Honam Mathematical J. **43** (2021), No. 1, pp. 88–99
https://doi.org/10.5831/HMJ.2021.43.1.88

ON THE GEOMETRY OF RATIONAL BÉZIER CURVES

Ayşe Yilmaz Ceylan*, Tunahan Turhan, and Gözde Özkan Tükel

Abstract. The purpose of this paper is to assign a movable frame to an arbitrary point of a rational Bézier curve on the 2−sphere S^2 in Euclidean 3−space R^3 to provide a better understanding of the geometry of the curve. Especially, we obtain the formula of geodesic curvature for a quadratic rational Bézier curve that allows a curve to be characterized on the surface. Moreover, we give some important results and relations for the Darboux frame and geodesic curvature of a such curve. Then, in specific case, given characterizations for the quadratic rational Bézier curve are illustrated on a unit 2-sphere.

1. Introduction

Geometry of curves and surfaces is very remarkable because it has many important applications in different areas. Therefore, various curves and surfaces have been studied by many authors for many years. Recently, due to its different structure, Bézier curves have attracted the attention of many researchers. Bézier curves are introduced firstly by an engineer Pierre Bézier in 1968. Such curves play a significant role in many areas such as computer graphics, computer aided geometric design, etc. For example, the concept of Bézier curves on Riemannian manifolds has been presented by Park and Ravani in [8]. They gave a suitable generalization of De Casteljau's algorithm to curved spaces and illustrated this generalization with an application in kinematic (1995). In [10], the author generalised the classical de Casteljau construction of Bézier curves to 3−sphere by replacing line segments with the minimal geodesics of a sphere. Then, Bézier curves on sphere have been studied by Popiel and Noakes (2006) by using spherical interpolation. Also, in computational geometry, a new classification by using angular curvature for Bézier control polygon has gotten by Tantay and Taş (2011). On the other hand, Erkan and Yüce (2018) give the Serret-Frenet frame and curvatures of a planar and spatial Bézier curves in Euclidean 3−space R^3. Bézier curves can represent a wide variety of curves; however, the conic sections cannot be represented in Bézier form. In order to

Received September 30, 2020. Revised February 1, 2021. Accepted February 13, 2021.
2020 Mathematics Subject Classification. 53A04; 53A05; 65D17
Key words and phrases. Darboux frame field; Geodesic curvature; Rational Bézier curve; 2-sphere.

*Corresponding author

be able to include conic sections in a Bézier form, rational Bézier curves are defined. In this way, this kind of Bézier form can be defined on sphere surface (for detail, see [3, 6]). Yılmaz Ceylan et. al. [12] construct Darboux frame field of the quadratic rational Bézier curves at the end points on S^2.
A curve in R^3 can be determined with respect to its curvature and torsion. However if this curve lies on an oriented surface, it has normal curvature, geodesic curvature and geodesic torsion with respect to the surface. These geometric entities are important tools for understanding characterizations of the curve. In such space, the calculation of ordinary curvatures for regular curves is well known. Moreover, if we get a curve lying on a surface, we can calculate the other curvatures [2, 7].
The rest part of the paper is given as follows. Firstly, we give basic notations and definitions for needed throughout the study. Then we construct the Darboux frame field and obtain geodesic curvature formula along a quadratic rational Bézier curve on 2-sphere S^2. Then, we give examples for a quadratic rational Bézier curve on the unit 2-sphere.

2. Geometrical Preliminaries

Let S be a regular orientable surface with unit normal vector field N in Euclidean 3−space R^3. If $\gamma : I \subset R \to S$ is a curve with arbitrary parameter, then we assign an orthogonal frame $\{T(t), V(t), N(t)\}$ at a point $p = \gamma(t)$ given by

$$T(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}, \tag{1}$$

$$V(t) = N(t) \times T(t) \tag{2}$$

and

$$N(t) = N(\gamma(t)), \tag{3}$$

where $T(t)$ is the unit tangent vector field to γ at p. These vector fields are known as the Darboux trihedron. The first derivatives of these vector fields at p can be shown as below

$$\begin{pmatrix} T' \\ V' \\ N' \end{pmatrix} = \begin{pmatrix} 0 & \upsilon\kappa_g & \upsilon\kappa_n \\ -\upsilon\kappa_g & 0 & \upsilon\tau_g \\ -\upsilon\kappa_n & -\upsilon\tau_g & 0 \end{pmatrix} \begin{pmatrix} T \\ V \\ N \end{pmatrix}$$

where $\upsilon(t) = \|\gamma'(t)\|$ is the speed of γ, $\kappa_g(t)$, $\kappa_n(t)$ and $\tau_g(t)$ are the geodesic curvature, the normal curvature and the geodesic torsion of γ at p, respectively (see [1, 2]). This frame field is similar to the Frenet frame of a regular Euclidean curve. However, in the case of a regular smooth curve on a surface, we prefer the Darboux frame field. This frame field is a reasonable way to establish an orthogonal trihedron to measure the change of a surface curve. It is possible

to establish some connections between curvatures by using transitions between these two frame fields.
The geodesic curvature is a value that results from own geometry of the curve and can be defined as the curvature of the projection of γ on the tangent plane of S at p. It is computed by the following equation [1, 2]:

$$\kappa_g = \frac{\det(\gamma', \gamma'', N)}{\|\gamma'\|^3}. \tag{4}$$

If a curve on the surface is geodesic, it has zero geodesic curvature.
The shape of a surface affects the shape of the curves that lie on it. Normal curvature gives information about the rate of change of a curve on the surface in the normal direction of the surface, that is, normal curvature measures the curvature of the curve resulting from own curvature of the surface. If the ordinary curvature of an Euclidean curve is κ, there is a relationship between κ, κ_g and κ_n at a point of the curve as follows.

$$\kappa^2 = \kappa_g^2 + \kappa_n^2. \tag{5}$$

The geodesic torsion is the torsion of the geodesic in direction of the curve. If the ordinary torsion of an Euclidean curve denoted by τ, the geodesic curvature at a point of the curve can be found in the following formula

$$\tau = \tau_g + \frac{\kappa_n' \kappa_g - \kappa_g' \kappa_n}{\kappa_g^2 + \kappa_n^2} \tag{6}$$

[13, 1].

We consider the sphere whose centre is the origin with radius r

$$S^2(r) = \{x \in R^3 :< x, x >= r^2, r \in R\}$$

and suppose that $\gamma(t)$ is a curve on S^2. By choosing a positive orientation on S^2, we have

$$N(\gamma(t)) = \frac{1}{r}\gamma(t) \tag{7}$$

and $\gamma' = rN'$. The normal sections through a point $p = \gamma(t)$ are circles with radius $\frac{1}{r}$ on S^2. All normal curvatures are equal to

$$\kappa_n = \frac{1}{r}. \tag{8}$$

We have also along γ on S^2

$$\tau_g = 0, \tag{9}$$

for every $t \in R$ ([1]). Thus all curves on the sphere are principle curves (see [7]). Taking into consideration (8) and (9), the derivative formulas of the Darboux frame field along γ reduces to

$$\begin{pmatrix} T' \\ V' \\ N' \end{pmatrix} = \begin{pmatrix} 0 & \upsilon\kappa_g & \frac{\upsilon}{r} \\ -\upsilon\kappa_g & 0 & 0 \\ -\frac{\upsilon}{r} & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ V \\ N \end{pmatrix}. \tag{10}$$

We will first give a general definition of the rational Bézier curve.

Definition 2.1. *A rational Bézier curve of degree n with control points $b_0, b_1, ..., b_n$ and corresponding scalar weights ω_i, $0 \leq i \leq n$, is defined to be*

$$B(t) = \frac{\sum\limits_{i=0}^{n} \omega_i b_i B_{i,n}(t)}{\sum\limits_{i=0}^{n} \omega_i B_{i,n}(t)}, \quad 0 \leq t \leq 1$$

where

$$B_{i,n}(t) = \binom{n}{i}(1-t)^{n-i}t^i \tag{11}$$

are called the Bernstein polynomials of degree n. If $\omega_i = 0$, then $\omega_i b_i$ is to be replaced by b_i. It is assumed that all the weights are non-zero. Rational Bézier curves are called quadratic for $n = 2$ [3, 6].

To construct a rational Bézier curve, the following algorithm is used.

Definition 2.2. *The de Casteljau algorithm for rational Bézier curves is defined by*

$$b_i^j = (1-t)\frac{\omega_i^{j-1}}{\omega_i^j} b_i^{j-1} + t\frac{\omega_{i+1}^{j-1}}{\omega_i^j} b_{i+1}^{j-1}$$

$$\omega_i^j = (1-t)\omega_i^{j-1} + t\omega_{i+1}^{j-1}$$

for $0 \leq t \leq 1$, $j = 1, ..., n$ and $i = 0, ..., n-j$ [6].

Lemma 2.3. *The first order derivative of*

$$(\sum_{i=0}^{2} \omega_i b_i B_{i,2}(t))' = 2\sum_{i=0}^{1} \Delta_i \omega_i b_i B_{i,1}(t) \tag{12}$$

where Δ_i is the difference equation defined by $\Delta_i \omega_i b_i = \omega_{i+1} b_{i+1} - \omega_i b_i$ [5].

3. Quadratic Rational Bézier Curves on the 2-sphere

Let a quadratic rational Bézier curve

$$B(t) = \frac{\sum\limits_{i=0}^{2} \omega_i b_i B_{i,2}(t)}{\sum\limits_{i=0}^{2} \omega_i B_{i,2}(t)}, \quad 0 \leq t \leq 1, \tag{13}$$

lies on 2-sphere S^2, that is $< B(t), B(t) >= r^2$. Now we will find the first and second derivative of (13). The first order derivative of the quadratic rational Bézier curve is given by the following equation

$$B'(t) = \frac{(\sum_{i=0}^{2} \omega_i b_i B_{i,2}(t))' - (\sum_{i=0}^{2} \omega_i B_{i,2}(t))' B(t)}{\sum_{i=0}^{2} \omega_i B_{i,2}(t)} \tag{14}$$

[5]. From (14) and the derivatives of (11), we derive

$$B'(t) = \frac{\sum_{i=0}^{2} a_i B_{i,2}(t)}{(\sum_{i=0}^{2} \omega_i B_{i,2}(t))^2} \tag{15}$$

where

$$a_0 = 2\omega_0\omega_1\Delta^1 b_0,$$
$$a_1 = \omega_0\omega_2(b_2 - b_0),$$
$$a_2 = 2\omega_1\omega_2\Delta^1 b_1$$

and $\Delta^1 b_0 = b_1 - b_0$, $\Delta^1 b_1 = b_2 - b_1$.
We calculate the derivative of (15) as follows

$$B''(t) = \frac{\sum_{i=0}^{5} c_i B_{i,5}(t)}{(\sum_{i=0}^{2} \omega_i B_{i,2}(t))^4}, \tag{16}$$

where

$$\begin{aligned}
c_0 &= 2\omega_0^3\omega_2(b_2 - b_0) + (4\omega_0^3\omega_1 - 8\omega_0^2\omega_1^2)\Delta^1 b_0, \\
c_1 &= \frac{1}{5}(6\omega_0^3\omega_2(b_2 - b_0) + (8\omega_0^2\omega_1^2 - 16\omega_0\omega_1^3 \\
&\quad -8\omega_0^2\omega_1\omega_2)\Delta^1 b_0 + 4\omega_0^2\omega_1\omega_2\Delta^1 b_1), \\
c_2 &= \frac{1}{10}((8\omega_0\omega_1^2\omega_2 + 8\omega_0^2\omega_1\omega_2)\Delta^1 b_1 \\
&\quad +(16\omega_0^2\omega_1\omega_2 - 8\omega_0\omega_1^2\omega_2 - 4\omega_0^2\omega_2^2)(b_2 - b_0) \\
&\quad -24\omega_0\omega_1^2\omega_2\Delta^1 b_0), \\
c_3 &= \frac{1}{10}(24\omega_0\omega_1^2\omega_2\Delta^1 b_1 - 8(\omega_0\omega_1^2\omega_2 \\
&\quad +\omega_0\omega_1\omega_2^2)(\Delta^1 b_0) + (-16\omega_0\omega_1\omega_2^2 \\
&\quad +8\omega_0\omega_1^2\omega_2 + 4\omega_0^2\omega_2^2)(b_2 - b_0)), \\
c_4 &= \frac{1}{5}((-8\omega_1^2\omega_2^2 + 16\omega_1^3\omega_2 + 8\omega_0\omega_1\omega_2^2)\Delta^1 b_1 \\
&\quad +(2\omega_0\omega_2^3 - 8\omega_0\omega_2^3)(b_2 - b_0)) - 4\omega_0\omega_1\omega_2^2\Delta^1 b_0)
\end{aligned}$$

and

$$c_5 \quad = \quad (-4\omega_1\omega_2^3 + 8\omega_1^2\omega_2^2)\Delta^1 b_1 - 2\omega_0\omega_2^3(b_2 - b_0).$$

In the following theorem, we give the Darboux frame field of a quadratic rational Bézier curve $B(t)$ for all $t \in R$.

Theorem 3.1. *The Darboux frame* $\{T(t), V(t), N(t)\}$ *along a quadratic rational Bézier curve* $B(t)$ *defined by (13) on 2-sphere* $S^2(r)$ *is given by*

$$T(t) = \frac{\sum_{i=0}^{2} a_i B_{i,2}(t)}{\|\sum_{i=0}^{2} a_i B_{i,2}(t)\|}, \tag{17}$$

$$V(t) = \frac{\sum_{i=0}^{2}\sum_{j=0}^{1} B_{i,2}(t)\, B_{j,1}(t) \left(\omega_i b_i \times \triangle^1 \omega_j b_j\right)}{\|\sum_{i=0}^{2}\sum_{j=0}^{1} B_{i,2}(t)\, B_{j,1}(t) \left(\omega_i b_i \times \triangle^1 \omega_j b_j\right)\|} \tag{18}$$

and

$$N(t) = \frac{\sum_{i=0}^{2} \omega_i b_i B_{i,2}(t)}{r \sum_{i=0}^{2} \omega_i B_{i,2}(t)}. \tag{19}$$

Proof. Suppose that $B(t)$ is a rational Bézier curve defined by (13) on 2-sphere $S^2(r)$. From (1) and (15), we obtain

$$T(t) = \frac{B'(t)}{\|B'(t)\|} = \frac{\sum_{i=0}^{2} a_i B_{i,2}(t)}{\|\sum_{i=0}^{2} a_i B_{i,2}(t)\|}.$$

Taking into consideration (13) and (7), we get the normal vector field

$$N(B(t)) = \frac{1}{r} B(t) = \frac{\sum_{i=0}^{2} \omega_i b_i B_{i,2}(t)}{r \sum_{i=0}^{2} \omega_i B_{i,2}(t)}.$$

Now we focus on finding third vector field of the Darboux frame. From (2), we equivalently have

$$V(t) = \frac{B(t) \times B'(t)}{\|B(t) \times B'(t)\|}. \tag{20}$$

By using (13), (14) and Lemma 2.3, we calculate

$$B(t) \times B'(t) = \frac{\sum_{i=0}^{2} \omega_i b_i B_{i,2}(t)}{\sum_{i=0}^{2} \omega_i B_{i,2}(t)} \times \frac{(\sum_{i=0}^{2} \omega_i b_i B_{i,2}(t))'}{\sum_{i=0}^{2} \omega_i B_{i,2}(t)}$$

$$= \frac{\sum_{i=0}^{2} \omega_i b_i B_{i,2}(t)}{\sum_{i=0}^{2} \omega_i B_{i,2}(t)} \times \frac{2(\sum_{i=0}^{1} \Delta^1 \omega_i b_i B_{i,1}(t))}{\sum_{i=0}^{2} \omega_i B_{i,2}(t)}$$

$$= 2\frac{\sum_{i=0}^{2}\sum_{j=0}^{1} B_{i,2}(t) B_{j,1}(t)(\omega_i b_i \times \Delta^1 \omega_j b_j)}{(\sum_{i=0}^{2} \omega_i B_{i,2}(t))^2} \tag{21}$$

where $\Delta^1 \omega_i b_i = \omega_{i+1} b_{i+1} - \omega_i b_i$. Substituting (21) into (20), we obtain (18). We can get the geodesic curvature of a quadratic rational Bézier curve $B(t)$ in the following theorem.

Theorem 3.2. *The geodesic curvature $\kappa_g(t)$ of a quadratic rational Bézier curve $B(t)$ defined by (13) on 2-sphere $S^2(r)$ is given by*

$$\kappa_g = \frac{\sum_{i=0}^{2}\sum_{j=0}^{5}\sum_{k=0}^{2} B_{i,2}(t) B_{j,5}(t) B_{k,2}(t) < a_i \times c_j, \omega_k b_k >}{r\left(\sum_{i=0}^{2} \omega_i B_{i,2}(t)\right) \left\| \sum_{i=0}^{2} a_i B_{i,2}(t) \right\|^3}. \tag{22}$$

Proof. Taking into consideration (7) in (4), we obtain

$$\kappa_g = \frac{\det(B', B'', B)}{r \left\| B' \right\|^3}. \tag{23}$$

Substituting (14), (16) and (13) into (23), we get the formula (22).

The ordinary curvature and torsion of a rational Bezier curve given as above can be obtained using (8), (9) and (22) in (5) and (6), respectively.

Corollary 3.3. *Any quadratic rational Bézier curve $B(t)$ defined by (13) on S^2 is a geodesic of the sphere (a great circle) when*

$$< a_i \times c_j, \omega_k b_k >= 0 \tag{24}$$

for $0 \le i \le 2$, $0 \le j \le 5$ and $0 \le k \le 2$.

The following theorems give the Darboux frame and the geodesic curvature for a quadratic rational Bézier curve at the point $B(0) = b_0$. The proofs are clear.

Theorem 3.4. *The Darboux frame $\{T, V, N\}$ and geodesic curvature κ_g of the quadratic rational Bézier curve $B(t)$ defined by (13) at $t = 0$ on S^2 are defined by*

$$T|_{t=0} = \frac{\Delta^1 b_0}{\|\Delta^1 b_0\|}, \tag{25}$$

$$V|_{t=0} = \frac{b_0 \times b_1}{\|b_0 \times b_1\|}, \tag{26}$$

$$N|_{t=0} = \frac{b_0}{r} \tag{27}$$

and

$$\kappa_g|_{t=0} = \frac{\omega_0\omega_2 < b_1 \times b_2, b_0 >}{2r\omega_1^2\|\Delta^1 b_0\|^3}. \tag{28}$$

Theorem 3.5. *Any quadratic rational Bézier curve $B(t)$ defined by (13) at $t = 0$ on S^2 lies on a geodesic arc of the sphere (a great circle) when*

$$< b_1 \times b_2, b_0 >= 0. \tag{29}$$

The following theorems give the Darboux frame and the geodesic curvature for a quadratic rational Bézier curve at the point $B(1) = b_2$. The proofs are clear.

Theorem 3.6. *The Darboux frame $\{T, V, N\}$ and geodesic curvature κ_g of the quadratic rational Bézier curve $B(t)$ defined by (13) at $t = 1$ on S^2 are given by*

$$T|_{t=1} = \frac{\Delta^1 b_1}{\|\Delta^1 b_1\|}, \tag{30}$$

$$V|_{t=1} = \frac{b_1 \times b_2}{\|b_1 \times b_2\|}, \tag{31}$$

$$N|_{t=1} = \frac{b_2}{r} \tag{32}$$

and

$$\kappa_g|_{t=1} = -\frac{\omega_0\omega_2 < b_1 \times b_0, b_2 >}{2r\omega_1^2\|\Delta^1 b_1\|^3}. \tag{33}$$

Theorem 3.7. *Any quadratic rational Bézier curve $B(t)$ defined by (13) at $t = 1$ on S^2 lies on a geodesic arc of the sphere (a great circle) when*

$$< b_1 \times b_0, b_2 >= 0. \tag{34}$$

Example 3.8. *We consider a quadratic rational Bézier curve $B(t)$ with control points*

$$b_0 = (1,0,0), b_1 = (1, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), b_2 = (0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$$

and the corresponding weights

$$\omega_0 = 1, \omega_1 = \frac{\sqrt{2}}{2}, \omega_2 = 1.$$

Then we can write the curve as follows

$$\begin{aligned} B(t) &= \frac{1}{g}(1 + (\sqrt{2}-2)t + (1-\sqrt{2})t^2, \\ & t + (\frac{1}{\sqrt{2}} - 1)t^2, t + (\frac{1}{\sqrt{2}} - 1)t^2) \end{aligned} \tag{35}$$

where $g = 1+(\sqrt{2}-2)t+(2-\sqrt{2})t^2$. One can easily see that $< B(t), B(t) >= 1$, that is, $B(t)$ lies on S^2 with radius 1 (see Fig. 1). From (17), we have the unit tangent vector field

$$\begin{aligned} T(t) &= \frac{1}{e}(1 - t + \frac{\sqrt{2}}{2}t^2, \frac{1}{2} + (\frac{\sqrt{2}}{2} - 1)t \\ & -\frac{1}{2}(1+\sqrt{2})t^2, \frac{1}{2} + \\ & (\frac{\sqrt{2}}{2} - 1)t - \frac{1}{2}(1+\sqrt{2})t^2), \end{aligned} \tag{36}$$

where

$$e = \sqrt{\begin{array}{l} 3t^2 + 2(\sqrt{2}-1)t^3 + 3(1-\sqrt{2})t^2 \\ + \left(\sqrt{2}-2\right)t + \frac{1}{2} \end{array}}.$$

The unit normal vector field is

$$N(t) = B(t) \tag{37}$$

and we can find the third orthogonal vector field $V(t)$ from the cross product of (37) and (36). From (25-27), the Darboux frame of $B(t)$ at the point $B(0)$ is given by

$$T|_{t=0} = b_2, V|_{t=0} = (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), N|_{t=0} = b_0$$

We calculate from (29)

$$< b_1 \times b_2, b_0 >= 0. \tag{38}$$

Similarly, one can see from (30-32)

$$T|_{t=1} = -b_0, V|_{t=1} = (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), N|_{t=1} = b_2.$$

From (34), we have

$$< b_1 \times b_0, b_2 >= 0. \tag{39}$$

Thus, from Eqs. (38) and (39) the quadratic rational Bézier curve $B(t)$ lies on a geodesic arc (a great circle) at the end points.

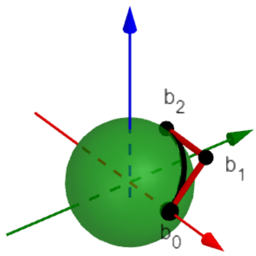

FIGURE 1. The Bézier curve corresponding to Example 3.8.

Example 3.9. *For given control points*

$$b_0 = (\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}), b_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}),$$
$$b_2 = (0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$$

and the corresponding weights

$$\omega_0 = 1, \omega_1 = \frac{\sqrt{2}}{2}, \omega_2 = 1,$$

we have the following quadratic rational Bézier curve $B(t)$ on S^2

$$\begin{aligned} B(t) &= \frac{1}{f}(\frac{1}{2} + (\sqrt{2}-2)t + (4-3\sqrt{2})t^2 \\ &\quad +(3\sqrt{2}-4)t^3 + (\frac{3}{2}-\sqrt{2})t^4, t^2 \\ &\quad +(\sqrt{2}-2)t^3 + (\frac{3}{2}-\sqrt{2})t^4, \\ &\quad \frac{1}{2} + (\sqrt{2}-2)t + (5-3\sqrt{2})t^2 \\ &\quad +(4\sqrt{2}-6)t^3 + (3-2\sqrt{2})t^4) \end{aligned}$$

where

$$\begin{aligned} f &= 1 + (2\sqrt{2}-4)t + (10-6\sqrt{2})t^2 \\ &\quad +(8\sqrt{2}-12)t^3 + (6-4\sqrt{2})t^4. \end{aligned}$$

We calculate from (29) and (34)

$$\kappa_g|_{t=0} = \kappa_g|_{t=1} = 1. \tag{40}$$

Thus, from (40), the quadratic rational Bézier curve $B(t)$ lies on a small circle at the end points.

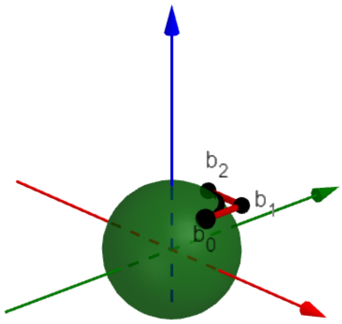

FIGURE 2. The Bézier curve corresponding to Example 3.9.

References

[1] G. Brunnet and P. E. Crouch, *Elastic curves on the sphere*, Advances in Computational Mathematics, **2(1)** (1994), 23-40.

[2] M. P. Do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall, Englewood Cliffs, NJ. 1976.

[3] G. E. Farin, *Curves and surfaces for CAGD:. a practical guide*, Morgan Kaufmann Publishers., 2002.

[4] E. Erkan and S. Yüce, *Serret-Frenet frame and curvatures of Bézier curves*, Mathematics **6(12)** (2018), 321.

[5] M. S. Floater, *Derivatives of rational Bézier curves*, Computer Aided Geometric Design **9(3)** (1993), 161-174.

[6] D. Marsh, *Applied geometry for computer graphics and CAD.*, Springer 2006.

[7] B. O'Neill, *Elementary differential geometry*, Elsevier 2006.

[8] F. C. Park and B. Ravani, *Bézier curves on Riemannian manifolds and Lie groups with kinematics applications*, Journal of Mechanical Design **117** (1995), 36-40.

[9] T. Popiel and L. Noakes, *C^2 spherical Bézier splines*, Computer aided geometric design **23(3)** (2006), 261-275.

[10] K. Shomake, *Animating rotations with quaternion curves*, ACM SIGGRAPH Computer Graphics **19** (1985), 245–254.

[11] B. Tantay and F. Taş, *The Curvature of a Bézier Control Polyline*, Mathematical and Computational Applications **16(2)** (2011), 350-358.

[12] A. Yılmaz Ceylan, T. Turhan and G. Ö. Tükel, *Darboux frame fields of rational Bézier curves on the two dimensional sphere*, BILTEK CONFERENCE-III Current Studies on Science, Technology Social Sciences Haziran 19-20, Adana, Turkey, (2020), 243-248.

[13] G. Özkan and A. Yücesan, *Generalized relaxed elastic line on an oriented surface*, Ukrainian Journal of Mathematics **64(8)** (2012), 1121-1131.

Ayşe Yılmaz Ceylan*
Department of Mathematics,
Akdeniz University, Turkey.
E-mail: ayilmazceylan@akdeniz.edu.tr

Tunahan Turhan
Department of Mathematics and Science Education,
Süleyman Demirel University, Turkey.
E-mail: tunahanturhan@sdu.edu.tr

Gözde Özkan Tükel
Isparta University of Applied Sciences, Turkey.
E-mail: gozdetukel@isparta.edu.tr