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k-ALMOST YAMABE SOLITONS ON KENMOTSU MANIFOLDS

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Abstract. In this current article, we intend to investigate k-almost Yamabe and gradient k-almost Yamabe solitons inside the setting of three-dimensional Kenmotsu manifolds.

1. Introduction

In [11] several years ago, Hamilton publicized the concept of *Yamabe soliton.* According to the author, a Riemannian metric g of a complete Riemannian manifold (M^n, g) is called a *Yamabe soliton* if it obeys

$$\frac{1}{2}\pounds_W \, g = (r - \lambda) \, g, \tag{1}$$

where W, λ, r and $\pounds$ indicates a smooth vector field, a real number, the well-known scalar curvature and Lie-derivative respectively. Here, W is termed as the soliton field of the *Yamabe soliton.* A *Yamabe soliton* is called shrinking or expanding according as $\lambda > 0$ or $\lambda < 0$, respectively whereas *steady* if $\lambda = 0$. Yamabe solitons have been investigated by several geometers in various contexts (see, [2], [3], [10], [17], [20]). The so called *Yamabe soliton* becomes the *almost Yamabe soliton* if λ is a C^∞ function. In [1], Barbosa and Ribeiro introduced the above notion which was completely classified by Seko and Maeta in [16] on hypersurfaces in Euclidean spaces.

The Yamabe soliton reduces to a *gradient Yamabe soliton* if the soliton field W is gradient of a C^∞ function $\gamma : M^n \to \mathbb{R}$. In this occasion, from (1) we have

$$\nabla^2\gamma = (r - \lambda)g, \tag{2}$$

where $\nabla^2\gamma$ indicates the Hessian of γ. The idea of gradient Yamabe soliton was generalized by Huang and Li [12] and named as *quasi-Yamabe gradient soliton.*

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According to Huang and Li, g (Riemannian metric) obeys the equation

$$\nabla^2\gamma = \frac{1}{m}\, d\gamma \otimes d\gamma + (r\ - \lambda)\, g, \tag{3}$$

where $\lambda \in \mathbb{R}$ and m is a positive constant. If $m = \infty$, the foregoing equation reduces to Yamabe gradient soliton.

A few years ago in [14], taking λ as a C^∞ function, Pirhadi and Razavi investigated an *almost quasi-Yamabe gradient soliton.* They got a few fascinating formulas and produce a necessary and sufficient condition under which an arbitrary compact almost Yamabe soliton is necessarily gradient. Recently, Chen [5] has studied almost quasi-Yamabe solitons within the context of almost Cosymplectic manifolds.

According to Chen [4], a Riemannian metric is said to be a *k-almost Yamabe soliton* if there exists a smooth vector field W, a C^∞ function λ and a nonzero function k such that

$$\frac{k}{2}\pounds_W\, g = (r - \lambda)\, g \tag{4}$$

holds. We denote the k-almost Yamabe soliton by (g, W, k, λ). If $W = D\gamma$, the previous equation reduces to gradient k-almost Yamabe soliton (g, γ, k, λ). The k-almost Yamabe soliton is called *closed* if the 1-form $W^\flat$ is closed. The k-almost Yamabe soliton becomes trivial if $W \equiv 0$, otherwise nontrivial. Furthermore, when $\lambda = \mathit{constant}$, the previous equation gives the k- Yamabe soliton.

The above works motivate us to study k-almost Yamabe soliton in 3-dimensional Kenmotsu manifolds. Precisely, we prove the following results:

Theorem 1.1. *There does not exist k-almost Yamabe soliton with soliton field pointwise collinear with the characteristic vector field in a Kenmotsu manifold M^3.*

For W being orthogonal to the characteristic vector field, we have

Theorem 1.2. *If the metric of a three-dimensional Kenmotsu manifold M^3 is a k-Yamabe soliton with W being orthogonal to ξ, then the manifold is of constant sectional curvature -1 and the k-Yamabe soliton is expanding with $\lambda = -6$.*

Theorem 1.3. *If a Kenmotsu manifold M^3 admits a closed k-almost Yamabe soliton, then M^3 is isometric to a Euclidean space $\mathbb{R}^3$.*

Theorem 1.4. *There does not exist nontrivial k-almost gradient Yamabe soliton on a Kenmotsu manifold M^3.*

2. Preliminaries

Let M^{2n+1} be a connected almost contact metric manifold endowed with an almost contact metric structure (ϕ,ξ,η,g), that is, ϕ is an (1,1)-tensor field,

ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric such that

$$\phi^2(E) = -E + \eta(E)\xi, \eta(\xi) = 1, \phi\xi = 0, \eta\phi = 0,$$
$$g(\phi E, \phi F) = g(E, F) - \eta(E)\eta(F),$$
$$g(E, \xi) = \eta(E)$$

for all $E, F \in \Gamma(TM)$.
If the following condition is fulfilled in an almost contact metric manifold

$$(\nabla_E \phi)F = g(\phi E, F)\xi - \eta(F)\phi E,$$

then M is called a *Kenmotsu manifold* [13], where ∇ denotes the Levi-Civita connection of g. From the antecedent equation it is clear that

$$\nabla_E \xi = E - \eta(E)\xi \tag{5}$$

and

$$(\nabla_E \eta)F = g(E, F) - \eta(E)\eta(F).$$

In addition, the curvature tensor R and the Ricci tensor S satisfy

$$R(E, F)\xi = \eta(E)F - \eta(F)E,$$
$$R(\xi, E)F = \eta(F)E - g(E, F)\xi,$$
$$R(\xi, E)\xi = E - \eta(E)\xi,$$
$$S(E, \xi) = -2n\eta(E).$$

From [8], we know that for a Kenmotsu manifold M^3

$$\begin{aligned} R(E, F)Z &= \frac{r+4}{2}[g(F, Z)E - g(E, Z)F] \\ &\quad -\frac{r+6}{2}[g(F, Z)\eta(E)\xi - g(E, Z)\eta(F)\xi \\ &\quad +\eta(F)\eta(Z)E - \eta(E)\eta(Z)F], \end{aligned} \tag{6}$$

$$S(E, F) = \frac{1}{2}[(r+2)g(E, F) - (r+6)\eta(E)\eta(F)], \tag{7}$$

where S, R and r are the Ricci tensor, the curvature tensor and the scalar curvature of the manifold respectively. Kenmotsu manifolds have been studied by several authors such as Pitis [15], De and De ([6], [7]) De, Yildiz and Yaliniz [9] and many others.

Definition 2.1. *A vector field W on an n dimensional Riemannian manifold (M, g) is said to be conformal if*

$$\pounds_W g = 2\rho g, \tag{8}$$

ρ being the conformal coefficient. If the conformal coefficient is zero then the conformal vector field is a Killing vector field.

In the following we write $\rho = \frac{r-\lambda}{k}$. Therefore we have the subsequent lemma:

Lemma 2.2 ([19]). *On an $2n+1$-dimensional Riemannian manifold endowed with a k-almost Yamabe soliton, the following relations are satisfied:*

$$\begin{aligned}(\pounds_W S)(E,F) &= -(2n-1)g(\nabla_E D\rho, F) - (\Delta\rho)g(E,F),\\ \pounds_W r &= -2\rho r - 4n\Delta\rho\end{aligned}$$

for $E, F \in \mathfrak{X}(M)$, D being the gradient operator and $\Delta = divD$ being the Laplacian operator of g.

3. Proofs of Theorems

3.1. Proof of Theorem 1.1

Here we suppose that the potential vector field W is pointwise collinear with the characteristic vector field ξ (i.e., $Z = c\xi$, where c is a function on M). Then from (4) we derive

(9) $$k\{g(\nabla_E c\xi, F) + g(\nabla_F c\xi, E)\} = 2(r-\lambda)g(E,F).$$

Utilizing (5) in (9), we get

(10) $$\begin{aligned}&2kc[g(E,F) - \eta(E)\eta(F)] + (Ec)\eta(F) + (Fc)\eta(E)\\ &-2(r-\lambda)g(E,F) = 0.\end{aligned}$$

Replacing F by ξ in (10) gives

(11) $$(Ec) + (\xi c)\eta(E) - 2(r-\lambda)\eta(E) = 0.$$

Putting $E = \xi$ in (11) yields

(12) $$\xi c = (r-\lambda).$$

Substituting the value of ξc in (11) we infer

(13) $$dc = (r-\lambda)\eta.$$

Applying d on (13) and using *Poincare lemma* $d^2 \equiv 0$, we have

(14) $$(r-\lambda)d\eta + (dr)\eta - (d\lambda)\eta = 0.$$

Taking wedge product of (14) with η, we obtain

(15) $$(r-\lambda)\eta \wedge d\eta = 0.$$

Since $\eta \wedge d\eta \neq 0$, we infer

(16) $$r - \lambda = 0.$$

Utilizing (16) in (13) gives $dc = 0$ i.e., c =constant. Then (4) yields $\pounds_\xi g = 0$ i.e., ξ is Killing vector field. But in a Kenmotsu manifold we know that ξ can never be a Killing vector field by (5). This finishes the proof.

3.2. Proof of Theorem 1.2

From (4) we have

$$\frac{k}{2}(g(\nabla_E W, F) + g(W, \nabla_F W)) = (r - \lambda)g(E, F) \quad \text{for all} \quad E, F.$$

Letting $E = F = \xi$ gives $r = \lambda$ since W is orthogonal to ξ and Equation (5) implies $\nabla_\xi \xi = 0$. Since λ is constant, by [20, Lemma 3.2] we know $r = \lambda = -6$. Further, it follows from (6) that $R(E, F)Z = -[g(F, Z)E - g(E, Z)F]$ for all vector fields E, F, Z. This means that M^3 is of constant sectional curvature -1. This completes the proof.

3.3. Proof of Theorem 1.3

Lie differentiating (7) along W and utilizing Lemma 2.1, we have

$$\begin{aligned}
&-2g(\nabla_E D\rho, F) - 2(\Delta\rho)g(E, F) \\
&= \pounds_W r[g(E, F) - \eta(E)\eta(F)] + (r + 2)(\pounds_W g)(E, F) \\
&-(r + 6)(\pounds_W \eta)(E)\eta(F) - (r + 6)\eta(E)(\pounds_W \eta)(F) \\
&= -(2\rho r + 4\Delta\rho)[g(E, F) - \eta(E)\eta(F)] + 2(r + 2)g(E, F) \\
&-(r + 6)[2\rho\eta(E) + g(E, \pounds_W \xi)]\eta(F) - (r + 6)[2\rho\eta(F) + g(F, \pounds_W \xi)]\eta(E) \\
&= -4(\Delta\rho - \rho)g(E, F) - (2\rho r - 4\Delta\rho + 24\rho)\eta(E)\eta(F) \\
&-(r + 6)[g(E, \pounds_W \xi)\eta(F) + g(F, \pounds_W \xi)\eta(E)],
\end{aligned}$$

from which we get

$$\begin{aligned}
\nabla_E D\rho \;&=\; (\Delta\rho - 2\rho)E - (\rho r - 2\Delta\rho + 12\rho)\eta(E)\xi \\
&\quad + (\frac{r}{2} + 3)[g(E, \pounds_W \xi)\xi + \eta(E)\pounds_W \xi]. \qquad (17)
\end{aligned}$$

Now setting $E = \xi$ yields

$$\nabla_\xi D\rho = (-\frac{\rho r}{2} - \Delta\rho + 7\rho)\xi + (\frac{r}{2} + 3)\pounds_W \xi. \qquad (18)$$

Let us assume that (g, W, k, λ) is a closed k-almost Yamabe soliton. Then from (4) we can easily get $\nabla_E W = \rho E$. Thus utilizing (5) we infer

$$\pounds_W \xi = W - \eta(W)\xi - \rho\xi. \qquad (19)$$

Moreover, for any vector fields E, F, we easily find that

$$R(E, F)W = E(\rho)F - F(\rho)E. \qquad (20)$$

Contracting the previous equation over E, we have $QW = -2D\rho$. Now differentiating the above expression along ξ gives $\nabla_\xi D\rho = \rho\xi$, since $Q\xi = -2\xi$ in a 3-dimensional Kenmotsu manifold. Hence from (18) we obtain

$$(\frac{r}{2} + 3)[F - \eta(F)\xi] = (-\Delta\rho + 3\rho)\xi. \qquad (21)$$

This gives $\Delta\rho = 3\rho$.

Substituting the above value and (19) into (17) yields

$$\nabla_E D\rho = \rho E. \tag{22}$$

Therefore, from [18, Theorem 2], we conclude that M^3 is isometric to a Euclidean space $\mathbb{R}^n$.

3.4. Proof of Theorem 1.4

Let us consider a k-almost Yamabe gradient soliton (g, γ, k, λ) on a Kenmotsu manifold M^3 . Then equation (4) can be written as

$$k\nabla_E D\gamma = (r - \lambda)F. \tag{23}$$

Executing the covariant derivative of (23) along E, we obtain

$$\begin{aligned} k\nabla_E\nabla_F D\gamma &= (E(r - \lambda))F + (r - \lambda)\nabla_E F \\ &\quad -\frac{1}{k}(Ek)(r - \lambda)F. \end{aligned} \tag{24}$$

Exchanging E and F in (24), we get

$$\begin{aligned} k\nabla_F\nabla_E D\gamma &= (F(r - \lambda))E + (r - \lambda)\nabla_F E \\ &\quad -\frac{1}{k}(Fk)(r - \lambda)E \end{aligned} \tag{25}$$

and

$$k\nabla_{[E,\ F]}\ D\gamma = (r - \lambda)[E,\ F]. \tag{26}$$

Utilizing (23)-(26) and together with $R(E,\ F)W = \nabla_E\nabla_F W - \nabla_F\ \nabla_E W - \nabla_{[E,F]}W$, we infer

$$\begin{aligned} k^2R(E,\ F)D\gamma &= k[\{E(r - \lambda)\}F] - k[\{F(r - \lambda)\}E] \\ &\quad -(Ek)(r - \lambda)F + (Fk)(r - \lambda)E. \end{aligned} \tag{27}$$

Executing the inner product of (27) with ξ yields

$$\begin{aligned} k^2g(R(E,\ F)\ D\gamma, \xi) &= k[\{E(r - \lambda)\}\eta(F)] - k[\{F(r - \lambda)\}\eta(E)] \\ &\quad -(Ek)(r - \lambda)\eta(F) + (Fk)(r - \lambda)\eta(E). \end{aligned} \tag{28}$$

Again, we obtain from relation (5) that

$$k^2g\ (R(E, F)D\gamma, \xi) = -k^2\{(F\gamma)\eta(E) - (E\gamma)\eta(F)\}. \tag{29}$$

Combining equation (28) and (29), we get

$$\begin{aligned} &k[\{E(r - \lambda)\}\eta(F)] - k[\{F(r - \lambda)\}\eta(E)] \\ &-(Ek)(r - \lambda)\eta(F) + (Fk)(r - \lambda)\eta(E) \\ &+k^2\{(F\gamma)\eta(E) - (E\gamma)\eta(F)\} = 0 \end{aligned} \tag{30}$$

Now replacing F by ξ, give that

$$k^2[E\gamma - (\xi\gamma)\eta(E)] = 0. \tag{31}$$

Since $k \neq 0$, we immediately have

$$D\gamma = (\xi\gamma)\xi. \tag{32}$$

From the preceding equation we can write $d\gamma = (\xi\gamma)\eta$, where d stands for the exterior differentiation. Taking exterior derivative of the previous equation yields $d^2\gamma = d(\xi\gamma)\wedge\eta + (\xi\gamma)d\eta$. Utilizing *Poincare lemma* $d^2 \equiv 0$ in the foregoing equation and then executing wedge product with η we obtain $(\xi\gamma)\eta \wedge d\eta = 0$. Hence $\xi\gamma = 0$, since $\eta \wedge d\eta \neq 0$ in a Kenmotsu manifold. Thus we conclude that $d\gamma = 0$ and therefore γ is constant. This finishes the proof.

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