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ON TRIGONOMETRICALLY QUASI-CONVEX FUNCTIONS

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Abstract. In this paper, we introduce and study the concept of trigonometrically quasi-convex function. We prove Hermite-Hadamard type inequalities for the newly introduced class of functions and obtain some new Hermite-Hadamard inequalities for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is trigonometrically quasi-convex convex. We also extend our initial results to functions of several variables. Next, we point out some applications of our results to give estimates for the approximation error of the integral the function in the trapezoidal formula.

1. Introduction

Let I be a non-empty interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a convex function. Then the following inequalities hold

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

for all $a, b \in I$ with $a < b$. This double inequality is well known as the Hermite-Hadamard inequality (for more information, see [4]). Since then, some refinements of the Hermite-Hadamard inequality for convex functions have been obtained [2, 12].

We recall that the notion of quasi-convex function generalizes the notion of convex function. More exactly, a function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be quasi-convex on $[a, b]$ if

$$f(tx + (1-t)y) \leq max\{f(x), f(y)\}$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex ([5]).
In [11], Kadakal gave the concept of trigonometrically convex function as follows:

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Definition 1.1 ([11]). *A non-negative function $f : I \to \mathbb{R}$ is called trigonometrically convex if for every $x, y \in I$ and $t \in [0, 1]$,*

$$f(tx + (1-t)y) \leq \left(\sin\frac{\pi t}{2}\right) f(x) + \left(\cos\frac{\pi t}{2}\right) f(y).$$

The class of all trigonometrically convex functions is denoted by $TC(I)$ on interval I.

Throughout this paper, we will use the following notation for brevity:

$$Q_f(x, y) = \max\{f(x), f(y)\}.$$

The main purpose of this paper is to introduce the concept of trigonometrically quasi-convex function which is connected with the concepts of quasi-convex function and trigonometrically convex convex function and establish some new Hermite-Hadamard type inequality for this class of functions. In recent years, for some related Hermite-Hadamard type inequalities, see [5, 6, 7, 8, 9, 10].

2. Main Results

In this section, we introduce a new concept, which is called trigonometrically quasi-convex function and we give by setting some algebraic properties for the trigonometrically quasi-convex functions.

Definition 2.1. *A function $f : I \to \mathbb{R}$ is called trigonometrically quasi-convex functions if for every $x, y \in I$ and $t \in [0, 1]$,*

$$f(tx + (1-t)y) \leq \left(\sin\frac{\pi t}{2} + \cos\frac{\pi t}{2}\right) \max\{f(x), f(y)\}.$$

We discuss some connections between the class of trigonometrically quasi-convex functions and other classes of generalized convex functions.

Remark 2.2. *Clearly, every nonnegative quasi-convex function is a trigonometrically quasi-convex function. Indeed, if $f : I \to \mathbb{R}$ is an arbitrary nonnegative quasi-convex function, then since $\sin\frac{\pi t}{2} + \cos\frac{\pi t}{2} \geq 1$ for all $t \in [0, 1]$, for every $x, y \in I$ and $t \in [0, 1]$ we have*

$$f(tx + (1-t)y) \leq Q_f(x, y) \leq \left(\sin\frac{\pi t}{2} + \cos\frac{\pi t}{2}\right) Q_f(x, y).$$

Moerever, Since every convex function is a quasi-convex function, we say that every nonnegative convex function is a trigonometrically quasi-convex function.

Proposition 2.3. *Every trigonometrically convex function is trigonometrically quasi-convex functions.*

Proof. Let $f : I \to \mathbb{R}$ be an arbitrary trigonometrically convex function, then we can write

$$\begin{aligned} f\left(tx+(1-t)y\right) &\leq \left(\sin\frac{\pi t}{2}\right)f(x)+\left(\cos\frac{\pi t}{2}\right)f(y) \\ &\leq \left(\sin\frac{\pi t}{2}\right)Q_f(x,y)+\left(\cos\frac{\pi t}{2}\right)Q_f(x,y) \\ &= \left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)Q_f(x,y). \end{aligned}$$

□

Theorem 2.4. *Let $f : [a,b] \to \mathbb{R}$ and $c \in \mathbb{R}$ ($c \geq 0$). If f is trigonometrically quasi-convex functions, then cf is trigonometrically quasi-convex function.*

Proof. Let f be trigonometrically quasi-convex function and $c \in \mathbb{R}$ *(* $c \geq 0$*), then*

$$\begin{aligned} (cf)\left(tx+(1-t)y\right) &\leq c\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)Q_f(x,y) \\ &= \left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)Q_{cf}(x,y). \end{aligned}$$

□

Theorem 2.5. *If $f : I \to J$ is convex and $g : J \to \mathbb{R}$ is trigonometrically quasi-convex function and nondecreasing then $g \circ f : I \to \mathbb{R}$ is a trigonometrically quasi-convex function.*

Proof. For $x, y \in I$ and $t \in [0,1]$, we get

$$\begin{aligned} (g\circ f)\left(tx+(1-t)y\right) &= g\left(f\left(tx+(1-t)y\right)\right) \\ &\leq g\left(tf(x)+(1-t)f(y)\right) \\ &\leq \left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)Q_g(f(x),f(y)) \\ &= \left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)Q_{g\circ f}\left(x,y\right). \end{aligned}$$

This completes the proof of theorem. □

3. Hermite-Hadamard inequality for trigonometrically quasi-convex functions

The goal of this paper is to establish some inequalities of Hermite-Hadamard type for trigonometrically quasi-convex functions.

We will denote by $L[a,b]$ the space of (Lebesgue) integrable functions on $[a,b]$.

Theorem 3.1. *Let $f : [a, b] \to \mathbb{R}$ be a trigonometrically quasi-convex function. If $a < b$ and $f \in L[a, b]$, then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{4}{\pi}Q_f(a, b).$$

Proof. By using the property of the trigonometrically quasi-convex function of the function f, if the variable is changed as $u = ta + (1-t)b$, then

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(u)du &= \int_0^1 f(ta + (1-t)b)\,dt \\ &\leq Q_f(a, b)\int_0^1 \left(\sin\frac{\pi t}{2} + \cos\frac{\pi t}{2}\right)dt \\ &= \frac{4}{\pi}Q_f(a, b).\end{aligned}$$

This completes the proof of theorem. □

Theorem 3.2. *Let the function $f : [a, b] \to \mathbb{R}$,be a trigonometrically quasi-convex function. If $a < b$ and $f \in L[a, b]$, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2\sqrt{2}}{b-a}\int_a^b f(x)dx.$$

Proof. From the propery of the trigonometrically P-function of the function f, we get

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &= f\left(\frac{[ta + (1-t)b] + [(1-t)a + tb]}{2}\right) \\ &= f\left(\frac{1}{2}[ta + (1-t)b] + \frac{1}{2}[(1-t)a + tb]\right) \\ &\leq \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right)Q_f(ta + (1-t)b, (1-t)a + tb) \\ &= \sqrt{2}Q_f(ta + (1-t)b, (1-t)a + tb).\end{aligned}$$

Now, if we take integral in the last inequality with respect to $t \in [0, 1]$, we deduce that

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &\leq \sqrt{2}\left[\int_0^1 f(ta + (1-t)b)\,dt + \int_0^1 f((1-t)a + tb)\,dt\right] \\ &= \sqrt{2}\left[\frac{1}{a-b}\int_b^a f(x)dx + \frac{1}{b-a}\int_a^b f(y)dy\right] \\ &= \frac{2\sqrt{2}}{b-a}\int_a^b f(x)dx.\end{aligned}$$

This completes the proof of theorem. □

4. Some new inequalities for trigonometrically quasi-convex function

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is trigonometrically quasi-convex function. Dragomir and Agrawal [1] used the following lemma:

Lemma 4.1. *Let $f : I^{\circ} \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$. If $f' \in L[a, b]$, then*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)\,dt.$$

Note that we will use the following integrals in this section:

$$\begin{aligned}\int_0^1 \sin\frac{\pi t}{2}dt &= \int_0^1 \cos\frac{\pi t}{2}dt = \frac{2}{\pi}, \quad \int_0^1 |1-2t|^p\,dt = \frac{1}{p+1}\\ \int_0^1 |1-2t|\sin\frac{\pi t}{2}dt &= \int_0^1 |1-2t|\cos\frac{\pi t}{2}dt = \frac{2}{\pi^2}\left(\pi - 4\left(\sqrt{2}-1\right)\right).\end{aligned}$$

Theorem 4.2. *Let $f : I \rightarrow \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$ and assume that $f' \in L[a, b]$. If $|f'|$ is trigonometrically quasi-convex function on interval $[a, b]$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq 2(b-a)\left[\frac{1}{\pi} - \frac{4}{\pi^2}\left(\sqrt{2}-1\right)\right]Q_{|f'|}(a,b).$$

Proof. Using Lemma 4.1 and the inequality

$$|f'(ta+(1-t)b)| \leq \left(\sin\frac{\pi t}{2} + \cos\frac{\pi t}{2}\right)Q_{|f'|}(a,b),$$

we get

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ & \frac{b-a}{2}\int_0^1 |1-2t|\,|f'(ta+(1-t)b)|\,dt\\ \leq\ & \frac{b-a}{2}Q_{|f'|}(a,b)\int_0^1 |1-2t|\left(\sin\frac{\pi t}{2} + \cos\frac{\pi t}{2}\right)dt\\ =\ & \frac{b-a}{2}Q_{|f'|}(a,b)\left[\int_0^1 |1-2t|\sin\frac{\pi t}{2}dt + \int_0^1 |1-2t|\cos\frac{\pi t}{2}dt\right]\\ =\ & 2(b-a)\left[\frac{1}{\pi} - \frac{4}{\pi^2}\left(\sqrt{2}-1\right)\right]Q_{|f'|}(a,b).\end{aligned}$$

This completes the proof of theorem. □

Theorem 4.3. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and assume that $f' \in L[a,b]$. If $|f'|^q$, $q > 1$, is an trigonometrically quasi-convex function on interval $[a,b]$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{4}{\pi}\right)^{\frac{1}{q}} Q_{|f'|}(a,b),$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 4.1, Hölder's integral inequality and the following inequality

$$|f'(ta+(1-t)b)|^q \leq \left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right) Q_{|f'|^q}(a,b)$$

which is the trigonometrically quasi-convex function of $|f'|^q$, we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{2}\left(\int_0^1 |1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 \left[\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right) Q_{|f'|^q}(a,b)\right] dt\right)^{\frac{1}{q}} \\
=\ & \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{4}{\pi}\right)^{\frac{1}{q}} Q_{|f'|}(a,b).
\end{aligned}$$

This completes the proof of theorem. □

Theorem 4.4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and assume that $f' \in L[a,b]$. If $|f'|^q, q \geq 1$, is an trigonometrically quasi-convex function on the interval $[a,b]$, then the following inequality holds*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{2^{2-\frac{2}{q}}} Q_{|f'|}(a,b)\left(\frac{1}{\pi}-\frac{4}{\pi^2}\left(\sqrt{2}-1\right)\right)^{\frac{1}{q}}
\end{aligned}$$

Proof. Assume first that $q > 1$. From Lemma 4.1, Hölder integral inequality and the property of the trigonometrically quasi-convex function of $|f'|^q$, we obtain

$$
\begin{aligned}
& \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \\
\leq \ & \frac{b-a}{2}\left(\int_0^1 |1-2t|\,dt\right)^{1-\frac{1}{q}} \left(\int_0^1 |1-2t|\left|f'\left(ta+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}} \\
= \ & \frac{b-a}{2}\left(\int_0^1 |1-2t|\,dt\right)^{1-\frac{1}{q}} \\
& \times \left(\int_0^1 |1-2t|\left[\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)Q_{|f'|^q}(a,b)\right]dt\right)^{\frac{1}{q}} \\
(1) \quad = \ & \frac{b-a}{2^{2-\frac{3}{q}}}Q_{|f'|}(a,b)\left(\frac{1}{\pi}-\frac{4}{\pi^2}\left(\sqrt{2}-1\right)\right)^{\frac{1}{q}}.
\end{aligned}
$$

It can be seen that

$$
(2) \quad \int_0^1 |1-2t|\sin\frac{\pi t}{2}dt = \int_0^1 |1-2t|\cos\frac{\pi t}{2}dt = \frac{2}{\pi^2}\left(\pi-4\left(\sqrt{2}-1\right)\right).
$$

By substituting (2) in (1), the desired result is obtained.

For $q = 1$ we use the estimates from the proof of Theorem 4.2, which also follow step by step the above estimates.

This completes the proof of theorem. □

Corollary 4.5. *Under the assumption of Theorem 4.4 with $q = 1$, we get the conclusion of Theorem 4.2.*

5. An extention of Theorem 4.2

In this section we will denote by A an open and convex set of $\mathbb{R}^n$ $(n \geq 1)$.

We say that a function $f : A \to \mathbb{R}$ is trigonometrically quasi-convex on A if

$$
f\left(tx+(1-t)y\right) \leq \left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)Q_f(x,y)
$$

for all $x, y \in A$ and $t \in [0,1]$.

Lemma 5.1. *Let $f : A \to \mathbb{R}$ be a function. Then f is trigonometrically quasi-convex on A if and only if for all $x, y \in A$ the function $\Phi : [0,1] \to \mathbb{R}$, $\Phi(t) = f\left(tx+(1-t)y\right)$ is trigonometrically quasi-convex on $[0,1]$.*

Proof. "$\Longleftarrow$"Let $x, y \in A$ be fixed. Assume that $\Phi : [0,1] \to \mathbb{R}$, $\Phi(t) = f\left(tx+(1-t)y\right)$ is trigonometrically quasi-convex on $[0,1]$.

Let $t \in [0,1]$ be arbitrary, but fixed. Clearly, $t = (1-t).0 + t.1$ and thus,

$$\begin{aligned} f(tx + (1-t)y) &= \Phi(t) = \Phi((1-t).0 + t.1) \\ &\le \left(\sin\frac{\pi t}{2} + \cos\frac{\pi t}{2}\right) Q_\Phi(0,1) \\ &= \left(\sin\frac{\pi t}{2} + \cos\frac{\pi t}{2}\right) Q_f(x,y). \end{aligned}$$

It follows that f is quasi-convex on A.

"$\Longrightarrow$" Assume that f is trigonometrically quasi-convex on A. Let $x, y \in A$ be fixed and define $\Phi : [0,1] \to \mathbb{R}$, $\Phi(t) = f(tx + (1-t)y)$. We must show that Φ is trigonometrically quasi-convex on $[0,1]$.

Let $u_1, u_2 \in [0,1]$ and $t \in [0,1]$. Then

$$\begin{aligned} \Phi(tu_1 + (1-t)u_2) &= f((tu_1 + (1-t)u_2)x + (1 - tu_1 - (1-t)u_2)y) \\ &= f(t(u_1x + (1-u_1)y + (1-t)(u_2x + (1-u_2)y) \\ &\le \left(\sin\frac{\pi t}{2} + \cos\frac{\pi t}{2}\right) Q_f(u_1x + (1-u_1)y, u_2x + (1-u_2)y) \\ &= \left(\sin\frac{\pi t}{2} + \cos\frac{\pi t}{2}\right) Q_\Phi(u_1, u_2). \end{aligned}$$

We deduce that Φ is quasi-convex on $[0,1]$.

The proof of Lemma 5.1 is complete. □

Using the above lemma we will prove an extension of Theorem 4.2 to functions of several variables.

Proposition 5.2. *Assume $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^+$ is a trigonometrically quasi-convex function on A. Then for any $x, y \in A$ and any $u, v \in (0,1)$ with $u < v$ the following inequality holds true*

$$\begin{aligned} &\left| \frac{1}{2}\int_0^u f(sx + (1-s)y)\,ds + \frac{1}{2}\int_0^v f(sx + (1-s)y)\,ds \right. \\ &\left. -\frac{1}{v-u}\int_u^v \left(\int_0^\theta f(sx + (1-s)y)\,ds\right) d\theta \right| \\ &\le 2(v-u)\left[\frac{1}{\pi} - \frac{4}{\pi^2}\left(\sqrt{2} - 1\right)\right] Q_f(ux + (1-u)y, vx + (1-v)y). \end{aligned} \tag{3}$$

Proof. We fix $x, y \in A$ and $u, v \in (0,1)$ with $u < v$. Since f is trigonometrically quasi-convex, by Lemma 5.1 it follows that the function

$$\Phi : [0,1] \to \mathbb{R}, \Phi(t) = f(tx + (1-t)y),$$

is trigonometrically quasi-convex on $[0,1]$.

Define $\Psi : [0,1] \to \mathbb{R}$,

$$\Psi(t) = \int_0^t \Phi(s)ds = \int_0^t f(sx + (1-s)y)\,ds.$$

Obviously, $\Psi'(t) = \Phi(t)$ for all $t \in (0,1)$.

Since $f(A) \subseteq \mathbb{R}^{+}$ it results that $\Phi \geq 0$ on $[0,1]$ and thus, $\Psi' \geq 0$ on $(0,1)$.

Applying Theorem 4.2 to the function Ψ we obtain

$$\left|\frac{\Psi(u)+\Psi(v)}{2} - \frac{1}{v-u}\int_u^v \Psi(\theta)d\theta\right| \leq 2\,(v-u)\left[\frac{1}{\pi} - \frac{4}{\pi^2}\left(\sqrt{2}-1\right)\right] Q_{|\Psi'|}(u,v),$$

and we deduce that relation (3) holds true. □

Remark 5.3. *We point out that a similar result as those of Proposition 5.2 can be stated by using Theorem 4.3 and Theorem 4.4 .*

6. Applications to the trapezoidal formula

Assume $\wp$ is a division of the interval $[a,b]$ such that

$$\wp: \quad a = x_0 < x_1 < ... < x_{n-1} < x_n = b.$$

For a given function $f : [a,b] \to \mathbb{R}$ we consider the trapezoidal formula

$$T(f,\wp) = \sum_{i=0}^{n-1} \frac{f(x_i)+f(x_{i+1})}{2}(x_{i+1}-x_i).$$

It is well known that if f is twice differentiable on (a,b) and $M = \sup_{x\in(a,b)} |f''(x)| < \infty$ then

$$\int_a^b f(x)dx = T(f,\wp) + E(f,\wp),$$

where $E(f,\wp)$ is the approximation error of the integral $\int_a^b f(x)dx$ by the trapezoidal formula and satisfies,

$$|E(f,\wp)| \leq \frac{M}{12}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3. \tag{4}$$

Clearly, if the function f is not twice differentiable or the second derivative is not bounded on (a,b), then (4) does not hold true. In that context, the following results are important in order to obtain some estimates of $E(f,\wp)$.

Proposition 6.1. *Assume $a,b \in \mathbb{R}$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ is a differentiable function on (a,b). If $|f'|$ is trigonometrically quasi-convex on $[a,b]$ then for each division $\wp$ of the interval $[a,b]$ we have,*

$$|E(f,\wp)| \leq 2\sqrt{2}\left[\frac{1}{\pi} - \frac{4}{\pi^2}\left(\sqrt{2}-1\right)\right] Q_{|f'|}(a,b)\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2. \tag{5}$$

Proof. We apply Theorem 4.2 on the sub-intervals $[x_i, x_{i+1}]$, $i = 0, 1, ..., n-1$ given by the division $\wp$. Adding from $i = 0$ to $i = n-1$ we deduce

(6)

$$\left| T(f, \wp) - \int_a^b f(x)dx \right| \leq \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \, 2 \left[\frac{1}{\pi} - \frac{4}{\pi^2} \left(\sqrt{2} - 1 \right) \right] Q_{|f'|}(x_i, x_{i+1}).$$

On the other hand, for each $x_i \in [a, b]$ there exists $t_i \in [0, 1]$ such that $x_i = t_i a + (1 - t_i)b$. Since $|f'|$ is trigonometrically quasi-convex and $\sin\frac{\pi t}{2} + \cos\frac{\pi t}{2} \leq \sqrt{2}$ for all $t \in [0, 1]$, we deduce

$$|f'(x_i)| \leq \left(\sin\frac{\pi t}{2} + \cos\frac{\pi t}{2} \right) Q_{|f'|}(a, b) \leq \sqrt{2} Q_{|f'|}(a, b) \tag{7}$$

for each $i = 0, 1, ..., n-1$. Relations (6) and (7) imply that relation (5) holds true. Thus, Proposition 5.2 is completely proved. □

A similar method as that used in the proof of Proposition 6.1 but based on Theorem 4.3 and Theorem 4.4 shows that the following results are valid.

Proposition 6.2. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). If $|f'|^q$, $q > 1$, is an trigonometrically quasi-convex function on interval $[a, b]$, then for each division $\wp$ of the interval $[a, b]$ we have,*

$$|E(f, \wp)| \leq \frac{1}{\sqrt{2}} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{4}{\pi} \right)^{\frac{1}{q}} Q_{|f'|}(a, b) \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proposition 6.3. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). If $|f'|^q$, $q > 1$, is an trigonometrically quasi-convex function on interval $[a, b]$, then for each division $\wp$ of the interval $[a, b]$ we have,*

$$|E(f, \wp)| \leq \frac{1}{2^{\frac{3}{2} - \frac{2}{q}}} \left(\frac{1}{\pi} - \frac{4}{\pi^2} \left(\sqrt{2} - 1 \right) \right)^{\frac{1}{q}} Q_{|f'|}(a, b) \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2.$$

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