+\nu+\frac{\xi}{1-\omega},2)\end{matrix}\right|\frac{x^2}{4[p(1-\omega)]^2}\right].\end{aligned} \tag{21}$$

Proof. Let $\mathcal{P}_2$ be the left-hand side of (21). Then, using (3), we find that

$$\mathcal{P}_2 = \left(P_{0+}^{\xi,\omega}\ t^{\mu-1}\sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma(1+\nu+hm)m!}\left(\frac{t}{2}\right)^{\nu+2m}\right)(x).$$

On reversing the order of summation and integration, we obtain

$$\mathcal{P}_2 = \sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma(1+\nu+hm)m!}\left(\frac{1}{2}\right)^{\nu+2m}\left(P_{0+}^{\xi,\omega}\ t^{\mu+\nu+2m-1}\right)(x).$$

Using Lemma 2.1, we arrive at

$$\begin{aligned}\mathcal{P}_2 &= \sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma(1+\nu+hm)m!}\left(\frac{1}{2}\right)^{\nu+2m}\\ &\times\frac{x^{\xi+\mu+\nu+2m}}{[p(1-\omega)]^{\mu+\nu+2m}}\frac{\Gamma\left(\frac{\xi}{1-\omega}+1\right)\Gamma(\mu+\nu+2m)}{\Gamma\left(1+\mu+\nu+2m+\frac{\xi}{1-\omega}\right)}.\end{aligned} \tag{22}$$

By applying the Hadamard product (7) in (22) and in the view of (2) and (4), this leads to the right-hand side of (21). □

Now, one can prove the following theorem for $\omega > 1$ and using (12).

Theorem 2.5. *Let $\xi, \mu, \nu \in \mathbf{C}$, $h > 0$, $p \in \mathbf{R}^+$ and $\omega > 1$, such that $\Re(\xi) > 0$, $\Re(\mu) > 0$, $\Re(\nu) > -1$, $\Re(\mu+\nu) > 0$, $\Re\left(1 - \frac{\xi}{\omega-1}\right) > 0$. Then*

$$\begin{aligned}\left(P_{0+}^{\xi,\omega}\, t^{\mu-1}{}_hJ_\nu(t)\right)(x) &= \frac{x^{\xi+\mu}}{[-p(\omega-1)]^{\mu+\nu}}\left(\frac{x}{2}\right)^{\nu}\Gamma\left(1-\frac{\xi}{\omega-1}\right)\\ &\times J_\nu^h\left(\frac{x^2}{4[-p(\omega-1)]^2}\right)\cdot {}_2\Psi_1\left[\left.\begin{matrix}(1,1),(\mu+\nu,2)\\ (1+\mu+\nu-\frac{\xi}{\omega-1},2)\end{matrix}\right|\frac{x^2}{4[-p(\omega-1)]^2}\right].\end{aligned} \tag{23}$$

Proof. Proof of Theorem 2.5 is similar as given in Theorem 2.4. □

3. $\mathcal{P}$-transform or pathway transform of ${}_hJ_\nu(z)$

In this section, we determine the pathway transform of generalized Bessel function (3) in terms of Fox H-function defined in (8) and (9).

The $\mathcal{P}$-transform or pathway transform is a generalization of Krätzel transform and of many existing integral transforms. In 2011, Kumar [5] gave the pathway transform which is obtained by using the pathway model of Mathai [7] and further developed by Mathai and Haubold [8] which is represented as,

$$(\mathcal{P}_v^{\omega,\delta,\gamma} f)(x) = \int_0^\infty \mathcal{D}_{\omega,\delta}^{v,\gamma}(xt)\ f(t)\ dt,\ x > 0, \tag{24}$$

where $\mathcal{D}_{\omega,\delta}^{v,\gamma}(x)$ is given as

$$\mathcal{D}_{\omega,\delta}^{v,\gamma}(x) = \int_0^\infty u^{v-1}[1 + a(\gamma-1)u^\omega]^{-\frac{1}{\gamma-1}} e^{-xu^{-\delta}} du,\ x > 0, \tag{25}$$

where $v \in \mathbf{C}$, $a > 0$, $\delta > 0$ and $\omega \in \mathbf{R}$, $\omega \neq 0$, $\gamma > 1$. In this case, we say that (24) is a type-2 $\mathcal{P}$-transform. If we use kernel function

$$\mathcal{D}_{\omega,\delta}^{v,\gamma}(x) = \int_0^{[\frac{1}{a(1-\gamma)}]^{\frac{1}{\omega}}} u^{v-1}[1 - a(1-\gamma)u^\omega]^{\frac{1}{1-\gamma}} e^{-xu^{-\delta}} du,\ x > 0, \tag{26}$$

with $v \in \mathbf{C}$, $a > 0$, $\delta > 0$ and $\omega > 0$, $\gamma < 1$ then we say that (24) is type-1 $\mathcal{P}$-transform. The $\mathcal{P}$-transform of both type-1 and type-2 are defined in the space $\mathrm{L}_{v,r}$ consisting of the Lebesgue measurable complex-valued functions f for which

$$\|f\|_{v,r} = \left\{\int_0^\infty |t^v f(t)|^r\right\}^{\frac{1}{r}} < \infty \tag{27}$$

for $1 \leq r < \infty, v \in \mathbf{R}$.

When $\delta = 1, a = 1$ and $\gamma \to 1$, one can observe that

$$\lim_{\gamma\to 1} \mathcal{D}_{\omega,1}^{v,\gamma} = Z_\omega^v(x), \tag{28}$$

where $Z_\omega^v(x)$ is the kernel function of the Krätzel transform, introduced by Krätzel [4] which is defined below as,

$$\mathcal{K}_v^\omega f(x) = \int_0^\infty Z_\omega^v(xt)\ f(t)\ dt,\ x > 0, \tag{29}$$

where

$$Z_\omega^v(x) = \int_0^\infty u^{v-1} e^{-u^\omega - xu^{-1}} du. \tag{30}$$

Here, we find the type-2 $\mathcal{P}$-transform of generalized Bessel function by using the following lemma given in [5].

Lemma 3.1. *Let $v, \mu \in \mathbf{C}$, $a > 0$, $\delta > 0$, $\omega \in \mathbf{R}$, $\omega \neq 0$, $\gamma > 1$ and $\Re(\mu) > 0$, such that $\Re(v + \delta\mu) > 0$ and $\Re(1/(\gamma - 1) - (v + \delta\mu)/\omega) > 0$ when $\omega > 0$, $\Re(v + \delta\mu) < 0$ and $\Re(1/(\gamma - 1) - (v + \delta\mu)/\omega) > 0$ when $\omega < 0$. Then type-2 $\mathcal{P}$-transform of power function is given by*

$$(\mathcal{P}_v^{\omega,\delta,\gamma} x^{\mu-1})(x) = \frac{\Gamma(\mu)\Gamma(\frac{v+\delta\mu}{\omega})\Gamma(\frac{1}{\gamma-1} - \frac{v+\delta\mu}{\omega})}{|\omega| x^\mu [a(\gamma-1)]^{\frac{v+\delta\mu}{\omega}} \Gamma(\frac{1}{\gamma-1})}. \tag{31}$$

Remark 3.2. If $a = 1, \delta = 1$ with existing conditions of Lemma 3.1, then

$$\lim_{\gamma \to 1} (\mathcal{P}_v^{\omega,\delta,\gamma} x^{\mu-1})(x) = (\mathcal{K}_\omega^v x^{\mu-1})(x) = \frac{\Gamma(\mu)\Gamma(\frac{v+\mu}{\omega})}{|\omega| x^\mu}, \tag{32}$$

where $\Re(\mu) > 0$ and $\Re(v + \mu) > 0$ when $\omega > 0$ and $\Re(v + \mu) < 0$ when $\omega < 0$.

Theorem 3.3. *Let $v, \mu \in \mathbf{C}$, $a > 0$, $\delta > 0$, $\omega \in \mathbf{R}$, $\omega \neq 0$, $\gamma > 1, b > 0$ and $\Re(\mu + \tau(\nu - 2s)) > 0$, such that $\Re(v + \delta(\mu + \tau(\nu - 2s))) > 0$ and $\Re(1/(\gamma-1) - (v+\delta(\mu+\tau(\nu-2s)))/\omega) > 0$ when $\omega > 0$, $\Re(v+\delta(\mu+\tau(\nu-2s))) < 0$ and $\Re(1/(\gamma - 1) - (v + \delta(\mu + \tau(\nu - 2s)))/\omega) < 0$ when $\omega < 0$. Then, type-2 $\mathcal{P}$-transform of generalized Bessel function* (3) *is given by,*
if $\omega > 0$,

$$\begin{aligned}
[\mathcal{P}_v^{\omega,\delta,\gamma} x^{\mu-1} {}_hJ_\nu(bx^\tau)](x) &= \frac{1}{\omega x^{\mu+\nu\tau} [a(\gamma-1)]^{\frac{v+\delta(\mu+\tau\nu)}{\omega}} \Gamma(1/(\gamma-1))} \left(\frac{b}{2}\right)^\nu \\
&\times H_{2,3}^{2,2}\left[\frac{b^2}{4x^{2\tau}[a(\gamma-1)]^{\frac{2\delta\tau}{\omega}}} \middle| \begin{matrix} (1-\mu-\nu\tau, 2\tau), \left(\frac{\omega-v-\delta(\mu+\nu\tau)}{\omega}, \frac{2\delta\tau}{\omega}\right) \\ (0,1), (-\nu, h), \left(\frac{1}{\gamma-1} - \frac{v+\delta(\mu+\nu\tau)}{\omega}, \frac{2\delta\tau}{\omega}\right) \end{matrix}\right]
\end{aligned} \tag{33}$$

and if $\omega < 0$

$$\begin{aligned}
[\mathcal{P}_v^{\omega,\delta,\gamma} x^{\mu-1} {}_hJ_\nu(bx^\tau)](x) &= \frac{1}{\omega x^{\mu+\nu\tau} [a(\gamma-1)]^{\frac{v+\delta(\mu+\tau\nu)}{\omega}} \Gamma(1/(\gamma-1))} \left(\frac{b}{2}\right)^\nu \\
&\times H_{2,3}^{2,2}\left[\frac{b^2}{4x^{2\tau}[a(\gamma-1)]^{\frac{2\delta\tau}{\omega}}} \middle| \begin{matrix} (1-\mu-\nu\tau, 2\tau), \left(1 - \frac{1}{\gamma-1} + \frac{v+\delta(\mu+\nu\tau)}{\omega}, -\frac{2\delta\tau}{\omega}\right) \\ (0,1), (-\nu, h), \left(\frac{v+\delta(\mu+\nu\tau)}{\omega}, -\frac{2\delta\tau}{\omega}\right) \end{matrix}\right].
\end{aligned} \tag{34}$$

Proof. Let $\omega > 0$. Then from (24) and (25), we have

$$\begin{aligned}
[\mathcal{P}_v^{\omega,\delta,\gamma} x^{\mu-1} {}_hJ_\nu(bx^\tau)](x) &= \int_0^\infty \mathcal{D}_{\omega,\delta}^{v,\gamma}(xt)\ t^{\mu-1} \\
&\times \frac{1}{2\pi i} \int_L \frac{\Gamma(s)}{\Gamma(1+\nu-sh)} \left(\frac{bt^\tau}{2}\right)^{\nu-2s} ds\ dt.
\end{aligned} \tag{35}$$

Further simplification yields,

$$[\mathcal{P}_v^{\omega,\delta,\gamma} x^{\mu-1} {}_h J_\nu(bx^\tau)](x) = \frac{1}{2\pi i}\int_L \frac{\Gamma(s)}{\Gamma(1+\nu-sh)}\left(\frac{b}{2}\right)^{\nu-2s}$$

$$\times\int_0^\infty \mathcal{D}_{\omega,\delta}^{v,\gamma}(xt)\, t^{\mu+\tau(\nu-2s)-1} dt\, ds. \tag{36}$$

Using Lemma 3.1, we obtain

$$[\mathcal{P}_v^{\omega,\delta,\gamma} x^{\mu-1} {}_h J_\nu(bx^\tau)](x) \tag{37}$$

$$= \frac{1}{2\pi i}\int_L \frac{\Gamma(s)}{\Gamma(1+\nu-sh)}\left(\frac{b}{2}\right)^{\nu-2s}$$

$$\times \frac{\Gamma(\mu+\tau(\nu-2s))\Gamma\left(\frac{v+\delta(\mu+\tau(\nu-2s))}{\omega}\right)\Gamma\left(\frac{1}{\gamma-1}-\frac{v+\delta(\mu+\tau(\nu-2s))}{\omega}\right)}{\omega\, x^{\mu+\tau(\nu-2s)}[a(\gamma-1)]^{\frac{v+\delta(\mu+\tau(\nu-2s))}{\omega}}\Gamma(1/(\gamma-1))},$$

with the help of (8) and (9), this leads to the right-hand side of (33). Similarly, we can prove the result (34) for $\omega < 0$. $\square$

Corollary 3.4. *If $a = 1, \delta = 1$ with existing conditions of Theorem 3.3, then Krätzel transform of generalized Bessel function* (3) *is given by*

$$\lim_{\gamma\to 1}[\mathcal{P}_v^{\omega,\delta,\gamma} x^{\mu-1} {}_h J_\nu(bx^\tau)](x) \tag{38}$$

$$= (\mathcal{K}_\omega^v x^{\mu-1} {}_h J_\nu(bx^\tau))(x)$$

$$= \frac{1}{\omega x^{\mu+\nu\tau}}\left(\frac{b}{2}\right)^\nu H_{2,2}^{2,2}\left[\frac{b^2}{4x^{2\tau}}\left|\begin{matrix}(1-\mu-\nu\tau,2\tau),\left(\frac{\omega-v-(\mu+\nu\tau)}{\omega},\frac{2\tau}{\omega}\right)\\ (0,1),(-\nu,h)\end{matrix}\right.\right]$$

for $\omega > 0$ and

$$\lim_{\gamma\to 1}[\mathcal{P}_v^{\omega,\delta,\gamma} x^{\mu-1} {}_h J_\nu(bx^\tau)](x) \tag{39}$$

$$= (\mathcal{K}_\omega^v x^{\mu-1} {}_h J_\nu(bx^\tau))(x)$$

$$= \frac{1}{\omega x^{\mu+\nu\tau}}\left(\frac{b}{2}\right)^\nu H_{1,3}^{2,2}\left[\frac{b^2}{4x^{2\tau}}\left|\begin{matrix}(1-\mu-\nu\tau,2\tau)\\ (0,1),(-\nu,h),\left(\frac{v+(\mu+\nu\tau)}{\omega},-\frac{2\tau}{\omega}\right)\end{matrix}\right.\right]$$

for $\omega < 0$.

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References

[1] L. Galué, *A generalized Bessel function*, Integral Transforms Spec. Funct. **14** (2003), no. 5, 395–401. https://doi.org/10.1080/1065246031000074362

[2] A. A. Kilbas and M. Saigo, *H-transforms*, Analytical Methods and Special Functions, 9, Chapman & Hall/CRC, Boca Raton, FL, 2004. https://doi.org/10.1201/9780203487372

[3] V. Kiryakova, *On two Saigo's fractional integral operators in the class of univalent functions*, Fract. Calc. Appl. Anal. **9** (2006), no. 2, 159–176.
[4] E. Krätzel, *Integral transformations of Bessel type*, Proceeding of International Conference on Generalized Functions and Operational Calculus, Varna (1975), 148–155.
[5] D. Kumar, *$\mathcal{P}$-transforms*, Integral Transforms Spec. Funct. **22** (2011), no. 8, 603–616. https://doi.org/10.1080/10652469.2010.536410
[6] A. M. Mathai, *A handbook of generalized special functions for statistical and physical sciences*, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1993.
[7] ———, *A pathway to matrix-variate gamma and normal densities*, Linear Algebra Appl. **396** (2005), 317–328. https://doi.org/10.1016/j.laa.2004.09.022
[8] A. M. Mathai and H. J. Haubold, *Pathway model, superstatistics, Tsallis statistics, and a generalized measure of entropy*, Phys. A **375** (2007), no. 1, 110–122. https://doi.org/10.1016/j.physa.2006.09.002
[9] ———, *On generalized distributions and path-ways*, Phys. Lett. A **372**(2008), 2109–2113.
[10] A. M. Mathai and R. K. Saxena, *The H-Function with Applications in Statistics and Other Disciplines*, Halsted Press, New York, 1978.
[11] A. M. Mathai, R. K. Saxena, and H. J. Haubold, *The H-Function*, Springer, New York, 2010. https://doi.org/10.1007/978-1-4419-0916-9
[12] S. S. Nair, *Pathway fractional integration operator*, Fract. Calc. Appl. Anal. **12** (2009), no. 3, 237–252.
[13] T. Pohlen, *The Hadamard product and universal power series*, Ph.D. Thesis, Universität Trier, Trier, Germany, 2009.
[14] E. D. Rainville, *Special Functions*, The Macmillan Co., New York, 1960.
[15] E. M. Wright, *On the Coefficients of Power Series Having Exponential Singularities*, J. London Math. Soc. **8** (1933), no. 1, 71–79. https://doi.org/10.1112/jlms/s1-8.1.71
[16] ———, *The Asymptotic Expansion of the Generalized Bessel Function*, Proc. London Math. Soc. (2) **38** (1935), 257–270. https://doi.org/10.1112/plms/s2-38.1.257

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