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ASYMPTOTICAL INVARIANT AND ASYMPTOTICAL LACUNARY INVARIANT EQUIVALENCE TYPES FOR DOUBLE SEQUENCES VIA IDEALS USING MODULUS FUNCTIONS

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Abstract. In this study, we present some asymptotical invariant and asymptotical lacunary invariant equivalence types for double sequences via ideals using modulus functions and investigate relationships between them.

1. Introduction and Background

The concept of convergence for double sequences was firstly introduced by Pringsheim [46]. Then, this concept was extended to the concept of statistical convergence by Mursaleen and Edely [31], to the concept of lacunary statistical convergence by Patterson and Savaş [43], to the concept of $\mathcal{I}_2$-convergence by Das et al. [6] and to the concept of lacunary $\mathcal{I}_2$-invariant convergence [56].

Patterson [42] introduced the concept of asymptotical equivalence for double sequences. Then, this concept was extended to the concept of asymptotical double statistical equivalence by Esi and Açıkgöz [13], to the concept of asymptotical double lacunary statistical equivalence by Esi [14], to the concept of asymptotical $\mathcal{I}_2$-equivalence by Hazarika and Kumar [15] and to the concept of asymptotical lacunary $\mathcal{I}_2$-invariant equivalence [57].

Modulus function was firstly introduced by Nakano [33]. Maddox [25], Pehlivan [45] and many other authors used a modulus function f to define new concepts and to give new theorems. Kumar and Sharma [23] studied on asymptotically lacunary equivalent sequences via ideals using modulus functions. For double sequences, the concept of f-asymptotical lacunary $\mathcal{I}_2$-equivalence was studied by Dündar and P. Akın [10].

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Several authors studied on the concepts of invariant mean, invariant convergence, asymptotical invariant equivalence of sequences and investigated some properties of these concepts (see [5, 9–12, 35, 39, 41, 47–49, 52, 56, 57]).

More study on the concepts of convergence or asymptotical equivalence for sequences can be found in [1–3,7,8,16–21,24,26–32,36–38,40,41,44,51,54,55,58].

In this study, we present some asymptotical invariant and asymptotical lacunary invariant equivalence types for double sequences via ideals using modulus functions and investigate relationships between them.

Now, before giving the main part of the study, we recall some basic concepts (see [4, 9–11, 22, 25, 26, 34, 42, 43, 45, 50, 53]).

Two non-negative sequences $x = (x_{kj})$ and $y = (y_{kj})$ are said to be asymptotically equivalent if

$$\lim_{k,j\to\infty} \frac{x_{kj}}{y_{kj}} = 1$$

(denoted by $x \sim y$).

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and a non-trivial ideal $\mathcal{I}$ is called admissible if $\{n\} \in \mathcal{I}$, for each $n \in \mathbb{N}$.

A non-trivial ideal $\mathcal{I} \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ is called strongly admissible ideal if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$, for each $i \in N$.

Throughout the study, we will consider the $\mathcal{I}_2$ as a strongly admissible ideal in $2^{\mathbb{N}\times\mathbb{N}}$.

It is evident that a strongly admissible ideal is admissible also.

Let $\mathcal{I}_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i, j) \notin A)\}$. Then, $\mathcal{I}_2^0$ is a strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

Two non-negative sequences $x = (x_{kj})$ and $y = (y_{kj})$ are said to be strong asymptotically $\mathcal{I}_2$-equivalent of multiple L if for every $\varepsilon > 0$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1,1}^{m,n} \left|\frac{x_{kj}}{y_{kj}} - L\right| \geq \varepsilon\right\} \in \mathcal{I}_2$$

(denoted by $x_{kj} \overset{[\mathcal{I}_2^L]}{\sim} y_{kj}$) and simply strong asymptotically $\mathcal{I}_2$-equivalent if $L = 1$.

Two non-negative sequences $x = (x_{kj})$ and $y = (y_{kj})$ are said to be asymptotically $\mathcal{I}_2$-statistical equivalent of multiple L if for every $\varepsilon > 0$ and each $\gamma > 0$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}\left|\left\{k, j \leq m, n : \left|\frac{x_{kj}}{y_{kj}} - L\right| \geq \varepsilon\right\}\right| \geq \gamma\right\} \in \mathcal{I}_2$$

(denoted by $x_{kj} \overset{\mathcal{I}_2^L(S)}{\sim} y_{kj}$) and simply asymptotically $\mathcal{I}_2$-statistical equivalent if $L = 1$.

A function $f : [0, \infty) \to [0, \infty)$ is called a modulus if

1. $f(x) = 0$ if and if only if $x = 0$,
2. $f(x + y) \leq f(x) + f(y)$,
3. f is increasing,
4. f is continuous from the right at 0.

A modulus may be unbounded (for example $f(x) = x^p$, $0 < p < 1$) or bounded (for example $f(x) = \frac{x}{x+1}$).

Let f be a modulus function. Two non-negative sequences $x = (x_{kj})$ and $y = (y_{kj})$ are said to be f-asymptotically $\mathcal{I}_2$-equivalent of multiple L if for every $\varepsilon > 0$,

$$\left\{(k, j) \in \mathbb{N} \times \mathbb{N} : f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \geq \varepsilon\right\} \in \mathcal{I}_2$$

(denoted by $x_{kj} \overset{\mathcal{I}_2^L(f)}{\sim} y_{kj}$) and simply f-asymptotically $\mathcal{I}_2$-equivalent if $L = 1$.

Let f be a modulus function. Two non-negative sequences $x = (x_{kj})$ and $y = (y_{kj})$ are said to be strong f-asymptotically $\mathcal{I}_2$-equivalent of multiple L if for every $\varepsilon > 0$,

$$\left\{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1,1}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \geq \varepsilon\right\} \in \mathcal{I}_2$$

(denoted by $x_{kj} \overset{[\mathcal{I}_2^L(f)]}{\sim} y_{kj}$) and simply strong f-asymptotically $\mathcal{I}_2$-equivalent if $L = 1$.

The double sequence $\theta_2 = \{(k_r, j_u)\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that

$$k_0 = 0,\ h_r = k_r - k_{r-1} \to \infty \text{ and } j_0 = 0,\ \bar{h}_u = j_u - j_{u-1} \to \infty \text{ as } r, u \to \infty.$$

We use the following notations in the sequel:

$$h_{ru} = h_r \bar{h}_u, \quad I_{ru} = \{(k, j) : k_{r-1} < k \leq k_r \ \ and \ \ j_{u-1} < j \leq j_u\}.$$

Throughout the study, we will consider the $\theta_2 = \{(k_r, j_u)\}$ as a double lacunary sequence.

Two non-negative sequences $x = (x_{kj})$ and $y = (y_{kj})$ are said to be strong asymptotically lacunary $\mathcal{I}_2$-equivalent of multiple L if for every $\varepsilon > 0$,

$$\left\{(r, u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} \left|\frac{x_{kj}}{y_{kj}} - L\right| \geq \varepsilon\right\} \in \mathcal{I}_2$$

(denoted by $x_{kj} \overset{[\mathcal{I}_{\theta_2}^L]}{\sim} y_{kj}$) and simply strong asymptotically lacunary $\mathcal{I}_2$-equivalent if $L = 1$.

The two non-negative sequences $x = (x_{kj})$ and $y = (y_{kj})$ are said to be asymptotically lacunary $\mathcal{I}_2$-statistical equivalent of multiple L if for every $\varepsilon > 0$

and each $\gamma > 0$,

$$\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \left| \left\{(k,j) \in I_{ru} : \left| \frac{x_{kj}}{y_{kj}} - L \right| \geq \varepsilon \right\} \right| \geq \gamma \right\} \in \mathcal{I}_2$$

(denoted by $x_{kj} \overset{\mathcal{I}_{\theta_2}^L(S)}{\sim} y_{kj}$) and simply asymptotically lacunary $\mathcal{I}_2$-statistical equivalent if $L = 1$.

Let f be a modulus function. Two non-negative sequences $x = (x_{kj})$ and $y = (y_{kj})$ are said to be f-asymptotically lacunary $\mathcal{I}_2$-equivalent of multiple L if for every $\varepsilon > 0$,

$$\left\{(k,j) \in I_{ru} : f\left(\left| \frac{x_{kj}}{y_{kj}} - L \right| \right) \geq \varepsilon \right\} \in \mathcal{I}_2$$

(denoted by $x_{kj} \overset{\mathcal{I}_{\theta_2}^L(f)}{\sim} y_{kj}$) and simply f-asymptotically lacunary $\mathcal{I}_2$-equivalent if $L = 1$.

Let f be a modulus function. Two non-negative sequences $x = (x_{kj})$ and $y = (y_{kj})$ are said to be strong f-asymptotically lacunary $\mathcal{I}_2$-equivalent of multiple L if for every $\varepsilon > 0$,

$$\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} f\left(\left| \frac{x_{kj}}{y_{kj}} - L \right| \right) \geq \varepsilon \right\} \in \mathcal{I}_2$$

(denoted by $x_{kj} \overset{[\mathcal{I}_{\theta_2}^L(f)]}{\sim} y_{kj}$) and simply strong f-asymptotically lacunary $\mathcal{I}_2$-equivalent if $L = 1$.

Let σ be a mapping such that $\sigma : \mathbb{N} \to \mathbb{N}$. A continuous linear functional ψ on ℓ_∞ is said to be an invariant mean or a σ-mean if it satisfies the following conditions:

1. $\psi(x_n) \geq 0$, when the sequence (x_n) has $x_n \geq 0$ for all n,
2. $\psi(e) = 1$, where $e = (1, 1, 1, \ldots)$ and
3. $\psi(x_{\sigma(n)}) = \psi(x_n)$ for all $(x_n) \in \ell_\infty$.

The mappings σ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all $m, n \in \mathbb{N}$, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus ψ extends the limit functional on c, in the sense that $\psi(x_n) = \lim x_n$ for all $(x_n) \in c$.

Let $E \subseteq \mathbb{N} \times \mathbb{N}$ and

$$s_{mk} = \min_{i,j} \left| E \cap \left\{ \big(\sigma(i), \sigma(j)\big), \big(\sigma^2(i), \sigma^2(j)\big), \ldots, \big(\sigma^m(i), \sigma^k(j)\big) \right\} \right|$$

and

$$S_{mk} = \max_{i,j} \left| E \cap \left\{ \big(\sigma(i), \sigma(j)\big), \big(\sigma^2(i), \sigma^2(j)\big), \ldots, \big(\sigma^m(i), \sigma^k(j)\big) \right\} \right|.$$

If the limits $\underline{V_2}(E) = \lim_{m,k \to \infty} \frac{s_{mk}}{mk}$ and $\overline{V_2}(E) = \lim_{m,k \to \infty} \frac{S_{mk}}{mk}$ exists, then they are called a lower σ-uniform density and an upper σ-uniform density of the set E, respectively. If $\underline{V_2}(E) = \overline{V_2}(E)$, then $V_2(E) = \underline{V_2}(E) = \overline{V_2}(E)$ is called σ-uniform density of the set E.

Denote by $\mathcal{I}_2^\sigma$ the class of all $E \subseteq \mathbb{N} \times \mathbb{N}$ with $V_2(E) = 0$. Obviously $\mathcal{I}_2^\sigma$ is a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Two non-negative sequences $x = (x_{kj})$ and $y = (y_{kj})$ are said to be asymptotically $\mathcal{I}_2$-invariant equivalent of multiple L if for every $\varepsilon > 0$,

$$\left\{(k,j) \in \mathbb{N} \times \mathbb{N} : \left|\frac{x_{kj}}{y_{kj}} - L\right| \geq \varepsilon\right\} \in \mathcal{I}_2^\sigma$$

(denoted by $x_{kj} \overset{\mathcal{I}_{\sigma_2}^L}{\sim} y_{kj}$) and simply asymptotically $\mathcal{I}_2$-invariant equivalent if $L = 1$.

Let $\theta_2 = \{(k_r, j_u)\}$ be a double lacunary sequence, $C \subseteq \mathbb{N} \times \mathbb{N}$ and

$$s_{ru} = \min_{m,n} \left| C \cap \left\{ \left(\sigma^k(m), \sigma^j(n)\right) : (k,j) \in I_{ru} \right\} \right|$$

and

$$S_{ru} = \max_{m,n} \left| C \cap \left\{ \left(\sigma^k(m), \sigma^j(n)\right) : (k,j) \in I_{ru} \right\} \right|.$$

If the limits $\underline{V_2^\theta}(C) = \lim_{r,u\to\infty} \frac{s_{ru}}{h_{ru}}$ and $\overline{V_2^\theta}(C) = \lim_{r,u\to\infty} \frac{S_{ru}}{h_{ru}}$ exist, then they are called a lower lacunary σ-uniform density and an upper lacunary σ-uniform density of the set C, respectively. If $\underline{V_2^\theta}(C) = \overline{V_2^\theta}(C)$, then $V_2^\theta(C) = \underline{V_2^\theta}(C) = \overline{V_2^\theta}(C)$ is called the lacunary σ-uniform density of the set C.

Denote by $\mathcal{I}_2^{\sigma\theta}$ the class of all $C \subseteq \mathbb{N} \times \mathbb{N}$ with $V_2^\theta(C) = 0$. Obviously $\mathcal{I}_2^{\sigma\theta}$ is a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Two non-negative sequences $x = (x_{kj})$ and $y = (y_{kj})$ are said to be asymptotically lacunary $\mathcal{I}_2$-invariant equivalent of multiple L if for every $\varepsilon > 0$,

$$\left\{(k,j) \in I_{ru} : \left|\frac{x_{kj}}{y_{kj}} - L\right| \geq \varepsilon\right\} \in \mathcal{I}_2^{\sigma\theta}$$

(denoted by $x_{kj} \overset{\mathcal{I}_{\sigma_2}^{\theta L}}{\sim} y_{kj}$) and simply asymptotically lacunary $\mathcal{I}_2$-invariant equivalent if $L = 1$.

Lemma 1.1. *[45] Let f be a modulus and $0 < \delta < 1$. Then, for each $x \geq \delta$ we have $f(x) \leq 2f(1)\delta^{-1}x$.*

2. MAIN RESULTS

Definition 2.1. *Two non-negative sequences $x = (x_{kj})$ and $y = (y_{kj})$ are strong asymptotically $\mathcal{I}_2$-invariant equivalent of multiple L if for every $\varepsilon > 0$,*

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1,1}^{m,n} \left|\frac{x_{kj}}{y_{kj}} - L\right| \geq \varepsilon\right\} \in \mathcal{I}_2^\sigma$$

(denoted by $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L]}{\sim} y_{kj}$*) and simply strong asymptotically* $\mathcal{I}_2$*-invariant equivalent if* $L = 1$*.*

Definition 2.2. *Let* f *be a modulus function. Two non-negative sequences* $x = (x_{kj})$ *and* $y = (y_{kj})$ *are* f*-asymptotically* $\mathcal{I}_2$*-invariant equivalent of multiple* L *if for every* $\varepsilon > 0$*,*

$$\left\{(k,j) \in \mathbb{N}\times\mathbb{N} : f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \geq \varepsilon\right\} \in \mathcal{I}_2^\sigma$$

(denoted by $x_{kj} \overset{\mathcal{I}_{\sigma_2}^L(f)}{\sim} y_{kj}$*) and simply* f*-asymptotically* $\mathcal{I}_2$*-invariant equivalent if* $L = 1$*.*

Definition 2.3. *Let* f *be a modulus function. Two non-negative sequences* $x = (x_{kj})$ *and* $y = (y_{kj})$ *are strong* f*-asymptotically* $\mathcal{I}_2$*-invariant equivalent of multiple* L *if for every* $\varepsilon > 0$*,*

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}\sum_{k,j=1,1}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \geq \varepsilon\right\} \in \mathcal{I}_2^\sigma$$

(denoted by $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L(f)]}{\sim} y_{kj}$*) and simply strong* f*-asymptotically* $\mathcal{I}_2$*-invariant equivalent if* $L = 1$*.*

Theorem 2.4. *Let* f *be a modulus function. Then,*

$$x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L]}{\sim} y_{kj} \Rightarrow x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L(f)]}{\sim} y_{kj}.$$

Proof. Suppose that $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L]}{\sim} y_{kj}$ and $\varepsilon > 0$ be given. Choose $0 < \delta < 1$ such that $f(t) < \varepsilon$, for $0 \leq t \leq \delta$. We can write

$$\frac{1}{mn}\sum_{k,j=1,1}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) = \frac{1}{mn}\sum_{\substack{k,j=1,1\\ \left|\frac{x_{kj}}{y_{kj}}-L\right|\leq\delta}}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) + \frac{1}{mn}\sum_{\substack{k,j=1,1\\ \left|\frac{x_{kj}}{y_{kj}}-L\right|>\delta}}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right)$$

and so, by Lemma 1.1, we have

$$\frac{1}{mn}\sum_{k,j=1,1}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) < \varepsilon + \left(\frac{2f(1)}{\delta}\right)\frac{1}{mn}\sum_{k,j=1,1}^{m,n}\left|\frac{x_{kj}}{y_{kj}} - L\right|.$$

Thus, for any $\gamma > 0$

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}\sum_{k,j=1,1}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \geq \gamma\right\}$$
$$\subseteq \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}\sum_{k,j=1,1}^{m,n} \left|\frac{x_{kj}}{y_{kj}} - L\right| \geq \frac{(\gamma-\varepsilon)\delta}{2f(1)}\right\}.$$

Since it is $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L]}{\sim} y_{kj}$, the second set in the above expression and therefore the first set also belongs to the $\mathcal{I}_2^\sigma$. This proves that $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L(f)]}{\sim} y_{kj}$. □

Theorem 2.5. *If* $\lim_{t\to\infty} \frac{f(t)}{t} = \alpha > 0$, *then*

$$x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L(f)]}{\sim} y_{kj} \Leftrightarrow x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L]}{\sim} y_{kj}.$$

Proof. We showed that $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L]}{\sim} y_{kj} \Rightarrow x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L(f)]}{\sim} y_{kj}$ in Theorem 2.4. Now, for proof, we must show $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L(f)]}{\sim} y_{kj} \Rightarrow x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L]}{\sim} y_{kj}$.

Assume that $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L(f)]}{\sim} y_{kj}$. Let $\lim_{t\to\infty} \frac{f(t)}{t} = \alpha > 0$, then $f(t) \geq \alpha t$ for all $t \geq 0$. So, we can write

$$\begin{aligned}\frac{1}{mn}\sum_{k,j=1,1}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) &\geq \frac{1}{mn}\sum_{k,j=1,1}^{m,n} \alpha\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \\ &= \alpha\left(\frac{1}{mn}\sum_{k,j=1,1}^{m,n} \left|\frac{x_{kj}}{y_{kj}} - L\right|\right),\end{aligned}$$

and it follows that for every $\varepsilon > 0$, we have

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}\sum_{k,j=1,1}^{m,n} \left|\frac{x_{kj}}{y_{kj}} - L\right| \geq \varepsilon\right\}$$
$$\subseteq \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}\sum_{k,j=1,1}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \geq \alpha\varepsilon\right\}.$$

Since it is $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L(f)]}{\sim} y_{kj}$, the second set in the above expression and therefore the first set also belongs to the $\mathcal{I}_2^\sigma$. This proves that $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L(f)]}{\sim} y_{kj} \Rightarrow x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L]}{\sim} y_{kj}$ and so $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L(f)]}{\sim} y_{kj} \Leftrightarrow x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L]}{\sim} y_{kj}$. □

Definition 2.6. *Two non-negative sequences* $x = (x_{kj})$ *and* $y = (y_{kj})$ *are asymptotically* $\mathcal{I}_2$*-invariant statistical equivalent of multiple* L *if for every* $\varepsilon > 0$

and each $\gamma > 0$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}\left|\left\{k,j \leq m,n : \left|\frac{x_{kj}}{y_{kj}} - L\right| \geq \varepsilon\right\}\right| \geq \gamma\right\} \in \mathcal{I}_2^{\sigma}$$

(denoted by $x_{kj} \overset{\mathcal{I}_{\sigma_2}^L(S)}{\sim} y_{kj}$*) and simply asymptotically* $\mathcal{I}_2$*-invariant statistical equivalent if* $L = 1$.

Theorem 2.7. *Let* f *be a modulus function. Then,*

$$x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L(f)]}{\sim} y_{kj} \Rightarrow x_{kj} \overset{\mathcal{I}_{\sigma_2}^L(S)}{\sim} y_{kj}.$$

Proof. Assume that $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L(f)]}{\sim} y_{kj}$ and $\varepsilon > 0$ be given. We can write

$$\begin{aligned}
\frac{1}{mn}\sum_{k,j=1,1}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) &\geq \frac{1}{mn}\sum_{\substack{k,j=1,1 \\ \left|\frac{x_{kj}}{y_{kj}}-L\right|\geq\varepsilon}}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \\
&\geq f(\varepsilon)\frac{1}{mn}\left|\left\{k \leq m, j \leq n : \left|\frac{x_{kj}}{y_{kj}} - L\right| \geq \varepsilon\right\}\right|.
\end{aligned}$$

Thus, for any $\gamma > 0$ we have

$$\begin{aligned}
&\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}\left|\left\{k \leq m, j \leq n : \left|\frac{x_{kj}}{y_{kj}} - L\right| \geq \varepsilon\right\}\right| \geq \frac{\gamma}{f(\varepsilon)}\right\} \\
&\qquad \subseteq \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}\sum_{k,j=1,1}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \geq \gamma\right\}.
\end{aligned}$$

Since it is $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L(f)]}{\sim} y_{kj}$, the second set in the above expression and therefore the first set also belongs to the $\mathcal{I}_2^{\sigma}$. This proves that $x_{kj} \overset{\mathcal{I}_{\sigma_2}^L(S)}{\sim} y_{kj}$. □

Theorem 2.8. *Let* f *be a modulus function. If* f *is bounded, then*

$$x_{kj} \overset{\mathcal{I}_{\sigma_2}^L(S)}{\sim} y_{kj} \Leftrightarrow x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L(f)]}{\sim} y_{kj}.$$

Proof. We showed that $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L(f)]}{\sim} y_{kj} \Rightarrow x_{kj} \overset{\mathcal{I}_{\sigma_2}^L(S)}{\sim} y_{kj}$ in Theorem 2.7. Now, for proof, we must show $x_{kj} \overset{\mathcal{I}_{\sigma_2}^L(S)}{\sim} y_{kj} \Rightarrow x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L(f)]}{\sim} y_{kj}$.

Assume that $x_{kj} \overset{\mathcal{I}_{\sigma_2}^L(S)}{\sim} y_{kj}$. Let f is bounded, then there exists a positive real number M such that $|f(x)| \leq M$, for all $x \geq 0$. So, we have

$$\begin{aligned}\frac{1}{mn}\sum_{k,j=1,1}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}}-L\right|\right) &= \frac{1}{mn}\sum_{\substack{k,j=1,1\\ \left|\frac{x_{kj}}{y_{kj}}-L\right|\geq\varepsilon}}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}}-L\right|\right)\\ &\quad+\frac{1}{mn}\sum_{\substack{k,j=1,1\\ \left|\frac{x_{kj}}{y_{kj}}-L\right|<\varepsilon}}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}}-L\right|\right)\\ &\leq \frac{M}{mn}\left|\left\{k\leq m, j\leq n : \left|\frac{x_{kj}}{y_{kj}}-L\right|\geq\varepsilon\right\}\right|\\ &\quad+f(\varepsilon).\end{aligned}$$

This proves that $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^L(f)]}{\sim} y_{kj}$. □

Definition 2.9. *Two non-negative sequences $x = (x_{kj})$ and $y = (y_{kj})$ are strong asymptotically lacunary $\mathcal{I}_2$-invariant equivalent of multiple L if for every $\varepsilon > 0$,*

$$\left\{(r,u)\in\mathbb{N}\times\mathbb{N} : \frac{1}{h_{ru}}\sum_{(k,j)\in I_{ru}}\left|\frac{x_{kj}}{y_{kj}}-L\right|\geq\varepsilon\right\}\in\mathcal{I}_2^{\sigma\theta}$$

(denoted by $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}]}{\sim} y_{kj}$) and simply strong asymptotically lacunary $\mathcal{I}_2$-invariant equivalent if $L = 1$.

Definition 2.10. *Let f be a modulus function. Two non-negative sequences $x = (x_{kj})$ and $y = (y_{kj})$ are f-asymptotically lacunary $\mathcal{I}_2$-invariant equivalent of multiple L if for every $\varepsilon > 0$,*

$$\left\{(k,j)\in I_{ru} : f\left(\left|\frac{x_{kj}}{y_{kj}}-L\right|\right)\geq\varepsilon\right\}\in\mathcal{I}_2^{\sigma\theta}$$

(denoted by $x_{kj} \overset{\mathcal{I}_{\sigma_2}^{\theta L}(f)}{\sim} y_{kj}$) and simply f-asymptotically lacunary $\mathcal{I}_2$-invariant equivalent if $L = 1$.

Definition 2.11. *Let f be a modulus function. Two non-negative sequences $x = (x_{kj})$ and $y = (y_{kj})$ are strong f-asymptotically lacunary $\mathcal{I}_2$-invariant equivalent of multiple L if for every $\varepsilon > 0$,*

$$\left\{(r,u)\in\mathbb{N}\times\mathbb{N} : \frac{1}{h_{ru}}\sum_{(k,j)\in I_{ru}} f\left(\left|\frac{x_{kj}}{y_{kj}}-L\right|\right)\geq\varepsilon\right\}\in\mathcal{I}_2^{\sigma\theta}$$

(denoted by $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}(f)]}{\sim} y_{kj}$*) and simply strong* f*-asymptotically lacunary* $\mathcal{I}_2$*-invariant equivalent if* $L = 1$.

Theorem 2.12. *Let* f *be a modulus function. Then,*

$$x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}]}{\sim} y_{kj} \Rightarrow x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}(f)]}{\sim} y_{kj}.$$

Proof. Suppose that $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}]}{\sim} y_{kj}$ and $\varepsilon > 0$ be given. Choose $0 < \delta < 1$ such that $f(t) < \varepsilon$, for $0 \leq t \leq \delta$. We can write

$$\begin{aligned}\frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) &= \frac{1}{h_{ru}} \sum_{\substack{(k,j) \in I_{ru} \\ \left|\frac{x_{kj}}{y_{kj}} - L\right| \leq \delta}} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \\ &\quad + \frac{1}{h_{ru}} \sum_{\substack{(k,j) \in I_{ru} \\ \left|\frac{x_{kj}}{y_{kj}} - L\right| > \delta}} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right)\end{aligned}$$

and so, by Lemma 1.1, we have

$$\frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) < \varepsilon + \left(\frac{2f(1)}{\delta}\right) \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} \left|\frac{x_{kj}}{y_{kj}} - L\right|.$$

Thus, for any $\gamma > 0$

$$\begin{aligned}&\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \geq \gamma\right\} \\ &\qquad \subseteq \left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} \left|\frac{x_{kj}}{y_{kj}} - L\right| \geq \frac{(\gamma - \varepsilon)\delta}{2f(1)}\right\}.\end{aligned}$$

Since it is $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}]}{\sim} y_{kj}$, the second set in the above expression and therefore the first set also belongs to the $\mathcal{I}_2^{\sigma\theta}$. This proves that $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}(f)]}{\sim} y_{kj}$. □

Theorem 2.13. *If* $\lim_{t \to \infty} \frac{f(t)}{t} = \alpha > 0$, *then*

$$x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}(f)]}{\sim} y_{kj} \Leftrightarrow x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}]}{\sim} y_{kj}.$$

Proof. We showed that $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}]}{\sim} y_{kj} \Rightarrow x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}(f)]}{\sim} y_{kj}$ in Theorem 2.12. Now, for proof, we must show $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}(f)]}{\sim} y_{kj} \Rightarrow x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}]}{\sim} y_{kj}$.

Assume that $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}(f)]}{\sim} y_{kj}$. Let $\lim_{t\to\infty} \frac{f(t)}{t} = \alpha > 0$, then $f(t) \geq \alpha t$ for all $t \geq 0$. So, we can write

$$\begin{aligned}\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) &\geq \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \alpha\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \\ &= \alpha\left(\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|\frac{x_{kj}}{y_{kj}} - L\right|\right),\end{aligned}$$

and it follows that for every $\varepsilon > 0$, we have

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|\frac{x_{kj}}{y_{kj}} - L\right| \geq \varepsilon\right\}$$

$$\subseteq \left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \geq \alpha\varepsilon\right\}.$$

Since it is $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}(f)]}{\sim} y_{kj}$, the second set in the above expression and therefore the first set also belongs to the $\mathcal{I}_2^{\sigma\theta}$. This proves that $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}(f)]}{\sim} y_{kj} \Rightarrow x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}]}{\sim} y_{kj}$ and so $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}(f)]}{\sim} y_{kj} \Leftrightarrow x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}]}{\sim} y_{kj}$. □

Definition 2.14. *Two non-negative sequences $x = (x_{kj})$ and $y = (y_{kj})$ are asymptotically lacunary $\mathcal{I}_2$-invariant statistical equivalent of multiple L if for every $\varepsilon > 0$ and each $\gamma > 0$,*

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{ru}}\left|\left\{(k,j) \in I_{ru} : \left|\frac{x_{kj}}{y_{kj}} - L\right| \geq \varepsilon\right\}\right| \geq \gamma\right\} \in \mathcal{I}_2^{\sigma\theta}$$

(denoted by $x_{kj} \overset{\mathcal{I}_{\sigma_2}^{\theta L}(S)}{\sim} y_{kj}$) and simply asymptotically lacunary $\mathcal{I}_2$-invariant statistical equivalent if $L = 1$.

Theorem 2.15. *Let f be a modulus function. Then,*

$$x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}(f)]}{\sim} y_{kj} \Rightarrow x_{kj} \overset{\mathcal{I}_{\sigma_2}^{\theta L}(S)}{\sim} y_{kj}.$$

Proof. Assume that $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}(f)]}{\sim} y_{kj}$ and $\varepsilon > 0$ be given. We can write

$$\begin{aligned}\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) &\geq \frac{1}{h_{ru}} \sum_{\substack{(k,j)\in I_{ru} \\ \left|\frac{x_{kj}}{y_{kj}} - L\right| \geq \varepsilon}} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \\ &\geq f(\varepsilon)\frac{1}{h_{ru}}\left|\left\{(k,j) \in I_{ru} : \left|\frac{x_{kj}}{y_{kj}} - L\right| \geq \varepsilon\right\}\right|.\end{aligned}$$

Thus, for any $\gamma > 0$, we have

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{ru}}\left|\left\{(k,j) \in I_{ru} : \left|\frac{x_{kj}}{y_{kj}} - L\right| \geq \varepsilon\right\}\right| \geq \gamma\right\}$$
$$\subseteq \left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \geq \gamma f(\varepsilon)\right\}.$$

Since it is $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}(f)]}{\sim} y_{kj}$, the second set in the above expression and therefore the first set also belongs to the $\mathcal{I}_2^{\sigma\theta}$. This proves that $x_{kj} \overset{\mathcal{I}_{\sigma_2}^{\theta L}(S)}{\sim} y_{kj}$. □

Theorem 2.16. *Let f be a modulus function. If f is bounded, then*

$$x_{kj} \overset{\mathcal{I}_{\sigma_2}^{\theta L}(S)}{\sim} y_{kj} \Leftrightarrow x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}(f)]}{\sim} y_{kj}.$$

Proof. We showed that $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}(f)]}{\sim} y_{kj} \Rightarrow x_{kj} \overset{\mathcal{I}_{\sigma_2}^{\theta L}(S)}{\sim} y_{kj}$ in Theorem 2.15. Now, for proof, we must show $x_{kj} \overset{\mathcal{I}_{\sigma_2}^{\theta L}(S)}{\sim} y_{kj} \Rightarrow x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}(f)]}{\sim} y_{kj}$.

Assume that $x_{kj} \overset{\mathcal{I}_{\sigma_2}^{\theta L}(S)}{\sim} y_{kj}$. Let f is bounded, then there exists a positive real number M such that $|f(x)| \leq M$, for all $x \geq 0$. So, we have

$$\begin{aligned}\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) &= \frac{1}{h_{ru}} \sum_{\substack{(k,j)\in I_{ru} \\ \left|\frac{x_{kj}}{y_{kj}}-L\right|\geq\varepsilon}} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \\ &\quad + \frac{1}{h_{ru}} \sum_{\substack{(k,j)\in I_{ru} \\ \left|\frac{x_{kj}}{y_{kj}}-L\right|<\varepsilon}} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \\ &\leq \frac{M}{h_{ru}}\left|\left\{(k,j) \in I_{ru} : \left|\frac{x_{kj}}{y_{kj}} - L\right| \geq \varepsilon\right\}\right| \\ &\quad + f(\varepsilon).\end{aligned}$$

This proves that $x_{kj} \overset{[\mathcal{I}_{\sigma_2}^{\theta L}(f)]}{\sim} y_{kj}$. □

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