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AMALGAMATED MODULES ALONG AN IDEAL

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Abstract. Let R and S be two commutative rings, J be an ideal of S and $f : R \to S$ be a ring homomorphism. The amalgamation of R and S along J with respect to f, denoted by $R \bowtie^f J$, is the special subring of $R \times S$ defined by $R \bowtie^f J = \{(a, f(a) + j) \mid a \in R, j \in J\}$. In this paper, we study some basic properties of a special kind of $R \bowtie^f J$-modules, called the amalgamation of M and N along J with respect to φ, and defined by $M \bowtie^\varphi JN := \{(m, \varphi(m) + n) \mid m \in M \text{ and } n \in JN\}$, where $\varphi : M \to N$ is an R-module homomorphism. The new results generalize some known results on the amalgamation of rings and the duplication of a module along an ideal.

1. Introduction

Throughout this paper, all rings are commutative with identity and all modules are unital. Let R and S be two rings, J be an ideal of S and $f : R \longrightarrow S$ be a ring homomorphism. D'Anna, Finocchiaro and Fontana in [3] and [4] introduced and studied the subring

$$R \bowtie^f J = \{(a, f(a) + j) \mid a \in R, j \in J\}$$

of $R \times S$ called the amalgamation of R and S along J with respect to f. Several classical constructions such as the $R+XS[X]$, $R+XS[[X]]$, the $D+M$ constructions, and the amalgamated duplication of a ring along an ideal can be considered as particular cases of the amalgamated algebra (see [3, Examples 2.5 and 2.6]). Let I be an ideal of R. The amalgamated duplication of R along the ideal I was defined by $R \bowtie I := \{(a, a + i) : a \in R, \ i \in I\}$ [6].

One of the key tools for studying $R \bowtie^f J$ is based on the fact that the amalgamation can be studied in the frame of pullback constructions [3, Section 4]. This point of view allows the authors in [3,4] to provide an ample description of various properties of $R \bowtie^f J$, in connection with the properties of R, J and f. Namely, in [3], the authors studied the basic properties of this construction (e.g., characterizations for $R \bowtie^f J$ to be a Noetherian ring, an integral domain,

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a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation.

Let R be a ring, I an ideal of R, and M an R-module. The authors of [2] recently introduced *the duplication of the R-module M along the ideal I* denoted by $M \bowtie I$ and gave the following definition

$$M \bowtie I = \{(m, m') \in M \times M \mid m - m' \in IM\}$$

which is an $R \bowtie I$-module with the multiplication given by

$$(r, r+i) \cdot (m, m') = (rm, (r+i)m'), \text{ where } r \in R,\ i \in I, \text{ and } (m, m') \in M \bowtie I.$$

If $M = R$, then the duplication of the R-module R along the ideal I coincides with the amalgamated duplication of the ring R along the ideal I. In their article, they studied some basic properties of the duplication of an R-module M along an ideal I. More precisely, they studied when $M \bowtie I$ is a Noetherian, an Artinian or an (Nil_*-)coherent $R \bowtie I$-module. They also investigated some basic homological properties of $M \bowtie I$: when $M \bowtie I$ is an injective module, a projective module or a flat module.

In this paper we introduce the generalization of the duplication of modules along an ideal. Let $f : R \to S$ be a ring homomorphism, J be an ideal of S, M be an R-module, N be an S-module (which is an R-module induced naturally by f) and $\varphi : M \to N$ be an R-module homomorphism. We define the amalgamation of M and N along J with respect to φ by

$$M \bowtie^{\varphi} JN := \{(m, \varphi(m) + n) \mid m \in M \text{ and } n \in JN\}.$$

It can be seen that $M \bowtie^{\varphi} JN$ is an $R \bowtie^{f} J$-module by the following scalar product

$$(r, f(r) + j)(m, \varphi(m) + n) := (rm, \varphi(rm) + f(r)n + j\varphi(m) + jn).$$

Note that $\varphi(rm) = f(r)\varphi(m)$, since φ is an R-module homomorphism. If $M = R$, $N = S$ and $\varphi = f$, then the amalgamation of the R-module R and the S-module S along J with respect to φ coincides with the amalgamation of rings R and S along J with respect to f. Also, if $S = R$, $N = M$ and $\varphi = Id_M$, then the amalgamation of M and N along J with respect to φ is exactly the duplication of the R-module M along the ideal J. Hence, the notion of amalgamation of modules is a generalization of all the notions already mentioned.

In this paper, we study some basic properties of the amalgamation of modules. More precisely, we study when $M \bowtie^{\varphi} JN$ is a Noetherian or a coherent $R \bowtie^{f} J$-module. The new results generalize some known results on the amalgamation of rings and the duplication of a module along an ideal.

Throughout the paper, $f : R \to S$ is a ring homomorphism, J an ideal of S, M an R-module, N an S-module and $\varphi : M \to N$ an R-module homomorphism.

2. Definition and basic properties

In this section, we present the basic properties of the amalgamation of M and N along J with respect to φ, $M \bowtie^{\varphi} JN$.

One can define $M \bowtie^{\varphi} JN$ by means of pullback of modules. Indeed, let $\pi : N \to N/JN$ be a natural homomorphism, $M \bowtie^{\varphi} JN \to N$ (respectively, $M \bowtie^{\varphi} JN \to M$) be the restriction to $M \bowtie^{\varphi} JN$ of the projection of $M \times N$ onto N (respectively, M). It can be seen that the following diagram is a pullback:

$$\begin{array}{ccc} M \bowtie^{\varphi} JN & \longrightarrow & N \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow[\pi\circ\varphi]{} & N/JN. \end{array}$$

Remark 2.1. (1) $f(R)+J$ is a subring of S. So, N is an $f(R)+J$-module. It is easy to see that $\varphi(M)+JN$ is an $f(R)+J$-submodule of N. Thus $\varphi(M)+JN$ is an $R \bowtie^{f} J$-module via $P_S : R \bowtie^{f} J \to f(R)+J$ where $P_S(r, f(r)+j) = f(r)+j$.

(2) $\pi_N : M \bowtie^{\varphi} NJ \to \varphi(M) + JN$ given by $\pi_N(m, \varphi(m) + n) = \varphi(m) + n$ is an $R \bowtie^{f} J$-module homomorphism.

(3) M is an $R \bowtie^{f} J$-module via the surjective homomorphism $P_R : R \bowtie^{f} J \to R$. It is easy to see that $\pi_M : M \bowtie^{\varphi} JN \to M$ given by $\pi_M(m, \varphi(m)+n) = m$ is an $R \bowtie^{f} J$-module homomorphism.

(4) It can be seen that JN is an $f(R) + J$-submodule of $\varphi(M) + JN$. Hence JN is an $R \bowtie^{f} J$-submodule of $\varphi(M) + JN$.

(5) We have the following exact sequence of $R \bowtie^{f} J$-modules and $R \bowtie^{f} J$-homomorphisms:

$$0 \to JN \overset{\iota}{\to} M \bowtie^{\varphi} JN \overset{\pi_M}{\to} M \to 0,$$

where $\iota : JN \to M \bowtie^{\varphi} JN$ given $\iota(n) = (0, n)$.

Proposition 2.2. *Let $f : R \to S$ be a ring homomorphism, J be an ideal of S, M be an R-module, N be an S-module and $\varphi : M \to N$ an R-module homomorphism. Then the following hold*:

(1) $\frac{M \bowtie^{\varphi} JN}{\{0\} \times JN} = M$.
(2) $\frac{M \bowtie^{\varphi} JN}{F \bowtie^{\varphi} JN} = \frac{M}{F}$, *where F is a submodule of M.*
(3) $\frac{M \bowtie^{\varphi} JN}{\varphi^{-1}(JN) \times \{0\}} = \varphi(M) + JN$.

Proof. Let $\pi_M : M \bowtie^{\varphi} JN \to M$; $(m, \varphi(m) + n) \mapsto m$, $\psi : M \bowtie^{\varphi} JN \to \frac{M}{F}$; $(m, \varphi(m) + n) \mapsto \overline{m}$ and $\pi_N : M \bowtie^{\varphi} JN \to \varphi(M) + JN$; $(m, \varphi(m) + n) \mapsto \varphi(m) + n$. The three homomorphisms are surjective with $\ker(\pi_M) = \{0\} \times JN$, $\ker(\psi) = F \bowtie^{\varphi} JN$ and $\ker(\pi_N) = \varphi^{-1}(JN) \times \{0\}$. Hence, we have the desired isomorphisms. □

Remark 2.3. If we consider R-modules in Proposition 2.2 as $R \bowtie^{f} J$-modules, then the isomorphisms are also $R \bowtie^{f} J$-isomorphisms.

We have the following results about localization.

Proposition 2.4. *With the notation of Proposition 2.2, the following statements hold*:

(1) *For* $\mathfrak{p} \in \mathrm{Spec}(R)$ *and* $\mathfrak{q} \in \mathrm{Spec}(S) \setminus \mathrm{V}(J)$*, set*

$$\mathfrak{p}'^f := \mathfrak{p} \bowtie^f J := \{(p, f(p) + j) \mid p \in \mathfrak{p}, j \in J\},$$

$$\overline{\mathfrak{q}}^f := \{(r, f(r) + j) \mid r \in R, j \in J, f(r) + j \in \mathfrak{q}\}.$$

Then, one has the following:

(a) *The prime ideals of* $R \bowtie^f J$ *are of the type* $\overline{\mathfrak{q}}^f$ *or* $\mathfrak{p}'^f$*, for* $\mathfrak{q}$ *varying in* $\mathrm{Spec}(S) \setminus \mathrm{V}(J)$ *and* $\mathfrak{p}$ *in* $\mathrm{Spec}(R)$.

(b) $\mathrm{Max}(R \bowtie^f J) = \{\mathfrak{p}'^f \mid \mathfrak{p} \in \mathrm{Max}(R)\} \cup \{\overline{\mathfrak{q}}^f \mid \mathfrak{q} \in \mathrm{Max}(S) \setminus \mathrm{V}(J)\}$.

(2) *The following formulas for localizations hold*:

(a) *For any* $\mathfrak{q} \in \mathrm{Spec}(S) \setminus \mathrm{V}(J)$*, the localization* $(M \bowtie^\varphi JN)_{\overline{\mathfrak{q}}^f}$ *is canonically isomorphic to* $N_\mathfrak{q}$*. This isomorphism maps the element* $(x, \varphi(x) + y)/(r, f(r) + j)$ *to* $(\varphi(x) + y)/(f(r) + j)$.

(b) *For any* $\mathfrak{p} \in \mathrm{Spec}(R) \setminus \mathrm{V}(f^{-1}(J))$*, the localization* $(M \bowtie^\varphi JN)_{\mathfrak{p}'^f}$ *is canonically isomorphic to* $M_\mathfrak{p}$*. This isomorphism maps the element* $(x, \varphi(x) + y)/(r, f(r) + j)$ *to* x/r.

(c) *For any* $\mathfrak{p} \in \mathrm{Spec}(R)$ *containing* $f^{-1}(J)$*, consider the multiplicative subset* $T_\mathfrak{p} := f(R \setminus \mathfrak{p}) + J$ *of* S *and set* $N_{T_\mathfrak{p}} := T_\mathfrak{p}^{-1}N$ *and* $J_{T_\mathfrak{p}} := T_\mathfrak{p}^{-1}J$*. If* $f_\mathfrak{p} : R_\mathfrak{p} \to S_{T_\mathfrak{p}}$ *is the ring homomorphism induced by* f *and* $\varphi_\mathfrak{p} : M_\mathfrak{p} \to N_{T_\mathfrak{p}}$ *is the* $R_\mathfrak{p}$*-homomorphism induced by* φ*, then the* $R_\mathfrak{p}$*-module* $(M \bowtie^\varphi JN)_{\mathfrak{p}'^f}$ *is canonically isomorphic to* $M_\mathfrak{p} \bowtie^{\varphi_\mathfrak{p}} J_{T_\mathfrak{p}} N_{T_\mathfrak{p}}$*. This isomorphism maps the element* $(x, f(x) + y)/(r, f(r) + j)$ *to* $(x/r, (f(x) + y)/(f(r) + j))$.

Proof. (1) is taken from [5, Corollaries 2.5 and 2.7]. For (2), we only prove (c). It is clear that $\psi : (M \bowtie^\varphi JN)_{\mathfrak{p}'^f} \to M_\mathfrak{p} \bowtie^{\varphi_\mathfrak{p}} J_{T_\mathfrak{p}} N_{T_\mathfrak{p}}$ which sends $(x, f(x)+y)/(r, f(r)+j)$ to $(x/r, (f(x)+y)/(f(r)+j))$ is a well defined $(R \bowtie^f J)_{\mathfrak{p}'^f}$-module homomorphism. To prove that ψ is one to one, assume that $(x/r, (f(x)+y)/(f(r)+j)) = 0$. Then $x/r = 0$ in $M_\mathfrak{p}$ and $(f(x)+y)/(f(r)+j) = 0$ in $N_{T_\mathfrak{p}}$. Hence there exist $r'' \in R \setminus \mathfrak{p}$ and $f(t)+u \in T_\mathfrak{p}$ such that $r''x = 0$ and $(f(t)+u)(\varphi(x)+y) = 0$. So we have $(r'', f(r''))(t, f(t)+u)(x, \varphi(x)+y) = (0, 0)$. This shows that $(x, f(x)+y)/(r, f(r)+j) = 0$ and so ψ is one to one. Assume finally that $(x/r, \varphi_\mathfrak{p}(x/r) + y/(f(r') + j)) \in M_\mathfrak{p} \bowtie^{\varphi_\mathfrak{p}} J_{T_\mathfrak{p}} N_{T_\mathfrak{p}}$. Then

$$\begin{aligned} &(x/r, \varphi_\mathfrak{p}(x/r) + y/(f(r') + j)) \\ =\ &(x/r, \varphi_\mathfrak{p}(x/r)) + (0, y/(f(r') + j)) \\ =\ &\psi((x, \varphi(x))/(r, f(r)) + \psi((0, y)/(r', f(r') + j)). \end{aligned}$$

It follows that ψ is surjective. □

3. Noetherian property

In [3, Propositions 5.6 and 5.7], the authors determined the Noetherian property of the amalgamated algebra $R \bowtie^f J$. We will now see when the amalgamation of two modules along an ideal is Noetherian.

Proposition 3.1. *With the notation of Proposition 2.2, the amalgamation $M \bowtie^{\varphi} JN$ is a Noetherian $R \bowtie^f J$-module if and only if $\varphi(M) + JN$ is a Noetherian $f(R) + J$-module and M is a Noetherian R-modules.*

Proof. ($\Rightarrow$) Using the homomorphism π_N, $\varphi(M)+JN$ is a Noetherian $R \bowtie^f J$-module. Hence $\varphi(M) + JN$ is a Noetherian $f(R) + J$-module. Also M is a Noetherian $R \bowtie^f J$-module using π_M. Hence M is a Noetherian R-module using P_R.

($\Leftarrow$) Since $\varphi(M) + JN$ is a Noetherian $f(R) + J$-module, then $\varphi(M) + JN$ is a Noetherian $R \bowtie^f J$-module. Hence JN is a Noetherian $R \bowtie^f J$-module. Also M is a Noetherian $R \bowtie^f J$-module. Therefore $M \bowtie^{\varphi} JN$ is a Noetherian $R \bowtie^f J$-module (cf. Part 5 of Remark 2.1). □

Proposition 3.2. *With above notation, assume that at least one of the following conditions holds*:

1. *JN is a Noetherian R-module (with the structure naturally induced by f).*
2. *$\varphi(M) + JN$ is a Noetherian R-module (with the structure naturally induced by f).*

Then $M \bowtie^{\varphi} JN$ is a Noetherian $R \bowtie^f J$-module if and only if M is a Noetherian R-module. In particular, if M is a Noetherian R-module and N is a Noetherian R-module (with the structure naturally induced by f), then $M \bowtie^{\varphi} JN$ is a Noetherian $R \bowtie^f J$-module for all ideals J of S.

Proof. Proposition 3.1 implies, without any extra assumption, that M is a Noetherian R-module if $M \bowtie^{\varphi} JN$ is a Noetherian $R \bowtie^f J$-module. Conversely, assume that M is a Noetherian R-module. One can easily see that (1) follows from (2). So assume that (1) holds. Since M and JN are Noetherian R-modules, they are Noetherian $R \bowtie^f J$-modules. Then by Remark 2.1(5), we obtain that $M \bowtie^{\varphi} JN$ is a Noetherian $R \bowtie^f J$-module. □

4. Coherent property

Let $f : R \to S$ be a ring homomorphism, J be an ideal of S, M be an R-modules, N be an S-module and $\varphi : M \to N$ be an R-module homomorphism and let n be a positive integer. Consider the function $\varphi^n : M^n \to N^n$ defined by $\varphi^n((m_i)_{i=1}^{i=n}) = (\varphi(m_i))_{i=1}^{i=n}$. Obviously, φ^n is an R-module homomorphism, and $JN^n = (JN)^n$ is a submodule of N^n. This allows us to define $M^n \bowtie^{\varphi^n} (JN)^n$.

We recall that the S-module N is an R-module induced by f, so it is the same for N^n. Hence, $rx = f(r)x$ for all $r \in R$ and $x \in N^n$ and so

$$\varphi^n(rm) = r\varphi^n(m) = f(r)\varphi^n(m) \text{ for all } r \in R \text{ and } x \in N^n.$$

We will use this in the proof of the following proposition.

Proposition 4.1. *Let F be a submodule of M^n. Then the following hold:*

(1) *Assume that F is a finitely generated R-module and JN is a finitely generated $(f(R)+J)$-module. Then $F \bowtie^{\varphi^n} (JN)^n$ is a finitely generated $R \bowtie^f J$-module.*

(2) *Suppose that $\varphi^n(F) \subseteq (JN)^n$. Then $F \bowtie^{\varphi^n} (JN)^n$ is a finitely generated $R \bowtie^f J$-module if and only if F is a finitely generated R-module and JN is a finitely generated $(f(R)+J)$-module.*

Proof. (1) Assume that $F = \sum_{i=1}^m Rf_i$, where $f_i \in F$ for all $i \in \{1, 2, \ldots, m\}$. Also let $(JN)^n = \sum_{i=1}^m (f(R)+J)n_i$ where $n_i \in (JN)^n$ since JN is a finitely generated $(f(R)+J)$-module. We claim that $F \bowtie^{\varphi^n} (JN)^n = \sum_{i=1}^m (R \bowtie^f J)(f_i, \varphi^n(f_i)) + \sum_{i=1}^m (R \bowtie^f J)(0, n_i)$. Indeed, $\sum_{i=1}^m (R \bowtie^f J)(f_i, \varphi^n(f_i)) + \sum_{i=1}^m (R \bowtie^f J)(0, n_i) \subseteq F \bowtie^{\varphi^n} (JN)^n$ since $(f_i, \varphi^n(f_i)) \in F \bowtie^{\varphi^n} (JN)^n$ and $(0, n_i) \in F \bowtie^{\varphi^n} (JN)^n$ for all $i \in \{1, 2, \ldots, m\}$. Let $(x, \varphi^n(x) + k) \in F \bowtie^{\varphi^n} (JN)^n$ where $x \in F$ and $k \in (JN)^n$. Then $x = \sum_{i=1}^m a_i f_i$ and $k = \sum_{i=1}^m (f(b_i) + j_i)n_i$ where $a_i \in R$ and $(f(b_i) + j_i) \in (f(R) + J)$ for all $i \in \{1, 2, \ldots, m\}$. Hence

$$\begin{aligned}
(x, \varphi^n(x) + k) &= (\sum_{i=1}^m a_i f_i, \varphi^n(\sum_{i=1}^m a_i f_i) + \sum_{i=1}^m (f(b_i) + j_i)n_i) \\
&= (\sum_{i=1}^m a_i f_i, \sum_{i=1}^m f(a_i)\varphi^n(f_i) + \sum_{i=1}^n (f(b_i) + j_i)n_i) \\
&= \sum_{i=1}^m (a_i f_i, f(a_i)\varphi^n(f_i)) + \sum_{i=1}^m (0, (f(b_i) + j_i)n_i) \\
&= \sum_{i=1}^m (a_i, f(a_i))(f_i, \varphi^n(f_i)) + \sum_{i=1}^m (b_i, f(b_i) + j_i)(0, n_i).
\end{aligned}$$

Therefore, $(x, \varphi^n(x) + k) \in \sum_{i=1}^m (R \bowtie^f J)(f_i, \varphi^n(f_i)) + \sum_{i=1}^m (R \bowtie^f J)(0, n_i)$ and so $F \bowtie^{\varphi^n} (JN)^n = \sum_{i=1}^m (R \bowtie^f J)(f_i, \varphi^n(f_i)) + \sum_{i=1}^m (R \bowtie^f J)(0, n_i)$ is a finitely generated $R \bowtie^f J$-module as desired.

(2) Suppose that $\varphi^n(F) \subseteq (JN)^n$. If F is a finitely generated R-module and JN is a finitely generated $(f(R)+J)$-module, then $F \bowtie^{\varphi^n} (JN)^n$ is a finitely generated $R \bowtie^f J$-module by (1). Conversely, assume that $F \bowtie^{\varphi^n} (JN)^n$ is a finitely generated $R \bowtie^f J$-module. Then, there exist $f_i \in F$ and $n_i \in (JN)^n$ for $i \in \{1, 2, \ldots, n\}$ such that $F \bowtie^{\varphi^n} (JN)^n = \sum_{i=1}^m (R \bowtie^f J)(f_i, \varphi^n(f_i) + n_i)$. So $F = \sum_{i=1}^m Rf_i$. On the other hand, we claim that $(JN)^n = \sum_{i=1}^m (f(R) + J)(\varphi^n(f_i) + n_i)$. Indeed, let $k \in (JN)^n$. Then $(0, k) = \sum_{i=1}^m (a_i, f(a_i) + j_i)(f_i, \varphi^n(f_i) + n_i)$ for some $a_i \in R$ and $j_i \in J$. Hence $k = \sum_{i=1}^m (f(a_i) + j_i)(\varphi^n(f_i) + n_i) \in \sum_{i=1}^m (f(R) + J)(\varphi^n(f_i) + n_i)$. So $(JN)^n \subseteq \sum_{i=1}^m (f(R) + J)(\varphi^n(f_i) + n_i)$. Since $\varphi^n(F) \subseteq (JN)^n$, then $\varphi^n(f_i) \in (JN)^n$ for all $i \in \{1, 2, \ldots, m\}$ and so $(\varphi^n(f_i) + n_i) \in (JN)^n$ for all $i \in \{1, 2, \ldots, m\}$. Hence

$\sum_{i=1}^{m}(f(R)+J)(\varphi^n(f_i)+n_i) \subseteq (JN)^n$. Therefore, $(JN)^n = \sum_{i=1}^{m}(f(R)+J)(\varphi^n(f_i)+n_i)$ is a finitely generated $f(R)+J$-module and so JN is a finitely generated $f(R)+J$-module. □

Recall that an R-module M is called a *coherent* R-module if it is finitely generated and every finitely generated submodule of M is finitely presented.

Theorem 4.2. (1) *Assume that JN is a finitely generated $f(R)+J$-module. If $M \bowtie^{\varphi} JN$ is a coherent $R \bowtie^f J$-module, then M is a coherent R-module.*

(2) *Assume that J is a finitely generated ideal of $f(R)+J$ and φ is surjective. If $M \bowtie^{\varphi} JN$ is a coherent $R \bowtie^f J$-module, then M is a coherent R-module.*

(3) *Assume that J and JN are finitely generated $f(R)+J$-modules and $\varphi^{-1}(JN)$ is a finitely generated R-module. Then $M \bowtie^{\varphi} JN$ is a coherent $R \bowtie^f J$-module if and only if M is a coherent R-module and $\varphi(M)+JN$ is a coherent $f(R)+J$-module.*

(4) *Assume that J is a finitely generated $f(R)+J$-module, φ is surjective and $\varphi^{-1}(JN)$ is a finitely generated R-module. Then $M \bowtie^{\varphi} JN$ is a coherent $R \bowtie^f J$-module if and only if M is a coherent R-module and $\varphi(M)+JN$ is a coherent $f(R)+J$-module.*

To prove Theorem 4.2, we need to establish the following lemmas. Before this, we recall the following well-known fact. Let $g : A \to B$ be a surjective ring morphism with kernel K. If T is an A-module annihilated by K, then T is canonically a B-module and $AX = BX$ for each subset X of T (let us say that the A-module structure on T is essentially the same as the B-module structure on T).

Lemma 4.3. (1) *$\{0\} \times JN$ is a finitely generated $R \bowtie^f J$-module if and only if JN is a finitely generated $f(R)+J$-module.*

(2) *$\varphi^{-1}(JN) \times \{0\}$ is a finitely generated $R \bowtie^f J$-module if and only if $\varphi^{-1}(JN)$ is a finitely generated R-module.*

(3) *If $M \bowtie^{\varphi} JN$ is a coherent $R \bowtie^f J$-module and $\varphi^{-1}(JN)$ is a finitely generated R-module, then $\varphi(M)+JN$ is a coherent $f(R)+J$-module.*

Proof. (1) Consider the second component projection from $R \bowtie^f J$ to $f(R)+J$. Then, for each subset X of the $R \bowtie^f J$-module $\{0\}\times JN$, one has $(R \bowtie^f J)X = (f(R)+J)X$.

(2) Consider the first component projection from $R \bowtie^f J$ to R. Then, for each subset X of the $R \bowtie^f J$-module $\varphi^{-1}(JN) \times \{0\}$, one has $(R \bowtie^f J)X = RX$.

(3) Suppose that $M \bowtie^{\varphi} JN$ is a coherent $R \bowtie^f J$-module and $\varphi^{-1}(JN)$ is a finitely generated R-module. Then $\varphi^{-1}(JN) \times \{0\}$ is a finitely generated $R \bowtie^f J$-module by (1). By Proposition 2.2, $\frac{M \bowtie^{\varphi} JN}{\varphi^{-1}(JN)\times\{0\}} = \varphi(M)+JN$ and so $\varphi(M)+JN$ is a coherent $R \bowtie^f J$-module by [7, Theorem 2.2.1], and since $f(R)+J$ is a finitely generated $R \bowtie^f J$-module ($f(R)+J \cong \{0\}\times(f(R)+J) = (R \bowtie^f J)(0,1)$), then $\varphi(M)+JN$ is a coherent $(f(R)+J)$-module by [7, Theorem 2.2.7]. □

Lemma 4.4. *Assume that J and $\varphi^{-1}(JN)$ are a finitely generated $(f(R)+J)$-module and an R-module, respectively. If M is a coherent R-module, then $\varphi^{-1}(JN)\times\{0\}$ is a coherent $R\bowtie^f J$-module.*

Proof. Since $\varphi^{-1}(JN)$ is a finitely generated R-module, then $\varphi^{-1}(JN)\times\{0\}$ is a finitely generated $R\bowtie^f J$-module. It remains to show that every finitely generated submodule of $\varphi^{-1}(JN)\times\{0\}$ is finitely presented. Assume that M is a coherent R-module and let K be a finitely generated submodule of $\varphi^{-1}(JN)\times\{0\}$. Then $K=F\times\{0\}$ where $F=\sum_{i=1}^{n}Rf_i$ for some positive integer n and $f_i\in F$. Consider the exact sequence of R-modules

$$0\to\ker v\to R^n\to F\to 0,$$

where $v((a_i)_{i=1}^{i=n})=\sum_{i=1}^{n}a_if_i$. We have $K=\sum_{i=1}^{n}(R\bowtie^f J)(f_i,0)$. Consider the exact sequence of $R\bowtie^f J$-module

$$0\to\ker u\to(R\bowtie^f J)^n\to K\to 0,$$

where $u((a_i,f(a_i)+j_i)_{i=1}^{i=n})=\sum_{i=1}^{n}(a_i,f(a_i)+j_i)(f_i,0)$. Then

$$\begin{aligned}\ker u&=\{(a_i,f(a_i)+j_i)_{i=1}^{i=n}\in(R\bowtie^f J)^n\mid\sum_{i=1}^{n}a_if_i=0\}\\&=\{((a_i)_{i=1}^{i=n},f^n((a_i)_{i=1}^{i=n})+(j_i)_{i=1}^{i=n})\in R^n\bowtie^{f^n}J^n\mid(a_i)_{i=1}^{i=n}\in\ker v\}\\&=\ker v\bowtie^{f^n}J^n.\end{aligned}$$

The second equality follows from the isomorphism $(R\bowtie^f J)^n\cong R^n\bowtie^{f^n}J^n$ [1, Page 3].

Since F is a submodule of M and M is coherent, then F is a finitely presented R-module and so $\ker v$ is finitely generated by the first sequence. Hence $\ker u=\ker v\bowtie^{f^n}J^n$ is a finitely generated $R\bowtie^f J$-module by [1, Lemma 2.4(1)]. Therefore, K is a finitely presented $R\bowtie^f J$-module by the second sequence and hence $\varphi^{-1}(JN)\times\{0\}$ is a coherent $R\bowtie^f J$-module. □

Proof of Theorem 4.2. (1) Assume that JN is a finitely generated $f(R)+J$-module. Suppose that $M\bowtie^{\varphi}JN$ is a coherent $R\bowtie^f J$-module. We have $\frac{R\bowtie^f J}{0\times J}=R$ by [4, Proposition 2.1]. Hence, using [7, Theorem 2.4.1], to prove that M is a coherent R-module it suffices to prove that it is coherent as an $R\bowtie^f J$-module. Since JN is a finitely generated $f(R)+J$-module, then $0\times JN$ is a finitely generated $R\bowtie^f J$-module. Hence, using Proposition 2.2 and [7, Theorem 2.2.1] our result follows immediately.

(2) Assume that J is a finitely generated ideal of $f(R)+J$. Suppose that $M\bowtie^{\varphi}JN$ is a coherent $R\bowtie^f J$-module. As in (1), using [7, Theorem 2.4.1], to prove that M is a coherent R-module it suffices to prove that it is coherent as an $R\bowtie^f J$-module. We first show that $(\{0\}\times J)(M\bowtie^{\varphi}JN)=0\times JN$. Indeed, it is clear that $(\{0\}\times J)(M\bowtie^{\varphi}JN)\subseteq\{0\}\times JN$. Now, let $(0,n)\in\{0\}\times JN$. Then $n=\sum_{i=1}^{n}j_in_i$ where $j_i\in J$ and $n_i\in N$. Since φ is surjective, then for each n_i there exists m_i such that $\varphi(m_i)=n_i$. Hence $(0,n)=$

$\sum_{i=1}^{n}(0,j_i)(m_i,\varphi(m_i)) \in (\{0\}\times J)(M \bowtie^{\varphi} JN)$; so that $(\{0\}\times J)(M \bowtie^{\varphi} JN) = 0\times JN$. Thus, $0\times JN$ is a finitely generated $R \bowtie^f J$-module since $(\{0\}\times J)$ and $M \bowtie^{\varphi} JN$ are finitely generated $R \bowtie^f J$-modules. It follows from Proposition 2.2 and [7, Theorem 2.2.1] that M is a coherent $R \bowtie^f J$-module.

(3) Assume that J and JN are finitely generated $f(R)+J$-modules and $\varphi^{-1}(JN)$ a finitely generated R-module. Suppose that $M \bowtie^{\varphi} JN$ is a coherent $R \bowtie^f J$-module. Then M is a coherent R-module by (1) and $\varphi(M)+JN$ is a coherent $f(R)+J$-module by Lemma 4.3(2). Conversely, suppose that M is a coherent R-module and $\varphi(M)+JN$ is a coherent $f(R)+J$-module. Then $\varphi^{-1}(JN)\times\{0\}$ is a coherent $R \bowtie^f J$-module by Lemma 4.4. By Proposition 2.2, $\frac{M\bowtie^{\varphi} JN}{\varphi^{-1}(JN)\times\{0\}} = \varphi(M)+JN$. Hence, $M \bowtie^{\varphi} JN$ is coherent by [7, Theorem 2.2.1].

(4) Assume that J is a finitely generated $f(R)+J$-module, φ is surjective and $\varphi^{-1}(JN)$ a finitely generated R-module. Suppose that $M \bowtie^{\varphi} JN$ is a coherent $R \bowtie^f J$-module. Then, M is a coherent R-module by (2) and $\varphi(M)+JN$ is a coherent $f(R)+J$-module by Lemma 4.3(2). We prove the converse by the same arguments in (3). □

If we set $M = R$, $N = S$ and $\varphi = f$, then the above theorem recovers a known result for the amalgamation of rings.

Corollary 4.5 ([1, Theorem 2.2 (2)]). *Assume that J and $f^{-1}(J)$ are finitely generated ideals of $f(R)+J$ and R, respectively. Then $R \bowtie^f J$ is a coherent ring if and only if R and $f(R)+J$ are coherent rings.*

By letting $R = S$, $N = M$ and $\varphi = Id_M$, Theorem 4.2 recovers the special case of amalgamated duplication of a module along an ideal, as recorded in the next corollary.

Corollary 4.6 ([2, Proposition 2.6]). *Let I be a finitely generated ideal of R. Then $M \bowtie I$ is a coherent $R \bowtie I$-module if and only if M is a coherent R-module.*

Theorem 4.2 recovers also a known result for the duplication of a ring along an ideal by taking $R = S = M = N$ and $\varphi = Id_R$.

Corollary 4.7 ([1, Corollary 2.8]). *Let I be a finitely generated ideal of R. Then $R \bowtie I$ is a coherent ring if and only if R is a coherent ring.*

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