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ANNIHILATING PROPERTY OF ZERO-DIVISORS

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Abstract. We discuss the condition that every nonzero right annihilator of an element contains a nonzero ideal, as a generalization of the insertion-of-factors-property. A ring with such condition is called *right AP*. We prove that a ring R is right AP if and only if $D_n(R)$ is right AP for every $n \geq 2$, where $D_n(R)$ is the ring of n by n upper triangular matrices over R whose diagonals are equal. Properties of right AP rings are investigated in relation to nilradicals, prime factor rings and minimal order.

Throughout this note every ring is an associative ring with identity unless otherwise stated. Let R be a ring. We denote the center, the group of all units, and the set of all idempotents of R by $Z(R)$, $U(R)$, and $I(R)$, respectively. A nilpotent element is also said to be a *nilpotent* for short. We use $N(R)$, $J(R)$, $N_*(R)$, $N^*(R)$ and $W(R)$ to denote the set of all nilpotents, Jacobson radical, lower nilradical (i.e., prime radical), upper nilradical (i.e., the sum of all nil ideals) and the Wedderburn radical (i.e., the sum of all nilpotent ideals) of R, respectively. It is well-known that

$$W(R) \subseteq N_*(R) \subseteq N^*(R) \subseteq N(R) \text{ and } N^*(R) \subseteq J(R).$$

The polynomial (resp., power series) ring with an indeterminate x over R is denoted by $R[x]$ (resp., $R[[x]]$). $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n). Use $\mathbb{Q}$ for the field of rational numbers. Denote the n by n ($n \geq 2$) full (resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $T_n(R)$). Write $D_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$ and $N_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{ii} = 0 \text{ for all } i\}$. Use E_{ij} for the matrix with (i, j)-entry 1 and zeros elsewhere. I_n denotes the identity matrix in $Mat_n(R)$. An element u of R is right (resp., left) regular if $ur = 0$ (resp., $ru = 0$) for $r \in R$ implies $r = 0$. An element is regular if it is both left and right regular. The monoid of all regular elements in R is denoted by $C(R)$. The right (resp., left) annihilator of a in R is written by $r_R(a)$ (resp., $l_R(a)$), where $a \in R$.

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1. Introduction

In this section, we discuss a ring property that left zero divisors are annihilated by nonzero ideals. Due to Bell [1], a ring R (possibly without identity) is called *IFP* if $ab = 0$ for $a, b \in R$ implies $aRb = 0$, i.e., R satisfies the insertion-of-factors-property. Following the literature, a ring (possibly without identity) is called *reduced* if it contains no nonzero nilpotents; and a ring (possibly without identity) is called *Abelian* if every idempotent is central. A ring R is usually called *directly finite* (or *Dedekind finite*) if $ab = 1$ for $a, b \in R$ implies $ba = 1$. It is easily proved that reduced rings are IFP, IFP rings are Abelian and Abelian rings are directly finite. We will freely use the preceding facts. The following is evident from definition.

Remark 1.1. (1) For a ring R the following conditions are equivalent:
(i) R is IFP;
(ii) If $ab = 0$ for $a, b \in R$, then $aRbR = 0$;
(iii) If $ab = 0$ for $a, b \in R$, then $RaRb = 0$.
(2) If R is an IFP ring, then $W(R) = N_*(R) = N^*(R) = N(R)$ and R is Abelian.

The aim of this article is to study the structure of rings R when $aI = 0$ (resp., $Jb = 0$) for some nonzero ideals I (resp., J) in place of RbR (resp., RaR), in Remark 1.1. For our purpose, we first observe the following example.

Example 1.2. (1) Set

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mat_2(\mathbb{Z}_{16}) \,\middle|\, a + d, b, c \in 2\mathbb{Z} \right\},$$

which is a subring of $Mat_2(\mathbb{Z}_{16})$. First we see that R is not IFP. For, $\left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right) = 0$ but $\left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 8 \\ 0 & 0 \end{smallmatrix}\right) \neq 0$.

Let $I = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Mat_2(\mathbb{Z}_{16}) \mid a, b, c, d \in 2\mathbb{Z} \right\}$. Then I is an ideal of R such that $I^4 = 0$. Observing

$$R/I \cong \{(s, t) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \mid s + t \in 2\mathbb{Z}\} = \{(0, 0), (1, 1)\},$$

we see that $W(R) = N_*(R) = N^*(R) = N(R)$, and that if $\alpha\beta = 0$ for $\alpha, \beta \in R$, then $\alpha, \beta \in I$. Therefore $\alpha I^3 = 0$ and $I^3\beta = 0$, noting $I^3 = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Mat_2(\mathbb{Z}_{16}) \mid a, b, c, d \in 8\mathbb{Z} \right\} \neq 0$.

(2) The IFP property does not pass to polynomial rings by [6, Example 2], but the polynomial rings over IFP rings are able to satisfy a weaker property as follows.

Let R be an IFP ring and suppose that $f(x)g(x) = 0$ for $0 \neq f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$. Then, by [12, Lemma 3], there exist nonnegative integers $l_0, l_1, \ldots, l_n$ that satisfy, for each $k \in \{0, 1, \ldots, n\}$,

$$f(x)b_k^{l_k} b_{k-1}^{l_{k-1}} \cdots b_0^{l_0} \neq 0 \text{ and } f(x)b_k^{l_k+1} b_{k-1}^{l_{k-1}} \cdots b_0^{l_0} = 0.$$

Next we apply the proof of [12, Theorem 4]. Let $y = b_n^{l_n} b_{n-1}^{l_{n-1}} \cdots b_0^{l_0}$. Then $y \neq 0$. Assume $g(x)y \neq 0$. Then $b_s b_n^{l_n} b_{n-1}^{l_{n-1}} \cdots b_0^{l_0} \neq 0$ for some $s \in \{0, 1, \ldots, n\}$, y' say. But $f(x) b_s^{l_s+1} b_{s-1}^{l_{s-1}} \cdots b_0^{l_0} = 0$ by the argument above, from which we infer that

$$f(x)y' = f(x)b_s(b_n^{l_n} \cdots b_{s+1}^{l_{s+1}})b_s^{l_s} b_{s-1}^{l_{s-1}} \cdots b_0^{l_0} = 0$$

by the IFPness of R. Whence we have $f(x)Ry'R = 0$ by using the IFPness of R again. Note $Ry'R \neq 0$.

Following Hwang *et al.* [8], a ring R is *strongly right* (*resp., left*) *AB* if every nonzero right (resp., left) annihilator of R contains a nonzero ideal of R. As a generalization of this, we next consider the ring property below.

A ring R (possibly without identity) will be said to satisfy the *right annihilating property* (simply, is said to be *right AP*) provided that if $ab = 0$ for $a, 0 \neq b \in R$, then $aI = 0$ for some nonzero ideal I of R; equivalently, if $r_R(a) \neq 0$, then $r_R(a)$ contains a nonzero ideal of R. Left AP rings are defined by symmetry. As we see later, this new concept is not left-right symmetric. A ring is called *AP* if it is both right and left AP. IFP rings are clearly AP, but the converse need not hold by the ring R in Example 1.2(1). We construct another kind of such ring in Theorem 2.1 to follow. Recall that IFP rings are Abelian, but right AP rings need not be *Abelian* as we see in Remark 1.5(1) to follow.

Lemma 1.3. (1) [10, Proposition 2.8] *A ring R is reduced if and only if $D_3(R)$ is IFP.*

(2) [11, Example 1.3] *$D_n(R)$ is not an IFP ring for $n \geq 4$ over any ring R.*

(3) *The class of IFP rings is closed under subrings and direct products.*

Proof. (3) is clear from definition. □

Strongly right AB rings are clearly right AP, however we do not know of any example of a right AP ring that is not strongly right AB.

Question. If R is a right AP ring, then is R strongly right AB?

The following contains basic properties of right AP rings.

Lemma 1.4. (1) *The class of right* (*left*) *AP rings is closed under direct products.*

(2) *If a prime ring R is right* (*left*) *AP, then R is a domain.*

(3) *Right* (*left*) *AP rings are directly finite.*

(4) *A ring R is right* (*resp., left*) *AP if and only if $r_R(a) \neq 0$* (*resp., $l_R(a) \neq 0$*) *for $a \in R$ implies $aRc = 0$* (*resp., $cRa = 0$*) *for some $0 \neq c \in R$.*

Proof. (1) Let R_i be rings for $i \in I$ and $R = \prod_{i \in I} R_i$ be the direct product of R_i's. Suppose that every R_i is right AP and $(a_i)(b_i) = 0$ for $(a_i), 0 \neq (b_i) \in R$. Let $I_0 \subseteq I$ such that $b_j \neq 0$ for all $j \in I_0$. Then since $a_j b_j = 0$ for all $j \in I_0$, $a_j K_j = 0$ for some nonzero ideal K_j of R_j. Write $L = \prod_{i \in I} L_i$ where $L_i = K_i$

for $i \in I_0$ and $L_i = 0$ for $i \in I \backslash I_0$ (if any). Then L is a nonzero ideal of R such that $(a_i)L = 0$. Thus R is right AP.

(2) Let R be a prime right AP ring. Suppose $ab = 0$ for $a, b \in R$. Assume $a, b \neq 0$. Then since R is right AP, $aI = 0$ for some nonzero ideal I of R. Since R is prime and $I \neq 0$, $aRI = aI = 0$ implies $a = 0$, contradicting $a \neq 0$. Thus R is a domain. The left case is similarly proved

(3) Let R be a right AP ring and suppose that $ab = 1$ for $a, b \in R$. We apply the proof of [8, Proposition 3.15]. Assume on the contrary that $ba \neq 1$. Then since R is right AP and $ba(1 - ba) = 0$ with $1 - ba \neq 0$, we see that $baRc = 0$ for some $0 \neq c \in R$. This yields $c = a(babc) \in a(baRc) = 0$, a contradiction. Thus $ba = 1$.

(4) This is evident from definition.

The proofs of the left cases of the results above are similar. □

By Lemma 1.4(2), we can obtain the following results. Let R be a ring such that $N_*(R) = N(R)$, i.e., $R/N_*(R)$ is reduced (e.g., IFP rings). Then every prime factor ring of R is a domain. As a corollary of Lemma 1.4(3), we can obtain that strongly one-sided AB rings are directly finite [8, Proposition 3.15].

The following elaborates upon Lemma 1.4. Especially, the ring below shows that the AP property is not left-right symmetric and that right AP rings need not be Abelian.

Remark 1.5. (1) There exist right AP rings which are neither Abelian nor left AP. One can see such a ring in [8, Example 2.5(4)], but we provide a ring of other kind. Let $A = \mathbb{Z}_2\langle a, b\rangle$ be the free algebra generated by noncommuting indeterminates a, b over $\mathbb{Z}_2$. Consider the ideal G of A generated by

$$b^2 - b, a^2 \text{ and } abab.$$

Next set $R = A/G$. Identity elements in A with their images in R for simplicity. Then since $a^2 = 0$ and $abab = 0$, we see $(RaR)^3 = 0$. Observing $R/(RaR) \cong \mathbb{Z}_2 + \mathbb{Z}_2 b = \{0, 1, b, 1 + b\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, we see $W(R) = RaR = N(R)$. Moreover $b^2 = b$ and $ab \neq ba$, so that R is not Abelian.

Every element of R is expressed by

$$h_0 + h_1 b + h_2 a + f_1 b + f_2 a,$$

where $h_i \in \mathbb{Z}_2$, $f_i \in R$ such that every term of f_i is of degree ≥ 1, every term of f_1 contains a when nonzero, and every term of f_2 contains b when nonzero.

Note that $h_2 a + f_1 b + f_2 a \in N(R) = RaR$ in the preceding expression of elements, and hence $1 + h_2 a + f_1 b + f_2 a \in U(R)$. However

$$(1 + b + h_2 a + f_1 b + f_2 a)R(baba)R = 0,$$

observing $R(baba)R = \{0, baba\} \neq 0$. Furthermore, we have

$$(b + h_2 a + f_1 b + f_2 a)R((1 + b)aba)R = 0,$$

observing $R((1 + b)aba)R = \{0, (1 + b)aba\} \neq 0$.

Therefore R is a right AP ring that is not Abelian.

We claim that R is not left AP. Assume that $fR(1+b)=0$ for some $f=h_0+h_1b+h_2a+f_1b+f_2a\in R$. Then $h_0=0$ and $h_2=0$ through a simple computation, entailing $f=h_1b+f_1b+f_2a$. From $0=f(1+b)=f_2a(1+b)=f_2a+f_2ab$, we obtain $f_2a=0$, entailing $f=h_1b+f_1b$. Note that $f_1=k_1ab+k_2bab$ with $k_i\in\mathbb{Z}_2$. So, from

$$\begin{aligned}0&=fa(1+b)=(h_1b+k_1ab+k_2bab)a(1+b)\\&=h_1ba+h_1bab+k_1aba+k_1abab+k_2baba+k_2babab\\&=h_1ba+h_1bab+k_1aba+k_2baba,\end{aligned}$$

we decide that $h_1=0,k_1=0,k_2=0$. This implies $f=0$. Hence R is not left AP because $b(1+b)=0$.

Next letting G' be the ideal of A that is generated by

$$b^2-b,a^2\ \text{ and }\ baba,$$

we can prove that the factor ring A/G' is left AP but not right AP through a similar computation. Clearly, A/G' is also non-Abelian.

(2) From Lemma 1.4(2), we infer that for a prime ring R, R is right (left) AP if and only if R is a domain if and only if R is reduced if and only if R is IFP.

(3) Prime factor rings of IFP rings need not be right (left) AP. To see that, we apply the argument in [5, Example 3]. Let p be an odd prime and R_0 be the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$ of $\mathbb{Z}$. Set R be the quaternions over R_0. Then R is clearly a domain (hence IFP), but R/pR is isomorphic to $Mat_2(\mathbb{Z}_p)$ by the argument in [4, Exercise 2A]. $Mat_2(\mathbb{Z}_p)$ is neither right nor left AP by Lemma 1.4(2). Note that pR is a maximal ideal of R. Consequently, the class of right AP rings is not closed under prime factor rings.

(4) There exist many directly finite rings which are neither right nor left AP. For example, $T_n(R)$ and $Mat_n(R)$ over any finite ring R for $n\geq 2$ (refer to Example 2.2 to follow), and right Noetherian prime rings which are not domains (refer to Lemma 1.4(2) and [9, Theorem 1]). Here we provide another kind of such ring. Let $A=\mathbb{Z}_2\langle a,b\rangle$ be the free algebra generated by noncommuting indeterminates a,b over $\mathbb{Z}_2$. Consider the ideal H of A generated by

$$b^2-b,a-ba,a^2\ \text{ and }\ ab.$$

Next set $R=A/H$. In the argument below, we identity elements in A with their images in R for simplicity. Since $a^2=0$ and $aba=0$, we see $(RaR)^2=0$. Observing $R/(RaR)\cong\mathbb{Z}_2+\mathbb{Z}_2b\cong\mathbb{Z}_2\times\mathbb{Z}_2$, we see $W(R)=RaR=N(R)$.

Every element of R is expressed by

$$h_0+h_1b+h_2a,$$

where $h_i\in\mathbb{Z}_2$. Note $b(1+b)=0$ and suppose that $bI=0$ for some nonzero ideal I of R. Let $0\neq f=h_0+h_1b+h_2a\in I$. Then, from $bf=0$, we get $(h_0+h_1)b+h_2a=0$, hence $h_0+h_1=h_2=0$, so that $f=h_0+h_1b$. But $f\neq 0$, hence $h_0=h_1=1$, that is $f=1+b$, from which we infer

$0 \neq a = a + ab = af = b(af) \in bI = 0$, a contradiction. Thus R is not right AP.

Next suppose that $J(1+b) = 0$ for some nonzero ideal J of R. Let $0 \neq g = k_0 + k_1 b + k_2 a \in J$. Then, from $g(1+b) = 0$, we get $k_0 + k_0 b + k_2 a = 0$, hence $k_0 = k_2 = 0$. But $g \neq 0$, hence $k_1 = 1$, so that $g = b$, from which we infer that $0 \neq a = ba(1+b) = ga(1+b) \in J(1+b) = 0$, a contradiction. Thus R is not left AP.

However R is clearly a finite ring, hence directly finite.

(5) There exists an AP ring without identity that is not Abelian. Let $A = \mathbb{Z}_2\langle a, b\rangle$ be the free algebra generated by noncommuting indeterminates a, b over $\mathbb{Z}_2$. Consider the ideal K of A generated by

$$b^2 - b, a - ab, a^2 \text{ and } ba.$$

Next set $R = A/K$. In the argument below, we identity elements in A with their images in R for simplicity. Through a similar argument to one of (3), we see that $(RaR)^2 = 0$, $R/(RaR) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, and $W(R) = RaR = N(R)$. Next set

$$S = \{f \in R \mid \text{the constant term of } f \text{ is zero}\}.$$

Then every element of S is expressed by $h_1 b + h_2 a$, where $h_i \in \mathbb{Z}_2$. Suppose that $fg = 0$ for $0 \neq f = h_1 b + h_2 a, g = k_1 b + k_2 a \in S$. Then $h_1 k_1 = 0$, so that $h_1 = 0$ or $k_1 = 0$.

Let $h_1 = 1$. Then $k_1 = 0$, so that $f = b + k_2 a$ and $g = a$. Letting J be the ideal of R generated by a, we have $fJ = (b + k_2 a)J = 0$.

Let $h_1 = 0$. Then $h_2 = 1$, so that $f = a$ and $g = a$. We also have $fJ = 0$. Thus R is right AP, but non-Abelian as can be seen by $b \in I(R)$ and $ab = a \neq 0 = ba$.

Moreover R is left AP. For, $Ja = 0$ in any case of the preceding argument, whence R is left AP.

(6) The class of right (left) AP rings is not closed under direct limits as the ring below shows, which is compared with Lemma 1.4(1). There exist a chain of AP rings whose direct limit is neither right nor left AP. We apply the construction of [7, Example 1.2]. Let A be a domain and consider the rings $D_{2^n}(A)$ for $n = 1, 2, \ldots$. Define a map $\sigma : D_{2^n}(A) \to D_{2^{n+1}}(A)$ by $M \mapsto \left(\begin{smallmatrix} M & 0 \\ 0 & M \end{smallmatrix}\right)$. Then $D_{2^n}(A)$ can be considered as a subring of $D_{2^{n+1}}(A)$ via σ (i.e., $M = \sigma(M)$ for $M \in D_{2^n}(A)$). Set R be the direct limit of the direct system (D_n, σ_{ij}), where $\sigma_{ij} = \sigma^{j-i}$. Then $R = \cup_{i=1}^{\infty} D_{2^n}(A)$. Every $D_{2^n}(A)$ is AP by Theorem 2.1, and R can be shown to be a prime ring by the proof of [7, Proposition 1.3]. So if R is right or left AP, then R is a domain by Lemma 1.4(2), contrary to R being not a domain. Therefore R is neither right nor left AP.

The class of right AP rings is not closed under subrings by [8, Example 3.2]. But the right AP property passes to the following kind of subring.

Proposition 1.6. *Let R be a right AP ring and $e \in I(R)$ be such that $eIe \neq 0$ for any nonzero ideal I of R. Then eRe is right AP.*

Proof. Let $ab = 0$ for $0 \neq a, b \in eRe$. Since R is right AP, there is a nonzero ideal I of R such that $aI = 0$. Then $eIe \neq 0$ by hypothesis, and $aI = 0$ implies $aeIe = eaeIe = 0$. But eIe is an ideal of eRe, hence eRe is right AP. □

The preceding proposition is similar to [8, Proposition 4.1]. We next consider some properties of semiprime right AP rings. Recall that an ideal I of a ring R is called *essential* if $I \cap J \neq 0$ for any nonzero ideal J of R.

Proposition 1.7. (1) *Let R be a semiprime right AP ring and $a^2 = 0$ for $a \in R$. Then we have the following.*

(i) *$r_R(a)$ does not contain any nonzero ideal that is contained in RaR.*

(ii) *RaR is a non-essential ideal of R.*

(2) *Let R be a semiprime right AP ring and $a \in R$. If RaR is an essential ideal of R, then $a \notin N(R)$.*

Proof. (1) If $a = 0$, then we are done. So suppose $0 \neq a \in R$. Since R is right AP, $r_R(a)$ contains a nonzero ideal I of R. Then $aI = 0$, i.e., $I \subseteq r_R(a)$. Consider $RaR \cap I$. Since R is semiprime, $(RaR \cap I)^2 \subseteq RaRI = 0$ implies $RaR \cap I = 0$. This completes the proof of (i). The preceding argument also proves (ii).

(2) is obtained from (1)-(ii). □

Recall that there exists a (right) AP ring such that some prime factor ring of it is neither right nor left AP (see Remark 1.5(3)).

Proposition 1.8. (1) *Let R be a right AP ring. If R/P is right AP for any minimal prime ideal P of R, then $R/N_*(R)$ is a reduced ring.*

(2) *Let R be a semiprime right AP ring. If R/P is right AP for any minimal prime ideal P of R, then R is a reduced ring.*

Proof. (1) Suppose that R/P is right AP for any minimal prime ideal P of R. Then R/P is a domain by Lemma 1.4(2). Thus $R/N_*(R)$ is a subdirect product of domains, so that $R/N^*(R)$ is reduced. (2) is an immediate consequence of (1). □

2. Right AP rings

In this section we investigate several kinds of ring extensions which preserve the right AP property.

Considering the rings below, we can see that there exist many non-IFP right AP rings.

Theorem 2.1. (1) *A ring R is right AP if and only if $D_n(R)$ is right AP for every $n \geq 2$.*

(2) *A ring R is left AP if and only if $D_n(R)$ is left AP for every $n \geq 2$.*

Proof. Write $D = D_n(R)$ and let $n \geq 2$. (1) Let R be a right AP ring. Suppose that $AB = 0$ for $A = (a_{ij}), B = (b_{ij}) \in D_n(R)\backslash\{0\}$. Then $a_{ii}b_{ii} = 0$.

If $A \in N_n(R)$, then $A(DE_{1n}) = 0$.

Let $A \notin N_n(R)$ (i.e., $a_{ii} \neq 0$).

Assume $b_{ii} \neq 0$. Then since R is right AP and $a_{ii}b_{ii} = 0$, we see that $a_{ii}I = 0$ for some nonzero ideal I of R. So we have

$$AD[IE_{1n}] = [a_{ii}RI]E_{1n} = [a_{ii}I]E_{1n} = 0,$$

noting that $D[IE_{1n}] = IE_{1n}$ is a nonzero ideal of D.

Assume $b_{ii} = 0$. Let t be largest in $\{2, \ldots, n\}$ such that the t-th column of (b_{ij}) is nonzero. Next let s be largest in $\{1, \ldots, n-1\}$ such that $b_{st} \neq 0$. It then follows from $(a_{ij})(b_{ij}) = 0$ that

$$0 = a_{ss}b_{st} + a_{s(s+1)}b_{(s+1)t} + \cdots + a_{s(n-1)}b_{(n-1)t} + a_{sn}b_{nt} = a_{ss}b_{st},$$

noting $a_{ii} = a_{ss}$.

Since R is right AP and $b_{st} \neq 0$, $a_{ii}J = 0$ for some nonzero ideal J of R. So we have

$$AD[JE_{1n}] = [a_{ii}RJ]E_{1n} = [a_{ii}J]E_{1n} = 0,$$

noting that $D[JE_{1n}] = JE_{1n}$ is a nonzero ideal of D.

Therefore $D_n(R)$ is right AP.

Conversely let D be right AP and suppose that $ab = 0$ for $a, 0 \neq b \in R$. Consider two matrices $\alpha = aI_n$ and $\beta = bI_n$ in D. Then $\alpha\beta = 0$ with $\beta \neq 0$. Since D is right AP, $\alpha(D\gamma D) = 0$ for some $0 \neq \gamma = (c_{ij}) \in D$. Say $c_{pq} \neq 0$. Then

$$c_{pq}E_{1n} = E_{1p}(c_{ij})E_{qn} \in D\gamma D,$$

so that $D\gamma D$ contains $(Rc_{pq}R)E_{1n}$ which is a nonzero ideal of D. Moreover

$$\alpha[(Rc_{pq}R)E_{1n}] \subseteq \alpha(D\gamma D) = 0,$$

whence $a(Rc_{pq}R) = 0$. Therefore R is right AP because $Rc_{pq}R$ is a nonzero ideal of R.

(2) The proof is similar to (1), but contains several different parts; hence we write it for completeness. Let R be a left AP ring. Suppose that $AB = 0$ for $A = (a_{ij}) \neq 0, B = (b_{ij}) \in D_n(R)$. Then $a_{ii}b_{ii} = 0$.

If $B \in N_n(R)$, then $(DE_{1n})B = 0$.

Let $B \notin N_n(R)$ (i.e., $b_{ii} \neq 0$).

Assume $a_{ii} \neq 0$. Then since R is left AP and $a_{ii}b_{ii} = 0$, we see that $Hb_{ii} = 0$ for some nonzero ideal H of R. So we have

$$[HE_{1n}]DB = [HRb_{ii}]E_{1n} = [Hb_{ii}]E_{1n} = 0,$$

noting that $[HE_{1n}]D = HE_{1n}$ is a nonzero ideal of D.

Assume $a_{ii} = 0$. Let p be smallest in $\{1, \ldots, n-1\}$ such that the p-th row of (a_{ij}) is nonzero. Next let q be smallest in $\{2, \ldots, n\}$ such that $a_{pq} \neq 0$. It then follows from $(a_{ij})(b_{ij}) = 0$ that

$$0 = a_{pp}b_{pq} + a_{p(p+1)}b_{(p+1)q} + \cdots + a_{p(q-1)}b_{(q-1)q} + a_{pq}b_{qq} = a_{pq}b_{qq},$$

noting $b_{ii} = b_{qq}$.

Since R is left AP and $a_{pq} \neq 0$, $Kb_{ii} = 0$ for some nonzero ideal K of R. So we have

$$[KE_{1n}]DB = [KRb_{ii}]E_{1n} = [Kb_{ii}]E_{1n} = 0,$$

noting that $[KE_{1n}]D = KE_{1n}$ is a nonzero ideal of D.

Therefore $D_n(R)$ is left AP.

Conversely let D be left AP and suppose that $ab = 0$ for $a \neq 0, b \in R$. Consider two matrices $\alpha = aI_n$ and $\beta = bI_n$ in D. Then $\alpha\beta = 0$ with $\alpha \neq 0$. Since D is left AP, $(D\delta D)\beta = 0$ for some $0 \neq \delta = (d_{ij}) \in D$. Say $d_{uv} \neq 0$. Then

$$d_{uv}E_{1n} = E_{1u}(d_{ij})E_{vn} \in D\delta D,$$

so that $D\delta D$ contains $(Rd_{uv}R)E_{1n}$ which is a nonzero ideal of D. Moreover

$$[(Rd_{uv}R)E_{1n}]\beta \subseteq (D\delta D)\beta = 0,$$

whence $(Rd_{uv}R)b = 0$. Therefore R is left AP because $Rd_{uv}R$ is a nonzero ideal of R. $\square$

$D_n(R)$ is not IFP for all $n \geq 4$ over any ring R, by Lemma 1.3(2). But if R is right AP (e.g., R is IFP), then $D_n(R)$ is right AP by Theorem 2.1(1). As another example, consider $D_3(R)$ over a non-reduced IFP ring R. Then $D_3(R)$ is not IFP by Lemma 1.3(1) but right AP by Theorem 2.1(1).

Considering Theorem 2.1(1), one may ask whether $T_n(R)$ ($Mat_n(R)$) is right (resp., left) AP over right (resp., left) AP ring R. However the answer is negative by the example below. Compare this with [8, Example 2.3(1)].

Example 2.2. Let R be any ring and consider $T = T_n(R)$ for $n \geq 2$. Take $\alpha = E_{11}$ and $\beta = E_{22}$ in T. Then $\alpha\beta = 0$. Assuming that T is right AP, $\alpha T\gamma T = 0$ for some $0 \neq \gamma = (c_{ij}) \in T$. Say $c_{pq} \neq 0$. Then $T\gamma T$ contains $E_{1p}(c_{ij})E_{qn} = c_{pq}E_{1n}$. This yields a contradiction that

$$0 \neq c_{pq}E_{1n} = E_{11}E_{1p}(c_{ij})E_{qn} \in \alpha T\gamma T = 0.$$

Thus $T_n(R)$ cannot be right AP. Similarly it is shown that $T_n(R)$ cannot be left AP over any ring R. The same argument is applicable to show that $Mat_n(R)$ for $n \geq 2$ is neither right nor left AP over any ring R.

Next we observe noncommutative right AP rings of minimal order. The Galois field of order p^n is denoted by $GF(p^n)$, where p is a prime and $n \geq 1$.

Lemma 2.3. (1) [3, Proposition] *Let R be a finite noncommutative ring. If the order of R is p^3, with p a prime, then R is isomorphic to $T_2(GF(p))$.*

(2) [3, Theorem] *Let R be a finite ring of order m. If m has a cube-free factorization, then R is a commutative ring.*

Every noncommutative ring of minimal order is isomorphic to the $T_2(\mathbb{Z}_2)$ by Lemma 2.3. So $D_3(\mathbb{Z}_2)$ is a noncommutative right AP ring of minimal order by Theorem 2.1 and Example 2.2.

Proposition 2.4. (1) *Let R be a noncommutative finite ring. If R is right AP, then any nonzero right annihilator in R contains a nonzero ideal of R.*

(2) *Let R be a noncommutative right AP ring of minimal order. Then R is isomorphic to A_k for some $k \in \{1, 2, 3, 4, 5, 6\}$, where A_i is the ring B_i in* [8, Example 2.5(3)] *for $i = 1, 2, 3$, A_4 is $D_3(\mathbb{Z}_2)$, A_5 is the ring R in* [8, Example 2.5(2)], *and $A_6 = C_1$ with $p^n = 2$ in* [8, Example 2.5(4)].

Proof. (1) Since R is finite, $J(R)$ is nilpotent, so that $J(R) = W(R)$. Further, $R/J(R)$ is isomorphic to a finite direct product of $Mat_2(\mathbb{Z}_2)$'s (if any) and $\mathbb{Z}_2$'s because R is of minimal order. Assume $J(R) = 0$. Then R is a finite direct product of $\mathbb{Z}_2$'s by help of Example 2.2 since R is right AP, so that R is commutative, contradicting the hypothesis. So $J(R) \neq 0$, and especially R is a local ring. Since R is finite, there exists $k \geq 2$ such that $J(R)^k = 0$ and $J(R)^{k-1} \neq 0$. Suppose that $r_R(X)$ is nonzero for some $X \subseteq R$. Then $X \subseteq J(R)$, hence $XJ(R)^{k-1} = 0$.

(2) can be proved by (1), Lemma 2.3 and [8, Theorem 2.6(1)]. □

Let R be an algebra (possibly without identity) over a commutative ring S. Following Dorroh [2], the *Dorroh extension* of R by S is the Abelian group $R \times S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$, where $r_i \in R$ and $s_i \in S$.

Proposition 2.5. *Let R be a nil algebra over a field K. If R is right AP, then so is the Dorroh extension of R by K.*

Proof. Let D be the Dorroh extension of R by K and suppose that $\alpha\beta = 0$ for $0 \neq \alpha, \beta \in D$. Then $\alpha = (a, 0), \beta = (b, 0)$ for some $0 \neq a, b \in R$ because $(r, u) \in U(D)$ when $u \neq 0$. So $ab = 0$. Since R is right AP and $b \neq 0$, there exists a nonzero ideal I of R such that $aI = 0$. Set $J = \{(c, 0) \mid c \in I\}$. Then J is a nonzero ideal of D such that $\alpha J = (a, 0)(I, 0) = (aI, 0) = 0$. Thus D is right AP. □

One may ask whether nil rings are right (left) AP. However the answer is negative as follows.

Example 2.6. We apply the construction of [7, Example 1.2]. Let A be any ring and consider the rings $T_{2^n}(A)$ for $n = 1, 2, \ldots$. Define a map $\sigma : T_{2^n}(A) \to T_{2^{n+1}}(A)$ by $M \mapsto \left(\begin{smallmatrix} M & 0 \\ 0 & M \end{smallmatrix}\right)$. Then $T_{2^n}(A)$ can be considered as a subring of $T_{2^{n+1}}(A)$ via σ (i.e., $M = \sigma(M)$ for $M \in T_{2^n}(A)$). Set R_0 be the direct limit of the direct system (D_n, σ_{ij}), where $\sigma_{ij} = \sigma^{j-i}$. Then $R_0 = \cup_{i=1}^{\infty} T_{2^n}(A)$. Set

$$R = \{(a_{ij}) \in R_0 \mid a_{ii} = 0 \text{ for all } i\}.$$

Then R is a nil ring.

Let $\alpha = E_{12} = \beta \in R$. Then $\alpha\beta = 0$. Assume that R is right AP. Then $\alpha R\delta R = 0$ for some $0 \neq \delta = (d_{ij}) \in N_{2^n}(A) \subset R$. Say $d_{pq} \neq 0$.

Set s be smallest such that the s-th row of δ contains a nonzero entry, and t be smallest such that $d_{st} \neq 0$ in the s-th row. Note that $s < t$ and $(s + 2^k, t + 2^k)$-entry of δ in $N_{2^{n+1}}(A)$ is also d_{st} for $k = n, n + 1, n + 2, \ldots$.

Then $R\delta R$ contains $E_{2,s+2^k}(d_{ij})E_{t+2^k,2^{k+1}} = d_{st}E_{2,2^{k+1}}$. This yields a contradiction that

$$0 \neq d_{st}E_{1,2^{k+1}} = E_{12}(E_{2,s+2^k}(d_{ij})E_{t+2^k,2^{k+1}}) \in \alpha R\delta R = 0.$$

Therefore R is not right AP. R can be shown to be not left AP by a similar method.

By the result below, we can always construct right AP ring extensions from given any right AP rings.

Proposition 2.7. *Let R be a ring and M be a multiplicatively closed subset of R that consists of central non-zero-divisors. Then R is right AP if and only if $M^{-1}R$ is right AP.*

Proof. The proof is almost similar to one of [8, Proposition 4.7], however we write it for completeness. Let R be right AP and suppose that $\alpha\beta = 0$ for $0 \neq \alpha = au^{-1}, \beta = bv^{-1} \in RM^{-1}$. Then $ab = 0$. Since R is right AP and $b \neq 0$, there exists a nonzero ideal I of R such that $aI = 0$. Set $J = IM^{-1}$. Then J is evidently a nonzero ideal of RM^{-1}. Moreover $\alpha J = au^{-1}IM^{-1} = u^{-1}M^{-1}aI = 0$. Thus RM^{-1} is right AP.

Conversely suppose that RM^{-1} is right AP and let $ab = 0$ for $0 \neq a, b \in R$. Since RM^{-1} is right AP, there exists a nonzero ideal K of RM^{-1} such that $aK = 0$. Take $0 \neq cw^{-1} \in K$. Then $aRcw^{-1}R \subseteq aK = 0$, so that $aRcR = 0$. But $c \neq 0$, hence R is right AP. □

This result is also valid for strongly right AB rings as we see in [8, Proposition 4.7]. The Laurent polynomial ring, with an indeterminate x over a ring R, consists of all formal sums $\sum_{i=k}^{n} m_i x^i$ with usual addition and multiplication, where $m_i \in R$ and k, n are (possibly negative) integers. This is denoted by $R[x; x^{-1}]$. Letting $M = \{x^i \mid i \geq 1\}$, we have $R[x]M^{-1} = R[x; x^{-1}]$. Thus, by Proposition 2.7, $R[x]$ is right AP if and only if $R[x; x^{-1}]$ is right AP.

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