Honam Mathematical J. **43** (2021), No. 1, pp. 141–151
https://doi.org/10.5831/HMJ.2021.43.1.141

GENERALIZATION OF LAGUERRE MATRIX POLYNOMIALS FOR TWO VARIABLES

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Abstract. The main object of the present paper is to introduce the generalized Laguerre matrix polynomials for two variables. We prove that these matrix polynomials are characterized by the generalized hypergeometric matrix function. An explicit representation, generating functions and some recurrence relations are obtained here. Moreover, these matrix polynomials appear as solution of a differential equation.

1. Introduction and preliminaries

Orthogonal matrix polynomials comprise an emerging fields of study, with important results in both Lie group theory and number theory its applications being still is contained to appear in the literature. Theory of classical orthogonal polynomials are extended to the orthogonal matrix polynomials ([9]). The study of functions of matrices is a very popular topic in the Matrix Analysis literature. Matrix generalization of special functions has become important in the last two decades. The reason of importance have many motivations. For instance, using special matrix functions provides solutions for some physical problems. Also, special matrix functions are in connection with different matrix functions ([4], [8], [14]). Jodar et al introduced Laguerre matrix polynomials in ([11]). Some important properties of Laguerre matrix polynomials such as asymptotic expressions relations between different matrix functions and generating matrix functions are studied ([8], [9], [11]). Indeed, in recent papers, matrix polynomials have significant emergent. Some results in the theory of classical orthogonal polynomials have been extended to orthogonal matrix polynomials ([1], [2], [4], [5], [7], [11], [13]) Throughout this paper, for a matrix A in $C^{r\times r}$, its spectrum $\sigma(A)$ denotes the set of all eigenvalues of A. The

Received November 4, 2020. Accepted February 9, 2021.
2020 Mathematics Subject Classification. 33C45, 42C05.
Key words and phrases. Laguerre matrix polynomials, generating functions, recurrence relations.
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two-norm will be denoted by $||A||_2$ and it is defined by ([10])

$$||A||_2 = \sup_{x \neq 0} \frac{||Ax||_2}{||x||_2}$$

where for a vector x in $C^{r\times r}$, $||x||_2 = (x^T x)^{\frac{1}{2}}$ is the Euclidean norm of x. Let us denote the real numbers $M(A)$ and $m(A)$ as in the following

$$(1) \qquad M(A) = max\left\{Re(z) : z \in \sigma(A)\right\}; m(A) = min\left\{Re(z) : z \in \sigma(A)\right\}.$$

If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set of the complex plane, and A, B are matrices in $C^{r\times r}$, with $\sigma(A) \subset \Omega$ and $\sigma(B) \subset \Omega$, such that $AB = BA$, then from the properties of the matrix functional calculus in ([6]) , it follows that

$$(2) \qquad f(A)g(B) = g(B)f(A).$$

Throughout this study, a matrix polynomial of degree n in x means an expression of the form n

$$(3) \qquad P_n(x) = A_n x^n + A_{n-1}x^{n-1} + ... + A_1 x^1 + A_0.$$

where x is a real variable or complex variable, A_j, for $0 \leq j \leq n$ and $A_n \neq 0$, where 0 is the null matrix or zero matrix in $C^{r\times r}$. We recall reciprocal gamma function denoted by $\Gamma^{-1}(z) = \frac{1}{\Gamma(z)}$ is an entire function of the complex variable z. Then, for any matrix A in $C^{r\times r}$, the image of $\Gamma^{-1}(A)$ acting on A, denoted by $\Gamma^{-1}(A)$ is a well defined matrix. Furthermore, if A is a matrix such that

$$(4) \qquad A + nI \; is \; invertible \; for \; every \; integer \; n \geq 0,$$

where I is the identity matrix in $C^{r\times r}$, then from ([11]) it follows that

$$(5) \quad (A)_n = A(A+I)...(A+(n-1)I) = \Gamma(A+nI)\Gamma^{-1}(A); n \geq 1; (A)_0 = 1.$$

If A is a positive stable matrix in $C^{r\times r}$, then the gamma matrix function, $\Gamma(A)$, is defined ([12]) by

$$(6) \qquad \Gamma(A) = \int_0^\infty e^{-t} t^{A-1} dt, \;\; Re(A) > 0.$$

And if A, B is a positive stable matrices in $C^{r\times r}$, then the beta matrix function, $\beta(A, B)$, are defined ([12]) by

$$(7) \qquad \beta(A, B) = \int_o^1 t^{A-1}(1-t)^{B-1} dt, \;\; Re(A) >, Re(B) > 0.$$

The generalized hypergeometric matrix function ${}_pF_q$ $(p,\, q \in \mathbb{N})$ given in ([13])

$$(8) \qquad {}_pF_q\left[\begin{matrix} A_1, \, \ldots, \, A_p\, ; \\ B_1, \, \ldots, \, B_q\, ; \end{matrix}\; x\right] = \sum_{n=0}^{\infty} \frac{(A_1)_n \cdots (A_p)_n}{(B_1)_n \cdots (B_q)_n} \frac{x^n}{n!}$$
$$= {}_pF_q(A_1, \, \ldots, \, A_p; \, B_1, \, \ldots, \, B_q; \, x).$$

Where A_i and B_j are matrices in $C^{r\times r}$ such that B_j; $1 \leq j \leq q$ satisfy condition. With p = 1 and q = 0, one gets the following relation by ([13])

$$ {}_1F_0(A; -; x) = (1-x)^{-A} = \sum_{n=0}^{\infty} \frac{(A)_n x^n}{n!}, \tag{9} $$

In ([12], [14], [15]) if λ is a complex number with $Re(\lambda) > 0$ and A is a positive stable matrix in $\mathbb{C}^{r\times r}$, with $A + nI$ invertible for every integer $n \geq 1$, then the nth Laguerre matrix polynomials $L_n^{(\lambda,A)}(x, y)$ for two variables are defined by the following generating matrix relation:

$$ \sum_{n=0}^{\infty} L_n^{(\lambda,A)}(x, y)t^n = (1-yt)^{-I-A} e^{\left(\frac{-\lambda xt}{1-yt}\right)}. \tag{10} $$

From the above equation we have

$$ L_n^{(\lambda,A)}(x, y) = \sum_{k=o}^{n} \frac{(I + A)_n(-x\lambda)^k(y)^{n-k}}{(n-k)!(I + A)_k k!}, \lambda \geq 0. \tag{11} $$

The second-order matrix differential equations of the form

$$ \left(xI\frac{d^2}{dx^2} + \left((1+A) - \lambda\frac{x}{y}\right) I\frac{d}{dx} + \frac{nI}{y}\right) L_{1,n}^{(\lambda,A)}(x, y) = 0. \tag{12} $$

where $n \in \mathbb{N}$, and $A \in \mathbb{C}^{r\times r}$. And $\mathbb{C}^{r\times r}$ denotes the vector space containing all square matrices with r rows and r columns with entries in the complex number $\mathbb{C}$.

2. Generalized Laguerre Matrix polynomials for two variables

We begin by defining generalized Laguerre matrix polynomials $L_{p,n}^{(\lambda,A)}(x, y)$ for two variables by the following generating function:

$$ \frac{1}{(1-yt)^{I+A}} \exp\left(\frac{-\lambda x^p t^p}{(1-yt)^p}\right) = \sum_{n=0}^{\infty} L_{n,p}^{(\lambda,A)}(x, y)\, t^n \tag{13} $$

$$ \left(p \in \mathbb{N};\ x, y \in \mathbb{C},\, A \in \mathbb{C}^{r\times r}\right). $$

Here and elsewhere, let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of positive integers, real number and complex numbers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Obviously $L_{n,1}^{(\lambda,A)}(x, y) = L_n^{(\lambda,A)}(x, y)$.

Hereafter we explore certain formulas and properties involving the generalized Laguerre Matrix polynomials in (13). Throughout, let $F(p; \lambda, x, y, t)$ be the left-handed generating function in (13).

Explicit representation.

We give an explicit expression of the generalized Laguerre matrix polynomials $L_{n,p}^{(\lambda,A)}(x,y)$ for two variables in the following theorem.

Theorem 2.1. *Let $x, \in \mathbb{C}$, $p \in \mathbb{N}$, $n \in \mathbb{N}_0$ and A is positive stable matrix in $\mathbb{C}^{r\times r}$. Then*

$$L_{n,p}^{(\lambda,A)}(x,y) = (I+A)_n \sum_{k=0}^{[n/p]} \frac{(-\lambda)^k (y)^{n-pk}}{k!\,(I+A)_{pk}\,(n-pk)!}\, x^{pk} \tag{14}$$

$$= \frac{(I+A)_n}{n!} \sum_{k=0}^{[n/p]} \frac{(\lambda)^k (-1)^{(p+1)k}\,(-nI)_{pk} (y)^{n-pk}}{k!\,(I+A)_{pk}}\, x^{pk}. \tag{15}$$

Here and throughout, $[m]$ denotes the greatest integer less than or equal to $m \in \mathbb{R}$. Or, equivalently,

$$L_{n,p}^{(\lambda,A)}(x,y) = \frac{(I+A)_n y^n}{n!}\, {}_pF_p \left[\begin{array}{l} \frac{-n}{p}I,\ \frac{-n+1}{p}I,\ \ldots,\ \frac{-n-1+p}{p}I\,; \\ \quad \frac{A+I}{p},\ \frac{A+2I}{p},\ \ldots,\ \frac{A+pI}{p}\,; \end{array} (-1)^{p+1}\lambda \left(\frac{x}{y}\right)^p \right]. \tag{16}$$

Proof. Expanding the exponential in the left-hand side of (13), we find

$$F(p;\lambda,x,y,t) = \frac{1}{(1-yt)^{I+A+pk}} \sum_{k=0}^{\infty} \frac{\lambda^k (-1)^k\, x^{pk}\, t^{pk}}{k!}.$$

Employing the binomial theorem

$$(1-x)^{-A} = {}_1F_0(A;-;x) = \sum_{n=0}^{\infty} \frac{(A)_n x^n}{n!}, \quad (A \in \mathbb{C}^{r\times r};\ |x|<1), \tag{17}$$

we obtain the following double series

$$F(p;\lambda,x,y,t) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k\,(I+A+pk)_n\, x^{pk} y^n}{k!\,n!}\, t^{n+pk}. \tag{18}$$

Recall ([11]) if $A(n,k)$ and $B(n,k)$ are matrices in $\mathbb{C}^{r\times r}$ and satisfying the spectral condition (4) for $n \geq 0$, $k \geq 0$, then it follows that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(n,k) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(n,n-k), \tag{19}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(n,k) = \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{p}]} A(n,n-pk), \quad (p \in \mathbb{N}), \tag{20}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{p}]} A(n,k) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(n, n+pk), \quad (p \in \mathbb{N}), \tag{21}$$

where $A_{x,y}$ denotes a function of two variables x and y and the involved double series is assumed to be absolutely convergent.

Applying (20) in (18), we get

$$F(p;\lambda,x,y,t) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} \frac{(-1)^k \lambda^k \,(I+A+pk)_{n-pk}\, x^{pk} y^{n-pk}}{k!\,(n-pk)!}\, t^n. \tag{22}$$

Equating the coefficients of t^n in the right members of (13) and (22) yields

$$L_{n,p}^{(\lambda,A)}(x,y) = \sum_{k=0}^{[n/p]} \frac{(-1)^k \lambda^k \,(I+A+pk)_{n-pk}}{k!\,(n-pk)!}\, x^{pk} y^{n-pk}. \tag{23}$$

Using ([12]) a known identity

$$\frac{1}{(n-k)!} I = \frac{(-1)^k (-nI)_k}{n!} \quad (k,\, n \in \mathbb{N}_0;\; 0 \le k \le n)\,, \tag{24}$$

we derive

(25)

$$(I+A+pk)_{n-pk} = \frac{(I+A)_n}{(I+A)_{pk}} \quad \text{and} \quad \frac{1}{(n-pk)!} I = \frac{(-1)^{pk}(-nI)_{pk}}{n!}; 0 \le pk \le n.$$

Hence, using (25) in (23) leads to the desired identity (15).

Finally, applying the multiplication formula

$$\frac{(-1)^{pk}(-nI)_{pk}}{n!} = \frac{(-1)^{pk}(p)^{pk}}{n!} \prod_{j=1}^{p} \left(\frac{j-n-1}{p} I\right)_n ;\; 0 \le pk \le n. \tag{26}$$

$$(\, p \in \mathbb{N};\; n \in \mathbb{N}_0)$$

to (15) gives the equivalent expression (16).

□

Generating function.

We establish two generating functions for the generalized Laguerre matrix polynomials $L_{n,p}^{(\lambda,A)}(x,y)$ for two variables in the following theorem.

Theorem 2.2. *Let $\lambda, t,\, x,\, y\, c \in \mathbb{C}$, $p \in \mathbb{N}$ and A is positive stable matrix in $\mathbb{C}^{r\times r}$ and satisfying the spectral condition (4). Then*

$$\begin{aligned} &e^{yt}\, {}_0F_p \left(\text{—} ;\, \frac{A+I}{p},\, \frac{A+2I}{p},\, \ldots,\, \frac{A+pI}{p}\, ;\, -\lambda \left(\frac{xzt}{p}\right)^p \right) \\ &= \sum_{n=0}^{\infty} \frac{L_{n,p}^{(\lambda,A)}(x,y)\, t^n}{(I+A)_n} \end{aligned} \tag{27}$$

and

$$\frac{1}{(1-yt)^c}\,{}_pF_p\left(\begin{array}{l}\frac{c}{p},\ \frac{c+1}{p},\ \ldots,\ \frac{c+p-1}{p};\\ \frac{A+I}{p},\ \frac{A+2I}{p},\ \ldots,\ \frac{A+pI}{p};\end{array}\ -\lambda\left(\frac{xt}{1-yt}\right)^p\right) = \sum_{n=0}^{\infty}\frac{(c)_n\,L_{n,p}^{(\lambda,A)}(x,y)\,t^n}{(I+A)_n} \quad (|t|<1). \tag{28}$$

Proof. Using (14), (21), and (26), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{L_{n,p}^{(\lambda,A)}(xz,y)\,t^n}{(I+A)_n} &= \sum_{n=0}^{\infty}\frac{(yt)^n}{n!}\sum_{k=0}^{\infty}\frac{(-\lambda x^p z^p t^p)^k}{k!\,(I+A)_{pk}}\\ &= e^{yt}\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!\prod_{j=1}^{p}\left(\frac{A+jI}{p}\right)_k}\left(\frac{xzt}{p}\right)^{pk}.\end{aligned} \tag{29}$$

In view of (8), the rightmost term of (29) can be expressed as the left-hand side of (27).

Employing (14), (17), and (21), we find

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(c)_n\,L_{n,p}^{(\lambda,A)}(x,y)\,t^n}{(I+A)_n} &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(c+pk)_n\,(yt)^n}{n!}\cdot\frac{(c)_{pk}\,\{-\lambda(xt)^p\}^k}{k!\,(I+A)_{pk}}\\ &= \frac{1}{(1-yt)^c}\sum_{k=0}^{\infty}\frac{(c)_{pk}}{k!\,(I+A)_{pk}}\left\{-\lambda\left(\frac{xt}{1-yt}\right)^p\right\}^k,\end{aligned}$$

which, upon using (26) and (8), leads to the left-hand member of (28).

□

It is noted that the case $c = I + A$ of (28) yields the generating function (13).

Recurrence relation.

We give some recurrence relations involving the generalized Laguerre matrix polynomials $L_{n,p}^{(\lambda,A)}(x,y)$ for two variables and their derivative in the following theorem.

Theorem 2.3. *Let $\lambda, t, x, c \in \mathbb{C}$, $p \in \mathbb{N}$ and A is positive stable matrix in $\mathbb{C}^{r\times r}$ and satisfying the spectral condition (4). Also let $D = \frac{d}{dx}$. Then*

$$x\,\lambda D L_{n,p}^{(\lambda,A)}(x,y) - n\,L_{n,p}^{(\lambda,A)}(x,y) + y(A+(n+1)I)\,L_{n-1,p}^{(\lambda,A)}(x,y) = 0; \tag{30}$$

$$DL_{n,p}^{(\lambda,A)}(x,y) = \begin{cases} 0 & (0 \le n \le p-1)\\ -p\,\lambda x^{p-1}\,L_{n-p,p}^{(\lambda,A+p)}(x,y) & (n \ge p);\end{cases} \tag{31}$$

(32)

$$y(A+(n+1)I)\, L_{n-1,p}^{(\lambda,A)}(x,y) - n\, L_{n,p}^{(\lambda,A)}(x,y) = p\,\lambda^2 x^p\, L_{n-p,p}^{(\lambda,A+p)}(x,y) \quad (n \geq p).$$

Proof. From (29), we can set

$$G(p;\lambda,x,y,t) := \sum_{n=0}^{\infty} \frac{L_{n,p}^{(\lambda,A)}(x)\, t^n}{(I+A)_n} = e^{yt}\,\Phi\left(\frac{-\lambda x^p t^p}{p^p}\right), \tag{33}$$

where the function

$$\Phi\left(\frac{-\lambda\, x^p\, t^p}{p^p}\right) = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!\, \prod_{j=1}^{p} \left(\frac{A+jI}{p}\right)_k} \left(\frac{xt}{p}\right)^{pk}.$$

Differentiating $G(p;\lambda,x,y,t)$ with respect to x and t, respectively, gives

$$G_x(p;\lambda,x,y,t) = e^{yt}\,\Phi'\left(\frac{-\lambda\, x^p\, t^p}{p^p}\right)\cdot \frac{-\lambda}{p^{p-1}} x^{p-1}\, t^p$$

and

$$G_t(p;\lambda,x,y,t) = ye^{yt}\Phi\left(\frac{-\lambda\, x^p\, t^p}{p^p}\right) + e^{yt}\,\Phi'\left(\frac{-\lambda\, x^p\, t^p}{p^p}\right)\cdot \frac{-\lambda}{p^{p-1}} x^p\, t^{p-1}.$$

Combining $G_x(p;\lambda,x,y,t)$ and $G_t(p;\lambda,x,y,t)$ yields

$$\lambda x\, G_x(p;\lambda,x,y,t) - t\, G_t(p;\lambda,x,y,t) + yt\, G(p;\lambda,x,y,t) = 0. \tag{34}$$

Applying the series in (33) to (34), we obtain

(35)

$$\sum_{n=1}^{\infty} \frac{\lambda x\, DL_{n,p}^{(\lambda,A)}(x,y)\, t^n}{(I+A)_n} - \sum_{n=1}^{\infty} \frac{n\, L_{n,p}^{(\lambda,A)}(x,y)\, t^n}{(I+A)_n} + y\sum_{n=1}^{\infty} \frac{L_{n-1,p}^{(\lambda,A)}(x,y)\, t^n}{(I+A)_{n-1}} = 0.$$

We find from (35) that each coefficient of t^n should be zero, which gives (30).

Differentiating both sides of (13) provides

$$\begin{aligned}\sum_{n=1}^{\infty} DL_{n,p}^{(\lambda,A)}(x,y)\, t^n &= \frac{1}{(1-yt)^{I+A+p}} \exp\left(\frac{-\lambda x^p t^p}{(1-yt)^p}\right)\cdot \left(-\lambda p\, x^{p-1}\, t^p\right) \\ &= -p\lambda\, x^{p-1} \sum_{n=0}^{\infty} L_{n,p}^{(\lambda,A+p)}(x,y)\, t^{n+p} \\ &= -p\lambda\, x^{p-1} \sum_{n=p}^{\infty} L_{n-p,p}^{(\lambda,A+p)}(x)\, t^n,\end{aligned}$$

which, upon equating the coefficients of t^n $(n \geq p)$ in the leftmost and rightmost members, produces (31).

Setting (31) in (30) provides (32).

□

Differential equation.

We provide a differential equation which is satisfied by the generalized Laguerre polynomials $L_{n,p}^{(\lambda,A)}(x,y)$ for two variables in the following theorem (for differential equation whose solution is ${}_pF_q$, see, e.g., [15, Section 47]).

Theorem 2.4. *Let $\lambda, t, x, c \in \mathbb{C}$, $p \in \mathbb{N}$ and A is positive stable matrix in $C^{r\times r}$ and satisfying the spectral condition (4). Also let $\theta = x\frac{d}{dx}$. Then*

$$\begin{aligned}\Bigg[&\frac{1}{p}\theta\prod_{j=0}^{p}\left(\frac{1}{p}(\theta I - I + A + jI)\right)\\ &+\lambda(-1)^{p-1}\left(\frac{x}{y}\right)^{p}\prod_{j=0}^{p}\frac{1}{p}(\theta + j - n - 1)I\Bigg] L_{p,n}^{(\lambda,A)}(x,y) = 0.\end{aligned} \tag{36}$$

Proof. We find from (15) that

$$\phi = \frac{(I+A)_n(y)^n}{n!}{}_pF_p\left(\frac{-n}{p}I, \frac{-n+1}{p}I \ldots \frac{-n+p-1}{p}I; \frac{I+A}{p}, \frac{2I+A}{p} \ldots \frac{pI+A}{p}; \lambda(-1)^{p-1}\left(\frac{x}{y}\right)^p\right).$$

$$= \frac{(I+A)_n}{n!}\sum_{k=o}^{[\frac{n}{p}]}\frac{\prod_{j=1}^{p}\left(\frac{j-n-1}{p}I\right)_k(-1)^{(p-1)k}\lambda^k x^{pk}(y)^{n-pk}}{\prod_{j=1}^{p}\left(\frac{jI+A}{p}\right)_k k!},$$

Since $\frac{1}{p}\theta x^{pk} = kx^{pk}$, it follows that

$$\left[\frac{1}{p}\theta\prod_{j=0}^{p}\frac{I}{p}(\theta - 1 + A + j)\right] = \frac{(I+A)_n}{n!}\sum_{k=o}^{[\frac{n}{p}]}\frac{\prod_{j=1}^{p}\left(\frac{jn-1}{p}I\right)_k\left(\frac{j+A+k-1}{p}I\right)(-1)^{(p-1)k}\lambda^k x^{pk}(y)^{n-pk}}{\prod_{i=1}^{p}\left(\frac{jI+A}{p}\right)_k(k-1)!},$$

But the last factor in $\left(\frac{jI+A}{p}\right)_k$ is $\left(\frac{jI+A+k-1}{p}\right)$ so that

$$= \frac{(I+A)_n}{n!}\sum_{k=o}^{[\frac{n}{p}]}\frac{\prod_{j=1}^{p}\left(\frac{j-n-1}{p}I\right)_k(-1)^{(p-1)k}\lambda^k x^{pk}(y)^{n-pk}}{\prod_{j=1}^{p}\left(\frac{jI+A}{p}\right)_{k-1}(k-1)!},$$

Now we replace k by (k+1) and have

$$\left[\frac{1}{p}\theta\prod_{j=0}^{p}\frac{1}{p}(\theta - 1 + A + j)I\right] = \frac{(I+A)_n}{n!}\sum_{k=o}^{[\frac{n}{p}]}\frac{\prod_{j=1}^{p}\left(\frac{j-n-1}{p}I\right)_{k+1}(-1)^{(p-1)(k+1)}\lambda^{k+1}x^{p(k+1)}(y)^{n-p(k+1)}}{\prod_{j=1}^{p}\left(\frac{jI+A}{p}\right)_k k!},$$

$$= \lambda(-1)^{p-I}\left(\frac{x}{y}\right)^p \frac{(I+A)_n}{n!}\sum_{k=o}^{[\frac{n}{p}]}\frac{\prod_{j=1}^{p}\left(\frac{j-n-1}{p}I\right)_k \prod_{j=1}^{p}\left(\frac{k+j-n-1}{p}I\right)(-1)^{(p-1)k}\lambda^k x^{pk}(y)^{n-pk}}{\prod_{j=1}^{p}\left(\frac{jI+A}{p}\right)_k k!},$$

$$= \lambda(-1)^{p-1}\left(\frac{x}{y}\right)^p\left[\prod_{j=0}^{p}\frac{1}{p}(\theta+j-n-1)I\right]\phi.$$

□

Some other properties.

We provide some other identities involving the generalized Laguerre polynomials $L_{n,p}^{(\lambda,A)}(x,y)$ for two variables in the following theorem.

Theorem 2.5. *Let $\lambda, t, x, y\, c \in \mathbb{C}$, $p, n \in \mathbb{N}$ and A, B is positive stable matrix in $\mathbb{C}^{r\times r}$ and satisfying the spectral condition (4). Then*

$$L_{n,p}^{(\lambda,A)}(x,y) = \sum_{k=0}^{n}\frac{(A-B)_k\, L_{n-k,p}^{(\lambda,B)}(x,y)}{k!}; \tag{37}$$

$$L_{n,p}^{(\lambda,A+B+I)}(z,y) = \sum_{k=0}^{n} L_{k,p}^{(\lambda,A)}(x,y)\, L_{n-k,p}^{(\lambda,B)}(z,y), \tag{38}$$

where $x^p + z^p \in \mathbb{C}\setminus\{0\}$ and $w := (x^p+z^p)^{\frac{1}{p}}$ whose principal branch can be chosen;

$$L_{n,p}^{(\lambda,A)}(xz,y) = \sum_{k=0}^{n}\frac{(I+A)_n\,(1-z)^{n-k}\, z^k\, L_{k,p}^{(\lambda,A)}(x,y)}{(n-k)!\,(I+A)_k}. \tag{39}$$

Proof. From (13), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} L_{n,p}^{(\lambda,A)}(x,y)\, t^n &= (1-yt)^{-I-A}\exp\left(\frac{-\lambda x^p t^p}{(1-yt)^p}\right)\\
&= (1-t)^{-(A-B)}\cdot(1-yt)^{-I-B}\exp\left(\frac{-\lambda x^p t^p}{(1-yt)^p}\right)\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(A-B)_k}{k!}L_{n,p}^{(\lambda,B)}(x,y)t^{n+k}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(A-B)_k}{k!}L_{n-k,p}^{(\lambda,B)}(x,y)t^{n},
\end{aligned}$$

which, upon equating the coefficients of t^n, yields (37).

We find from (13) that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{k=0}^{n} L_{k,p}^{(\lambda,A)}(x,y)\, L_{n-k,p}^{(\lambda,B)}(z,y)\, t^n \\
&\quad = (1-yt)^{-I-A} \exp\left(\frac{-\lambda x^p t^p}{(1-yt)^p}\right) (1-t)^{-I-B} \exp\left(\frac{-\lambda z^p t^p}{(1-yt)^p}\right) \\
&\quad = (1-yt)^{-I-(A+B+I)} \exp\left(\frac{-\lambda w^p t^p}{(1-yt)^p}\right) \\
&\quad = \sum_{n=0}^{\infty} L_{n,p}^{(A+B+I)}(w,y)\, t^n,
\end{aligned}$$

which, upon equating the coefficients of t^n, gives (38).

We consider

$$\begin{aligned}
&e^{yt}\, {}_0F_p\left(\text{---}; \frac{A+I}{p}, \frac{A+2I}{p}, \ldots, \frac{A+pI}{p}; -\lambda\left(\frac{xzt}{p}\right)^p\right) \\
&= e^{(1-z)yt}\, e^{yzt}\, {}_0F_p\left(\text{---}; \frac{A+I}{p}, \frac{A+2I}{p}, \ldots, \frac{A+pI}{p}; -\lambda\left(\frac{x(zt)}{p}\right)^p\right),
\end{aligned}$$

which, in view of (27), produces

$$\sum_{n=0}^{\infty} \frac{L_{n,p}^{(\lambda,A)}(xz,y)\, t^n}{(I+A)_n} = \left(\sum_{n=0}^{\infty} \frac{(1-zt)^n\, y^n t^n}{n!}\right)\left(\sum_{k=0}^{\infty} \frac{L_{k,p}^{(\lambda,A)}(x,y)\, z^k t^k}{(I+A)_k}\right).$$

Then, from the last equality, we obtain (39).

□

3. Conclusion remarks

The generalized Laguerre matrix polynomials for two variables are introduced here and their properties and formulas presented are hoped to be potentially useful. Since $L_{n,1}^{(\lambda,A)}(x,y) = L_n^{(\lambda,A)}(x,y)$, the results in Section 2 reduce to yield certain properties and formulas for the classical Laguerre matrix polynomials $L_n^{(\lambda,A)}(x,y)$ for two variables.

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