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ON DUAL ZARISKI TOPOLOGY OVER GRADED COMULTIPLICATION MODULES

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Abstract. In this article, we deal with Zariski topology on graded comultiplication modules. The purpose of this article is obtaining some connections between algebraic properties of graded comultiplication modules and topological properties of dual Zariski topology on graded comultiplication modules.

1. Introduction

Throughout this article, G will be a group with identity e and R a commutative ring with a nonzero unity 1. R is said to be G-graded if $R = \bigoplus_{g\in G} R_g$ with $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$ where R_g is an additive subgroup of R for all $g \in G$. The elements of R_g are called homogeneous of degree g. Consider $supp(R, G) = \{g \in G : R_g \neq 0\}$. If $x \in R$, then x can be written as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. Also, $h(R) = \bigcup_{g\in G} R_g$. Moreover, it has been proved in [10] that R_e is a subring of R and $1 \in R_e$. Let I be an ideal of a graded ring R. Then I is said to be graded ideal if $I = \bigoplus_{g\in G}(I \cap R_g)$, i.e., for $x \in I$, $x = \sum_{g\in G} x_g$ where $x_g \in I$ for all $g \in G$. The following example shows that an ideal of a graded ring need not be graded.

Example 1.1. Consider $R = \mathbb{Z}[i] = \{a + ib : a, b \in \mathbb{Z}, i^2 = -1\}$ (where $\mathbb{Z}$ is the ring of integers) and $G = \mathbb{Z}_2$ (the group of integers modulo 2). Then R is G-graded by $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Now, $I = \langle 1 + i\rangle$ is an ideal of R with $1 + i \in I$. If I is graded, then $1 \in I$, so $1 = a(1 + i)$ for some $a \in R$, i.e., $1 = (x + iy)(1 + i)$ for some $x, y \in \mathbb{Z}$. Thus $1 = x - y$ and $0 = x + y$, i.e., $2x = 1$ and hence $x = \frac{1}{2}$ a contradiction. So, I is not graded ideal of R.

Let R be a G-graded ring and I is a graded ideal of R. Then R/I is G-graded by $(R/I)_g = (R_g + I)/I$ for all $g \in G$.

Assume that M is a left R-module. Then M is said to be G-graded if $M = \bigoplus_{g\in G} M_g$ with $R_gM_h \subseteq M_{gh}$ for all $g, h \in G$ where M_g is an additive

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subgroup of M for all $g \in G$. The elements of M_g are called homogeneous of degree g. Also, we consider $supp(M,G) = \{g \in G : M_g \neq 0\}$. It is clear that M_g is an R_e-submodule of M for all $g \in G$. Moreover, $h(M) = \bigcup_{g\in G} M_g$. Let N be an R-submodule of a graded R-module M. Then N is said to be graded R-submodule if $N = \bigoplus_{g\in G}(N \cap M_g)$, i.e., for $x \in N$, $x = \sum_{g\in G} x_g$ where $x_g \in N$ for all $g \in G$. The following example shows that an R-submodule of a graded R-module need not be graded.

Example 1.2. Consider $R = \mathbb{Z}$, $M = \mathbb{Z}[i]$ and $G = \mathbf{Z}_2$. Then R is G-graded by $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Also, M is G-graded by $M_0 = \mathbb{Z}$ and $M_1 = i\mathbb{Z}$. Similarly as in Example 1.1, $N = \langle 1+i \rangle$ is an R-submodule of M which is not graded.

Let M be a G-graded R-module and N be a graded R-submodule of M. Then M/N is a graded R-module by $(M/N)_g = (M_g + N)/N$ for all $g \in G$.

Lemma 1.3 ([8], Lemma 2.1). *Let R be a G-graded ring and M be a G-graded R-module.*

1. *If I and J are graded ideals of R, then $I + J$ and $I \bigcap J$ are graded ideals of R.*
2. *If N and K are graded R-submodules of M, then $N + K$ and $N \bigcap K$ are graded R-submodules of M.*
3. *If N is a graded R-submodule of M, $r \in h(R)$, $x \in h(M)$ and I is a graded ideal of R, then Rx, IN and rN are graded R-submodules of M. Moreover, $(N :_R M) = \{r \in R : rM \subseteq N\}$ is a graded ideal of R.*

Also, it has been proved in [9] that if N is a graded R-submodule of M, then $Ann_R(N) = \{r \in R : rN = \{0\}\}$ is a graded ideal of R.

A proper graded ideal P of R is said to be graded prime if whenever $x, y \in h(R)$ such that $xy \in P$, then either $x \in P$ or $y \in P$. Graded prime ideals have been admirably introduced and studied in [12]. The set of all graded prime ideals of R is denoted by $GSpec(R)$. Graded prime submodules have been introduced by Atani in [5]. A proper graded R-submodule N of M is said to be graded prime if whenever $r \in h(R)$ and $m \in h(M)$ such that $rm \in N$, then either $m \in N$ or $r \in (N :_R M)$. Let M and S be two G-graded R-modules. An R-homomorphism $f : M \to S$ is said to be graded R-homomorphism if $f(M_g) \subseteq S_g$ for all $g \in G$ (see [10]). Graded second submodules have been introduced by Ansari-Toroghy and Farshadifar in [4]. A nonzero graded R-submodule N of M is said to be graded second if for each $a \in h(R)$, the graded R-homomorphism $f : N \to N$ defined by $f(x) = ax$ is either surjective or zero. In this case, $Ann_R(N)$ is a graded prime ideal of R. Graded second submodules have been wonderfully studied by Çeken and Alkan in [7]. On the other hand, graded secondary modules have been introduced by Atani and Farzalipour in [6]. A nonzero graded R-module M is said to be graded secondary if for each $a \in h(R)$, the graded R-homomorphism $f : M \to M$ defined by $f(x) = ax$ is either surjective or nilpotent. Let K be a graded R-submodule of M. Then the graded second radical of K is denoted by $GSec(K)$ and is defined as the sum of all graded second R-submodules of M, which are contained in K.

In the graded module theory, the concept of graded second submodule is known as the dual concept of graded prime submodule. Similarly, Zariski topology on the set of all graded second submodules of M, denoted by $GSpec^s(M)$, is known as dual Zariski topology. Let K be a graded R-submodule of a graded R-module M and $V^s(K) = \{P \in GSpec^s(M) : P \subseteq K\}$. Then the following hold:

(1) $V^s(\{0\}) = \emptyset$ and $V^s(M) = GSpec^s(M)$.
(2) $\bigcap_{i\in\Lambda} V^s(K_i) = V^s\left(\bigcap_{i\in\Lambda} K_i\right)$ for every family of graded R-submodules K_i's of M.
(3) $V^s(K)\bigcup V^s(L) \subseteq V^s(K+L)$ where K and L are graded R-submodules of M.

Graded comultiplication modules have been introduced by H. A. Toroghy and F. Farshadifar in [3]. A graded R-module M is said to be graded comultiplication if for every graded R-submodule N of M, $N = (0 :_M I) = \{m \in M : Im = 0\}$ for some graded ideal I of R. The concept of graded comultiplication modules has been studied by several authors, for example, see [2] and [1].

Proposition 1.4. *Let M be a G-graded R-module and N a graded R-submodule of M. If I is a graded ideal of R, then $(N :_M I) = \{m \in M : Im \subseteq N\}$ is a graded R-submodule of M.*

Proof. Clearly, $(N :_M I)$ is an R-submodule of M. Let $m \in (N :_M I)$. Then $Im \subseteq N$. Now, $m = \sum_{g\in G} m_g$ where $m_g \in M_g$ for all $g \in G$. Let $x \in I$. Then $x_g \in I$ for all $g \in G$ since I is graded. Assume that $h \in G$. Then $x_h m_g \in M_{hg} \subseteq h(M)$ for all $g \in G$ such that $\sum_{g\in G} x_h m_g = x_h\left(\sum_{g\in G} m_g\right) = x_h m \in N$. Since N is graded, $x_h m_g \in N$ for all $g \in G$ which implies that $\sum_{h\in G} x_h m_g \in N$ for all $g \in G$, and then $x m_g \in N$ for all $g \in G$. So, $Im_g \subseteq N$ for all $g \in G$, and hence $m_g \in (N :_M I)$ for all $g \in G$. Therefore, $(N :_M I)$ is a graded R-submodule of M. □

Corollary 1.5. *Let M be a G-graded R-module and I a graded ideal of R. Then $(0 :_M I) = \{m \in M : Im = 0\}$ is a graded R-submodule of M.*

Let I and J be graded ideals of R. Then it is accessible that

$$\begin{aligned} V^s((0 :_M I))\bigcup V^s((0 :_M J)) &= V^s((0 :_M I) + (0 :_M J)) \\ &= V^s((0 :_M I\bigcap J)) = V^s((0 :_M IJ)). \end{aligned}$$

Let I be a graded ideal of R. Then $\Gamma^s(M) = \{V^s((0 :_M I))\}$ satisfies the axioms for closed sets of a topological space on $GSpec^s(M)$. If M is a graded comultiplication R-module, then there exists a topology, the collection of whose all closed sets is $\{V^s(K) : K \text{ is a graded } R\text{-submodule of } M\}$, on $GSpec^s(M)$.

In this article, we deal with Zariski topology on graded comultiplication modules, with purpose of obtaining some connections between algebraic properties of graded comultiplication modules and topological properties of dual Zariski topology on graded comultiplication modules.

2. On dual Zariski topology over graded comultiplication modules

In this section, we follow [11] to deal with Zariski topology on graded comultiplication modules in order to obtain some connections between algebraic properties of graded comultiplication modules and topological properties of dual Zariski topology on graded comultiplication modules.

Let M be a graded comultiplication R-module and K be a graded R-submodule of M. Then $K = (0 :_M I)$ for some graded ideal I of R. Assume that $X_K^s = GSpec^s(M) - V^s((0 :_M I))$ and $\widehat{V}^s((0 :_M J)) = V^s((0 :_M J)) - V^s((0 :_M I))$ where J is a graded ideal of R.

Proposition 2.1. *Under the above assumptions,*

$$\Gamma_K^s = \left\{\widehat{V}^s((0 :_M J)) : J \textit{ is a graded ideal of } R\right\}$$

satisfies the axioms for closed sets of a topological space on X_K^s.

Proof. (1) Let J be a graded ideal of R. If $J = \{0\}$, then $V^s((0 :_M J)) = GSpec^s(M)$, and then $\widehat{V}^s((0 :_M J)) = V^s((0 :_M J)) - V^s((0 :_M I)) = GSpec^s(M) - V^s((0 :_M I)) = X_K^s$. Hence, $X_K^s \in \Gamma_K^s$.

(2) Let $\left\{\widehat{V}^s((0 :_M J_i)) : i \in \Lambda\right\}$ be a family of Γ_K^s. Then

$$\begin{aligned}
\bigcap_{i\in\Lambda} \widehat{V}^s((0 :_M J_i)) &= \bigcap_{i\in\Lambda} \left(V^s((0 :_M J_i)) - V^s((0 :_M I))\right) \\
&= \left(\bigcap_{i\in\Lambda} V^s((0 :_M J_i))\right) - V^s((0 :_M I)) \\
&= V^s\left(\bigcap_{i\in\Lambda}(0 :_M J_i)\right) - V^s((0 :_M I)) \\
&= \widehat{V}^s\left(\bigcap_{i\in\Lambda}(0 :_M J_i)\right) \in \Gamma_K^s.
\end{aligned}$$

(3) Let $\widehat{V}^s((0 :_M J_1)), \widehat{V}^s((0 :_M J_2)) \in \Gamma_K^s$. Then

$$\begin{aligned}
&\widehat{V}^s((0 :_M J_1)) \bigcup \widehat{V}^s((0 :_M J_2)) \\
= &\left(V^s((0 :_M J_1)) \bigcup V^s((0 :_M J_2))\right) - V^s((0 :_M I)) \\
= &\left(V^s((0 :_M J_1) + (0 :_M J_2))\right) - V^s((0 :_M I)) \\
= &\widehat{V}^s((0 :_M J_1) + (0 :_M J_2)) \in \Gamma_K^s.
\end{aligned}$$

□

This topology will be called the complement dual Zariski topology of K in M. For the rest of our article, we fix the graded R-submodule K of M as $K = (0 :_M I)$, where I is a graded ideal of R, and the graded R-module M as a graded comultiplication R-module.

Lemma 2.2. *Let R be a graded ring and J be an ideal of R. Then J is a graded ideal of R if and only if J has a homogeneous generating set.*

Proof. Suppose that J is a graded ideal of R. Let

$$S = \{x \in J : x \text{ is homogeneous}\}.$$

Then $S = \{x \in J \bigcap h(R)\}$, and then since J is graded, S generates J. Conversely, let S be a homogeneous generating set for J. Assume that $x \in J$. Then $x = \sum_{g\in G} x_g$ where $x_g \in R_g$ for all $g \in G$. Now, $x = \sum_{i=1}^{n} r_i s_i$ where $s_i \in S$ is homogeneous and $r_i \in R$ for all $1 \le i \le n$. Let $s_i \in R_{g_i}$ for some $g_i \in G$ and $r_i = \sum_{g\in G} r_{i_g}$. Then $x = \sum_{i,g} r_{i_g} s_i$. For $h \in G$, $x_h = \sum_i r_{i_{h-g_i}} s_i \in J$ as $s_i \in J$. Hence, J is a graded ideal of R. □

Let J be a graded ideal of R. Then by Lemma 2.2, $J = \sum_{x_i \in J} Rx_i$ where $x_i \in h(R)$ for all i, and then $V^s((0 :_M J)) = V^s\left(\left(0 :_M \sum_{x_i\in J} Rx_i\right)\right) = V^s\left(\bigcap_{x_i\in J}(0 :_M Rx_i)\right) = \bigcap_{x_i\in J} V^s((0 :_M Rx_i))$.

Proposition 2.3. *Let K be a graded R-submodule of a graded comultiplication R-module M. Then for any graded ideal J of R, the set*

$$(X_K^s)^{(0:_M J)} = X_K^s - \widehat{V}^s((0 :_M J))$$

forms a base for the complement dual Zariski topology of K in M on X_K^s.

Proof. If $X_K^s = \emptyset$, then $(X_K^s)^{(0:_M J)} = \emptyset$. Let $A \subset X_K^s$ be an open set. Then $A = X_K^s - \widehat{V}^s((0 :_M J))$, and then

$$\begin{aligned} A &= X_K^s - \widehat{V}^s\left(\left(0 :_M \sum_{x_i\in J} Rx_i\right)\right) \\ &= X_K^s - \bigcap_{x_i\in J} \widehat{V}^s((0 :_M Rx_i)) \\ &= \bigcup_{x_i\in J} (X_K^s - \widehat{V}^s((0 :_M Rx_i))) \\ &= \bigcup_{x_i\in J} (X_K^s)^{(0:_M J)}. \end{aligned}$$

□

Proposition 2.4. *Let $K = (0 :_M I)$ be a graded R-submodule of a graded comultiplication R-module M. Then the following hold:*

(1) *$(X_K^s)^{(0:_M J)} = GSpec^s(M) - V^s((0 :_M IJ))$ for every graded ideal J of R.*
(2) *$(X_K^s)^{(0:_M J_1)} \bigcap (X_K^s)^{(0:_M J_2)} = (X_K^s)^{(0:_M J_1 J_2)}$ for every graded ideals J_1 and J_2 of R.*

Proof. (1) Let J be a graded ideal of R. Then

$$\begin{aligned}(X_K^s)^{(0:_M J)} &= X_K^s - \widehat{V}^s((0:_M J)) \\ &= GSpec^s(M) - \Big(V^s((0:_M I))\bigcup V^s((0:_M J))\Big) \\ &= GSpec^s(M) - V^s((0:_M IJ)).\end{aligned}$$

(2) Assume that J_1 and J_2 are graded ideals of R. Let

$$A \in (X_K^s)^{(0:_M J_1)} \bigcap (X_K^s)^{(0:_M J_2)}.$$

Then $A \in (X_K^s)^{(0:_M J_1)}$ and $A \in (X_K^s)^{(0:_M J_2)}$, which implies that $A \notin V^s((0:_M IJ_1))$ and $A \notin V^s((0:_M IJ_2))$, and then $A \notin V^s((0:_M IJ_1))\bigcup V^s((0:_M IJ_2)) = V^s((0:_M IJ_1J_2))$. Hence, $A \in (X_K^s)^{(0:_M J_1J_2)}$. Now, let $A \in (X_K^s)^{(0:_M J_1J_2)}$. Then $A \notin V^s((0:_M IJ_1J_2)) = V^s((0:_M IJ_1))\bigcup V^s((0:_M IJ_2))$, which implies that $A \notin V^s((0:_M IJ_1))$ and $A \notin V^s((0:_M IJ_2))$, and then $A \in GSpec^s(M) - V^s((0:_M IJ_1))$ and $A \in GSpec^s(M) - V^s((0:_M IJ_2))$. Hence, $A \in (X_K^s)^{(0:_M J_1)} \bigcap (X_K^s)^{(0:_M J_2)}$. □

The next proposition declares some connections between some properties of the complement dual Zariski topology and the graded second module.

Proposition 2.5. *Let $K = (0:_M I)$ be a graded R-submodule of a graded comultiplication R-module M. Then the following hold:*

1. *$(X_K^s)^{(0:_M J)} = \emptyset$ if and only if $GSec(M) \subseteq (0:_M IJ)$ for every graded ideal J of R.*
2. *$(X_K^s)^{(0:_M J_1)} = (X_K^s)^{(0:_M J_2)}$ if and only if*

$$GSec((0:_M IJ_1)) = GSec((0:_M IJ_2))$$

for every graded ideals J_1 and J_2 of R.

Proof. (1) Let J be a graded ideal of R. Suppose that $(X_K^s)^{(0:_M J)} = \emptyset$. Then $GSpec^s(M) = V^s((0:_M IJ))$, and since every graded second R-submodule of M is contained in $(0:_M IJ)$, we have that $GSec(M) \subseteq (0:_M IJ)$. Suppose that $GSec(M) \subseteq (0:_M IJ)$. Then $(0:_M IJ)$ contains every graded second R-submodule of M, and then $GSpec^s(M) = V^s((0:_M IJ))$, which implies that $(X_K^s)^{(0:_M J)} = \emptyset$.

(2) Let J_1 and J_2 be graded ideals of R. Suppose that $(X_K^s)^{(0:_M J_1)} = (X_K^s)^{(0:_M J_2)}$. Then $V^s((0:_M IJ_1)) = V^s((0:_M IJ_2))$, which implies that $GSec((0:_M IJ_1)) = GSec((0:_M IJ_2))$. Suppose that $GSec((0:_M IJ_1)) = GSec((0:_M IJ_2))$. Then $V^s((0:_M IJ_1)) = V^s((0:_M IJ_2))$, which implies that $(X_K^s)^{(0:_M J_1)} = (X_K^s)^{(0:_M J_2)}$. □

Corollary 2.6. *Let $K = (0:_M I)$ be a graded R-submodule of a graded comultiplication R-module M. Then $(X_K^s)^{(0:_M J)} = X_K^s$ if and only if $GSec((0:_M IJ)) = GSec((0:_M I))$ for every graded ideal J of R.*

Proof. Let J be a graded ideal of R. Suppose that $(X_K^s)^{(0:_M J)} = X_K^s$. Then $V^s((0:_M IJ)) = V^s((0:_M I))$, and then $GSec((0:_M IJ)) = GSec((0:_M I))$. Suppose that $GSec((0:_M IJ)) = GSec((0:_M I))$. Then $V^s((0:_M IJ)) = V^s((0:_M I))$, and then $(X_K^s)^{(0:_M J)} = X_K^s$. □

Definition. Let K be a proper graded R-submodule of a graded comultiplication R-module M. Then for graded R-submodule N of M, we define $GS_K(N) = \sum\{P \in GSpec^s(M) : P \subseteq N \text{ and } P \nsubseteq K\}$.

By Lemma 1.3, we have that $GS_K(N)$ is a graded R-submodule of M for every graded R-submodule N of M. Moreover, the new class $GS_K(N)$ has the following properties:

Proposition 2.7. *Let K be a proper graded R-submodule of a graded comultiplication R-module M. Then the following hold*:
(1) $GS_K(M) = GS_{GSec(K)}(M)$.
(2) $GS_{K/A}(N/A) = GS_K(N)/A$ *for every graded R-submodules N and A of M such that $A \subseteq N$.*

Proof. (1) Since every graded second R-submodule of M which is not contained in K is also not contained in $GSec(K)$, we have that $GS_K(M) = GS_{GSec(K)}(M)$.

(2) Let N and A be graded R-submodules of M such that $A \subseteq N$. Then

$$\begin{aligned} GS_{K/A}(N/A) &= \sum_{P_i/A \subseteq N/A,\, P_i/A \nsubseteq K/A} (P_i/A) = \left(\sum_{P_i \subseteq N,\, P_i \nsubseteq K} P_i\right)/A \\ &= GS_K(N)/A. \end{aligned}$$ □

Definition. A graded comultiplication R-module M is said to satisfy the graded S-condition for a proper graded R-submodule K, if for any chain $GS_K(N_1) \subseteq GS_K(N_2) \subseteq \cdots$, where N_i is a graded R-submodule of M, there is a positive integer n such that $GS_K(N_n) = GS_K(N_{n+j})$ for all positive integer j.

Proposition 2.8. *Let K be a proper graded R-submodule of a graded comultiplication R-module M. Then M satisfies the graded S-condition for K if and only if X_K^s is a graded Artinian topological space.*

Proof. Suppose that M satisfies the graded S-condition for K. Consider the chain $\widehat{V}^s(N_1) \subseteq \widehat{V}^s(N_2) \subseteq \cdots$, where N_i is a graded R-submodule of M. Then $GS_K(N_1) \subseteq GS_K(N_2) \subseteq \cdots$ and then there is a positive integer n such that $GS_K(N_n) = GS_K(N_{n+j})$ for all positive integer j. So, $\widehat{V}^s(N_n) = \widehat{V}^s(N_{n+j})$ for all positive integer j. Hence, X_K^s is a graded Artinian topological space. Conversely, consider the chain $GS_K(N_1) \subseteq GS_K(N_2) \subseteq \cdots$, where N_i is a graded R-submodule of M. Then $\widehat{V}^s(N_1) \subseteq \widehat{V}^s(N_2) \subseteq \cdots$ and then there is a positive integer n such that $\widehat{V}^s(N_n) = \widehat{V}^s(N_{n+j})$ for all positive integer j.

So, $GS_K(N_n) = GS_K(N_{n+j})$ for all positive integer j. Hence, M satisfies the graded S-condition for K. □

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