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PSEUDO JORDAN HOMOMORPHISMS AND DERIVATIONS ON MODULE EXTENSIONS AND TRIANGULAR BANACH ALGEBRAS

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Abstract. This paper considers pseudo Jordan homomorphisms on module extensions of Banach algebras and triangular Banach algebras. We characterize pseudo Jordan homomorphisms on module extensions of Banach algebras and triangular Banach algebras. Moreover, we define pseudo derivations on the above stated Banach algebras and characterize this new notion on those algebras.

1. Introduction and Preliminaries

Let A and B be rings (algebras) over $\mathbb{C}$, and let B be a right (left) A-module. An additive mapping $\varphi : A \longrightarrow B$ is called an n-Jordan homomorphism if $\varphi(a^n) = (\varphi(a))^n$, for all $a \in A$, it is an n-ring homomorphism if $\varphi(\prod_{i=1}^n a_i) = \prod_{i=1}^n \varphi(a_i)$, for all $a_i \in A$, where $i = 1, 2, \dots, n$. If $\varphi : A \longrightarrow B$ is an n-ring homomorphism; we say that φ is an n-homomorphism. The concept of n-homomorphism was introduced for (complex) Banach algebras by Hejazian, Mirzavaziri, and Moslehian [8] and studied by many authors in [3, 7, 9, 10, 6]. A 2-Jordan homomorphism is a Jordan homomorphism, in the usual sense. Every Jordan homomorphism is an n-Jordan homomorphism, for all $n \geq 2$.

Let A be a Banach algebra and X be a Banach A-bimodule. Throughout of this paper, all maps are continuous. As a generalization of Jordan and n-Jordan homomorphisms, pseudo n-Jordan homomorphisms were introduced by Ebadian, Jabbari and Kanzi [5]. A linear mapping $\varphi : A \longrightarrow B$ is called a pseudo n-Jordan homomorphism if there exists an element $w \in A$ such that $\varphi(a^n w) = \varphi(a)^n \cdot w$ ($\varphi(a^n w) = w \cdot \varphi(a)^n$), for all $a \in A$ and $n \geq 2$. We say that w is a Jordan coefficient of φ. A pseudo 2-Jordan homomorphism is called a pseudo Jordan homomorphism. A linear map $d : A \longrightarrow X$ is called a derivation if $d(aa') = a \cdot d(a') + d(a) \cdot a'$, for all $a, a' \in A$. The derivation $d : A \longrightarrow X$ is

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said to be inner, if there exists $x \in X$ such that $d(a) = a \cdot x - x \cdot a$, for every $a \in A$.

Let A be a Banach algebra and X be a Banach A-bimodule. By a module extension Banach algebra corresponding to A and X, we will mean the ℓ^1-direct sum of A and X i.e., $A \oplus_1 X$ with the following algebra product and norm:

$$(a_1, x_1)(a_2, x_2) = (a_1 a_2, a_1 \cdot x_2 + x_1 \cdot a_2),$$
$$\|(a, x)\| = \|a\|_A + \|x\|_X,$$

for all $a_1, a_2 \in A$ and $x_1, x_2 \in X$. These algebras have been studied by Zhang [22]. Some homological, cohomological results, results related to derivations on the second dual and module extension of dual Banach algebras are given in [1, 11, 17, 18]. Triangular Banach algebras are as examples of module extension Banach algebras that these algebras are considered extensively by Forrest and Marcoux in [12, 13, 14]. For more results related to homological and cohomological results of triangular Banach algebras, we refer to [2, 4, 15, 16, 19, 20, 21]. Let A and B be Banach algebras and M be a Banach A, B-module. We define triangular Banach algebra

$$\mathcal{T} = \begin{bmatrix} A & M \\ & B \end{bmatrix},$$

with the sum and product being giving by the usual 2×2 matrix operations and internal module actions. The norm on $\mathcal{T}$ is

$$\| \begin{bmatrix} a & m \\ & b \end{bmatrix} \| = \|a\|_A + \|m\|_M + \|b\|_B.$$

The triangular Banach algebra $\mathcal{T}$ as a Banach space is isomorphic to the ℓ^1-direct sum of A, B and M.

Let A be a Banach algebra, X and Y be two Banach A-bimodules. A linear operator $T : X \longrightarrow Y$ is called an A-bimodule morphism if $T(\alpha \cdot x \cdot \beta) = \alpha \cdot T(x) \cdot \beta$, for all $\alpha, \beta \in A$ and $x \in X$.

In the next section, we define pseudo Jordan Φ-derivations and pseudo Jordan module Φ-homomorphisms on Banach algebras, where Φ is a linear map. By these definitions we characterize pseudo Jordan homomorphisms on module extensions and triangular Banach algebras. In section 3, we characterize pseudo Jordan derivations on the above mentioned Banach algebras.

2. Pseudo Jordan homomorphisms on $A \oplus_1 X$

In this section, we characterize pseudo Jordan homomorphisms on $A \oplus_1 X$ and next we characterize this notion on triangular Banach algebras.

Definition 2.1. *Let A be a Banach algebra, X be a Banach A-bimodule, $\Phi : A \longrightarrow A$ and $D : A \longrightarrow X$ be a linear maps. We say that D is a pseudo Jordan Φ-derivation with Jordan coefficient $\alpha \in A$, if it satisfies the following:*

$$D(a^2\alpha) = (\Phi(a) \cdot D(a) + D(a) \cdot \Phi(a)) \cdot \alpha, \tag{1}$$

for all $a \in A$. Note that if Φ is the identity map, then we call D a pseudo Jordan derivation with Jordan coefficient $\alpha \in A$.

Example 2.2. *Let*

$$T_2(\mathbb{R}) = \left\{ \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \right\}$$

and

$$U_2(\mathbb{R}) = \left\{ \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix} : \alpha, \beta \in \mathbb{R} \right\}$$

be two algebras of 2×2 matrixes, $\Phi : T_2(\mathbb{R}) \longrightarrow T_2(\mathbb{R})$ and $D : T_2(\mathbb{R}) \longrightarrow U_2(\mathbb{R})$ be linear maps defined by

$$\Phi\left(\begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix}\right) = \begin{bmatrix} \alpha & 0 \\ 0 & \gamma \end{bmatrix}, \qquad D\left(\begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix}\right) = \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}.$$

It is easy to see that D is not a Jordan Φ-derivation. Let $\Delta = \begin{bmatrix} 0 & 0 \\ 0 & \theta \end{bmatrix}$. Then by easy calculations D is a pseudo Jordan Φ-derivation with Jordan coefficient Δ.

Definition 2.3. *Let A be a Banach algebra, X be a Banach A-bimodule and $\Phi_A : A \longrightarrow A$ be a pseudo Jordan homomorphism with Jordan coefficient $\alpha \in A$. A linear map $\Phi_X : X \longrightarrow X$ is called pseudo module Φ_A-homomorphism with Jordan coefficient α, if it satisfies the following*

$$\Phi_X(a\cdot x\cdot \alpha) = \Phi_A(a)\cdot \Phi_X(x)\cdot \alpha \qquad \text{and} \qquad \Phi_X(x\cdot a\alpha) = \Phi_X(x)\cdot \Phi_A(a)\alpha, \tag{2}$$

for all $a \in A$ and $x \in X$.

Example 2.4. *Let $T_2(\mathbb{R})$ and $U_2(\mathbb{R})$ be as in Example 2.2. Define $\Phi : T_2(\mathbb{R}) \longrightarrow T_2(\mathbb{R})$ and $D : U_2(\mathbb{R}) \longrightarrow U_2(\mathbb{R})$ by*

$$\Phi\left(\begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix}\right) = \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}, \qquad D\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} a & a+b \\ 0 & 0 \end{bmatrix}.$$

Clearly, D is not a Jordan derivation and Φ is a pseudo Jordan homomorphism with Jordan coefficient $\Delta = \begin{bmatrix} 0 & \theta \\ 0 & 0 \end{bmatrix}$. By a routine calculations one can see that D satisfies (2). Hence, D is a pseudo module Φ-homomorphism with Jordan coefficient Δ.

Now by the following result, we characterize pseudo Jordan homomorphism on $A \oplus_1 X$, where A is a without order Banach algebra.

Theorem 2.5. *Let $A \oplus_1 X$ be a module extension Banach algebra, where A is a without order Banach algebra. Then $\Phi : A \oplus_1 X \longrightarrow A \oplus_1 X$ is a pseudo Jordan homomorphism with Jordan coefficient $(\alpha, 0) \neq (0, 0)$ if and only if*

$$\Phi((a, x)) = (\Phi_A(a), \Phi_X(x) + \Phi_{A,X}(a)), \tag{3}$$

such that

(i) Φ_A *is a pseudo Jordan homomorphism with Jordan coefficient* α.
(ii) Φ_X *is a pseudo module* Φ_A*-homomorphism with Jordan coefficient* α.
(iii) $\Phi_{A,X}$ *is a pseudo Jordan* Φ_A*-derivation with Jordan coefficient* α.

Proof. Let Φ be a pseudo Jordan homomorphism with the Jordan coefficient $(\alpha, 0)$. Define the canonical injective maps $\imath_A : A \longrightarrow (A \oplus_1 X)$, $\imath_X : X \longrightarrow (A \oplus_1 X)$ by $\imath_A(a) = (a, 0)$, $\imath_X(x) = (0, x)$, for all $a \in A$, $x \in X$ and projective maps $\pi_A : (A \oplus_1 X) \longrightarrow A$ and $\pi_X : (A \oplus_1 X) \longrightarrow X$. Define $\Phi_A := \pi_A \circ \mathcal{D} \circ \imath_A : A \longrightarrow A$, $\Phi_X := \pi_X \circ \mathcal{D} \circ \imath_X : X \longrightarrow X$, $\Phi_{X,A} := \pi_A \circ \mathcal{D} \circ \imath_X : X \longrightarrow A$ and $\Phi_{A,X} := \pi_X \circ \mathcal{D} \circ \imath_A : A \longrightarrow X$. Since, Φ is a linear map, the defined above maps are linear maps. Then

$$\Phi((a, x)) = (\Phi_A(a) + \Phi_{X,A}(x), \Phi_X(x) + \Phi_{A,X}(a)), \tag{4}$$

for all $a \in A$ and $x \in X$. For any $a \in A$ and $x \in X$, (4) implies that

$$\begin{aligned}\Phi((a, x)^2(\alpha, 0)) &= \Phi((a^2\alpha, a \cdot x \cdot \alpha + x \cdot a\alpha)) \\ &= (\Phi_A(a^2\alpha) + \Phi_{X,A}(a \cdot x \cdot \alpha + x \cdot a\alpha), \\ &\qquad \Phi_X(a \cdot x \cdot \alpha + x \cdot a\alpha) + \Phi_{A,X}(a^2\alpha)) \\ &= (\Phi_A(a^2\alpha) + \Phi_{X,A}(a \cdot x \cdot \alpha) + \Phi_{X,A}(x \cdot a\alpha), \\ &\qquad \Phi_X(a \cdot x \cdot \alpha) + \Phi_X(x \cdot a\alpha) + \Phi_{A,X}(a^2\alpha)),\end{aligned} \tag{5}$$

for every $(a, x) \in A \oplus_1 X$. On the other hand,

$$\begin{aligned}\Phi((a, x))^2(\alpha, 0) &= (\Phi_A(a) + \Phi_{X,A}(x), \Phi_X(x) + \Phi_{A,X}(a))^2(\alpha, 0) \\ &= (\Phi_A(a)^2\alpha + \Phi_A(a)\Phi_{X,A}(x)\alpha + \Phi_{X,A}(x)\Phi_A(a)\alpha \\ &\quad + \Phi_{X,A}(x)^2\alpha, \Phi_A(a) \cdot \Phi_X(x) \cdot \alpha + \Phi_A(a) \cdot \Phi_{A,X}(a) \cdot \alpha \\ &\quad + \Phi_{X,A}(x) \cdot \Phi_X(x) \cdot \alpha + \Phi_{X,A}(x) \cdot \Phi_{A,X}(a) \cdot \alpha \\ &\quad + \Phi_X(x) \cdot \Phi_A(a)\alpha + \Phi_X(x) \cdot \Phi_{X,A}(x)\alpha \\ &\quad + \Phi_{A,X}(a) \cdot \Phi_A(a)\alpha + \Phi_{A,X}(a) \cdot \Phi_{X,A}(x)\alpha),\end{aligned} \tag{6}$$

for every $(a, x) \in A \oplus_1 X$. Letting $a = 0$ in (5) and (6) implies that $\Phi_{X,A}(x)^2\alpha = 0$. Since $\alpha \neq 0$ and A is without of order, $\Phi_{X,A}(x) = 0$, for all $x \in X$. This means that $\Phi_{X,A} = 0$. We now set $x = 0$. Again by (5) and (6) we have

$$\Phi_A(a^2\alpha) = \Phi_A(a)\alpha \tag{7}$$

and

$$\Phi_{A,X}(a^2\alpha) = (\Phi_A(a) \cdot \Phi_{A,X}(a) + \Phi_{A,X}(a) \cdot \Phi_A(a)) \cdot \alpha, \tag{8}$$

for all $a \in A$. Thus (7) implies that Φ_A is a pseudo Jordan homomorphism with the Jordan coefficient α. Moreover, (8) implies that $\Phi_{A,X}$ is a pseudo Jordan Φ_A-derivation with the Jordan coefficient α and Φ_X is a pseudo module homomorphism with Jordan coefficient α. □

As mentioned in the first section, triangular Banach algebras are special case of module extension Banach algebras. We denote a triangular Banach algebra $\mathcal{T} = \begin{bmatrix} A & M \\ & B \end{bmatrix}$ by $\mathcal{T} = \mathrm{Tri}(A, M, B)$. We now generalize Definition 2.3 for triangular Banach algebras as follows:

Definition 2.6. *Let $\mathcal{T} = \mathrm{Tri}(A, M, B)$ be a triangular Banach algebra, $\phi : A \longrightarrow A$ be a linear map and $\psi : B \longrightarrow B$ be a pseudo Jordan homomorphism with Jordan coefficient β. We say a linear map $\Lambda : M \longrightarrow M$ is a pseudo module (ϕ, ψ)-homomorphism with Jordan coefficient β, if it satisfies the following*

$$\Lambda(a \cdot m \cdot \beta) = \phi(a) \cdot \Lambda(m) \cdot \beta \qquad \text{and} \qquad \Lambda(m \cdot b\beta) = \Lambda(m) \cdot \psi(b)\beta, \tag{9}$$

for all $a \in A$, $b \in B$ and $m \in M$.

Proposition 2.7. *Let $\mathcal{T} = \mathrm{Tri}(A, M, B)$ be a triangular Banach algebra, $\phi : A \longrightarrow A$, $\psi : B \longrightarrow B$ be pseudo Jordan homomorphisms with Jordan coefficients α and β, respectively and $\Lambda : M \longrightarrow M$ be a pseudo module (ϕ, ψ)-homomorphism with Jordan coefficient β. Then $\Phi : \mathcal{T} \longrightarrow \mathcal{T}$ defined by*

$$\Phi\left(\begin{bmatrix} a & m \\ & b \end{bmatrix}\right) = \begin{bmatrix} \phi(a) & \Lambda(m) \\ & \psi(b) \end{bmatrix} \tag{10}$$

is a pseudo Jordan homomorphism with Jordan coefficient $\begin{bmatrix} \alpha & 0 \\ & \beta \end{bmatrix}$.

Proof. For any $\begin{bmatrix} a & m \\ & b \end{bmatrix} \in \mathrm{Tri}(A, M, B)$,

$$\begin{aligned}
\Phi\left(\begin{bmatrix} a & m \\ & b \end{bmatrix}^2 \begin{bmatrix} \alpha & 0 \\ & \beta \end{bmatrix}\right) &= \Phi\left(\begin{bmatrix} a^2\alpha & a \cdot m \cdot \beta + m \cdot b \cdot \beta \\ & b^2\beta \end{bmatrix}\right) \\
&= \begin{bmatrix} \phi(a^2\alpha) & \Lambda(a \cdot m \cdot \beta + m \cdot b \cdot \beta) \\ & \psi(b^2\beta) \end{bmatrix} \\
&= \begin{bmatrix} \phi(a)^2\alpha & \phi(a) \cdot \Lambda(m) \cdot \beta \\ & +\Lambda(m) \cdot \psi(b)\beta \\ & \psi(b)^2\beta \end{bmatrix} \qquad (11)
\end{aligned}$$

and

$$\begin{aligned}
\Phi\left(\begin{bmatrix} a & m \\ & b \end{bmatrix}\right)^2 \begin{bmatrix} \alpha & 0 \\ & \beta \end{bmatrix} &= \begin{bmatrix} \phi(a) & \Lambda(m) \\ & \psi(b) \end{bmatrix}^2 \begin{bmatrix} \alpha & 0 \\ & \beta \end{bmatrix} \\
&= \begin{bmatrix} \phi(a)^2 & \phi(a) \cdot \Lambda(m) \\ & +\Lambda(m) \cdot \psi(b) \\ & \psi(b^2\beta) \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ & \beta \end{bmatrix} \\
&= \begin{bmatrix} \phi(a)^2\alpha & \phi(a) \cdot \Lambda(m) \cdot \beta \\ & +\Lambda(m) \cdot \psi(b)\beta \\ & \psi(b)^2\beta \end{bmatrix} \qquad (12)
\end{aligned}$$

Thus, Φ is a pseudo Jordan homomorphism with Jordan coefficient $\begin{bmatrix} \alpha & 0 \\ & \beta \end{bmatrix}$. $\square$

Theorem 2.8. *Let $\mathcal{T} = \mathrm{Tri}(A, M, B)$ be a triangular Banach algebra and $\Phi : \mathcal{T} \longrightarrow \mathcal{T}$ be a linear map. If Φ is a pseudo Jordan homomorphism with Jordan coefficient $\begin{bmatrix} \alpha & \beta \\ & \gamma \end{bmatrix}$, then there are pseudo Jordan homomorphisms $\phi_A : A \longrightarrow A$, $\phi_B : B \longrightarrow B$ with Jordan coefficients α, γ, respectively, and a pseudo module (ϕ_A, ϕ_B)-homomorphism $\phi_M : M \longrightarrow M$ with Jordan coefficient γ such that $\phi_M(a^2 \cdot \beta) = \phi_A(a)^2 \cdot \beta$, for all $a \in A$.*

Proof. Similar to the proof of Theorem 2.5 consider the linear mappings $\pi_A : \mathcal{T} \longrightarrow A$, $\imath_A : A \longrightarrow \mathcal{T}$, $\pi_B : \mathcal{T} \longrightarrow B$, $\imath_B : B \longrightarrow \mathcal{T}$, $\pi_M : \mathcal{T} \longrightarrow M$, $\imath_M : M \longrightarrow \mathcal{T}$. Define the coordinate mappings $\phi_A := \pi_A \circ \Phi \circ \imath_A : A \longrightarrow A$, $\phi_B := \pi_B \circ \Phi \circ \imath_B : B \longrightarrow B$ and $\phi_M := \pi_M \circ \Phi \circ \imath_M : M \longrightarrow M$. For any $\begin{bmatrix} a & m \\ & b \end{bmatrix} \in \mathcal{T}$,

$$\begin{aligned}
&\Phi\left(\begin{bmatrix} a & m \\ & b \end{bmatrix}^2 \begin{bmatrix} \alpha & \beta \\ & \gamma \end{bmatrix}\right) \\
&\quad = \Phi\left(\begin{bmatrix} a^2\alpha & a^2 \cdot \beta + a \cdot m \cdot \gamma + m \cdot b\gamma \\ & b^2\gamma \end{bmatrix}\right)^2 \begin{bmatrix} \alpha & \beta \\ & \gamma \end{bmatrix} \\
&\quad = \begin{bmatrix} \phi_A(a^2\alpha) & \phi_M(a^2 \cdot \beta + a \cdot m \cdot \gamma + m \cdot b\gamma) \\ & \phi_B(b^2\gamma) \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
&\Phi\left(\begin{bmatrix} a & m \\ & b \end{bmatrix}\right)^2 \begin{bmatrix} \alpha & \beta \\ & \gamma \end{bmatrix} \\
&\quad = \begin{bmatrix} \phi_A(a)^2 & \phi_A(a) \cdot \phi_M(m) + \phi_M(m) \cdot \phi(b) \\ & \phi_B(b)^2 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ & \gamma \end{bmatrix} \\
&\quad = \begin{bmatrix} \phi_A(a)^2\alpha & \phi_A(a)^2 \cdot \beta + \phi_A(a) \cdot \phi_M(m) \cdot \gamma + \phi_M(m) \cdot \phi(b)\gamma \\ & \phi_B(b^2)\gamma \end{bmatrix}.
\end{aligned}$$

The above equalities imply that ϕ_A and ϕ_B are pseudo Jordan homomorphism with Jordan coefficients α and β, respectively. If we set $b = 0$ and $m = 0$, then we have $\phi_M(a^2 \cdot \beta) = \phi_A(a)^2 \cdot \beta$, for all $a \in A$. Letting $a = 0$ implies that $\phi_M(m \cdot b\gamma) = \phi_M(m) \cdot \phi(b)\gamma$, for all $b \in B$. Moreover, by setting $b = 0$ and the fact $\phi_M(a^2 \cdot \beta) = \phi_A(a)^2 \cdot \beta$, for all $a \in A$, we obtain that $\phi_M(a \cdot m \cdot \gamma) = \phi_M(m) \cdot \phi(b)\gamma$. Hence, ϕ_M is a pseudo module (ϕ_A, ϕ_B)-homomorphism. $\square$

3. Pseudo Jordan derivations on $A \oplus_1 X$

In this section, we characterize pseudo Jordan derivations that we have defined in the previous section on $A \oplus_1 X$ and we also extend this characterization to triangular Banach algebra $\mathrm{Tri}(A, M, B)$.

Theorem 3.1. *Let $A \oplus_1 X$ be a module extension Banach algebra where A is a without of order and let $D : A \oplus_1 X \longrightarrow A \oplus_1 X$ be a linear map. Then D is a pseudo Jordan derivation with Jordan coefficient $(0, 0) \neq (\alpha, 0) \in A \oplus_1 X$ if and only if, for any $(a, x) \in A \oplus_1 X$,*

$$D((a, x)) = (d_A(a) + d_{X,A}(x), d_X(x) + d_{A,X}(a)) \tag{13}$$

(i) *d_A and $d_{A,X}$ are pseudo Jordan derivations with Jordan coefficient α.*
(i) *$x \cdot d_{X,A}(x)\alpha + d_{X,A}(x) \cdot x \cdot \alpha = 0$, for all $x \in X$.*
(iii) *$d_{X,A}$ and d_X are A-bimodule morphisms.*

Proof. Let π_A, $\imath_A$, π_X and $\imath_X$ be as in the proof of Theorem 2.5. Define $d_A := \pi_A \circ D \circ \imath_A$, $d_X := \pi_X \circ D \circ \imath_X$, $d_{A,X} := \pi_X \circ D \circ \imath_A$ and $d_{X,A} := \pi_A \circ D\imath_X$. These maps are linear. For any $(a, x) \in A \oplus_1 X$,

$$\begin{aligned} D((a, x)^2(\alpha, 0)) &= D((a^2\alpha, a \cdot x \cdot \alpha + x \cdot a\alpha)) \\ &= (d_A(a^2\alpha) + d_{X,A}(a \cdot x \cdot \alpha + x \cdot a\alpha), \\ &\qquad d_X(a \cdot x \cdot \alpha + x \cdot a\alpha) + d_{A,X}(a^2\alpha)) \end{aligned} \tag{14}$$

and

$$\begin{aligned} &((a, x)D((a, x)) + D((a, x))(a, x))(\alpha, 0) \\ &\quad = ((a, x)(d_A(a) + d_{X,A}(x), d_X(x) + d_{A,X}(a)) \\ &\qquad + (d_A(a) + d_{X,A}(x), d_X(x) + d_{A,X}(a))(a, x))(\alpha, 0) \end{aligned} \tag{15}$$

Letting $a = 0$ in (14) and (15) implies that $x \cdot d_{X,A}(x)\alpha + d_{X,A}(x) \cdot x \cdot \alpha = 0$. Setting $x = 0$ in (14) and (15) implies that d_A and $d_{A,X}$ are pseudo Jordan derivations with Jordan coefficient α. These implies that

$$d_{X,A}(a \cdot x \cdot \alpha + x \cdot a\alpha) = a \cdot d_{X,A}(x) \cdot \alpha + d_{X,A}(x) \cdot a\alpha$$

and

$$d_X(a \cdot x \cdot \alpha + x \cdot a\alpha) = a \cdot d_X(x) \cdot \alpha + d_X(x) \cdot a\alpha,$$

for all $a \in A, x \in X$. Thus, $d_{X,A}$ and d_X are A-bimodule morphisms. The converse is clear. □

We now investigate pseudo Jordan derivations on triangular Banach algebras. Let $\mathcal{T} = \mathrm{Tri}(A, M, B)$ be a triangular Banach algebra, $\delta_A : A \longrightarrow A$, $\delta_B : B \longrightarrow B$ and $\delta_M : M \longrightarrow M$ be linear mappings. Moreover, suppose that δ_B is a pseudo Jordan derivation with coefficient γ. We say that δ_M is a pseudo generalized (δ_A, δ_B)-derivation if, for any $a \in A$, $m \in M$ and $b \in B$, we have

$$\delta_M(a \cdot m \cdot \gamma) = a \cdot \delta_M(m) \cdot +\delta_A(a) \cdot m \cdot \gamma$$

and

$$\delta_M(m\cdot b\gamma)=m\cdot\delta_B(b)\gamma+\delta_M(m)\cdot b\gamma.$$

Theorem 3.2. *Let $\mathcal{T}=\mathrm{Tri}(A,M,B)$ be a triangular Banach algebra and $D:\mathcal{T}\longrightarrow\mathcal{T}$ be a linear map. The D is a pseudo Jordan derivation with Jordan coefficient $\begin{bmatrix}\alpha & \beta\\ & \gamma\end{bmatrix}$ if and only if there are pseudo Jordan derivations $\delta_A:A\longrightarrow A$, $\delta_B:B\longrightarrow B$ with Jordan coefficients α, γ, respectively, and a pseudo generalized (δ_A,δ_B)-derivation $\delta_M:M\longrightarrow M$ with Jordan coefficient γ such that $\delta_M(a^2\cdot\beta)=a\cdot\delta_A(a)\cdot\beta+\delta_A(a)a\cdot\beta$, for all $a\in A$.*

Proof. Consider the coordinate linear mappings $\delta_A:A\longrightarrow A$, $\delta_B:B\longrightarrow B$ and $\delta_M:M\longrightarrow M$ such that for any $\begin{bmatrix}a & m\\ & b\end{bmatrix}$,

$$D\left(\begin{bmatrix}a & m\\ & b\end{bmatrix}\right)=\begin{bmatrix}\delta_A(a) & \delta_M(m)\\ & \delta_B(b)\end{bmatrix}.$$

Then

$$D\left(\begin{bmatrix}a & m\\ & b\end{bmatrix}^2\begin{bmatrix}\alpha & \beta\\ & \gamma\end{bmatrix}\right)=D\left(\begin{bmatrix}a^2\alpha & a^2\cdot\beta+a\cdot m\cdot\gamma+m\cdot b\gamma\\ & b^2\gamma\end{bmatrix}\right)$$

$$=\begin{bmatrix}\delta_A(a^2\alpha) & \delta_M(a^2\cdot\beta+a\cdot m\cdot\gamma+m\cdot b\gamma)\\ & \delta_B(b^2\gamma)\end{bmatrix}\tag{16}$$

and

$$\left(\begin{bmatrix}a & m\\ & b\end{bmatrix}D\left(\begin{bmatrix}a & m\\ & b\end{bmatrix}\right)+D\left(\begin{bmatrix}a & m\\ & b\end{bmatrix}\right)\begin{bmatrix}a & m\\ & b\end{bmatrix}\right)\begin{bmatrix}\alpha & \beta\\ & \gamma\end{bmatrix}$$

$$=\begin{bmatrix}a\delta_A(a)\alpha+\delta_A(a)a\alpha & \begin{matrix}a\cdot\delta_A(a)\cdot\beta+\delta_A(a)a\cdot\beta\\ +a\cdot\delta_M(m)\cdot\gamma+\delta_A(a)\cdot m\cdot\gamma\\ +m\cdot\delta_B(b)\gamma+\delta_M(m)\cdot b\gamma\end{matrix}\\ & \\ & b\delta_B(b)\gamma+\delta_B(b)b\gamma\end{bmatrix}.\tag{17}$$

Clearly, by (16) and (17), δ_A and δ_B are pseudo Jordan derivations $\delta_A:A\longrightarrow A$, $\delta_B:B\longrightarrow B$ with Jordan coefficients α, γ, respectively. By setting $a=0$ in (16) and (17), we get that $\delta_M(m\cdot b\gamma)=m\cdot\delta_B(b)\gamma+\delta_M(m)\cdot b\gamma$, for all $b\in B$ and $m\in M$. Moreover, by setting $m=0$ and $b=0$, we have $\delta_M(a^2\cdot\beta)=a\cdot\delta_A(a)\cdot\beta+\delta_A(a)a\cdot\beta$, for all $a\in A$. The above obtained results together with taking $b=0$ imply that $\delta_M(a\cdot m\cdot\gamma)=a\cdot\delta_M(m)\cdot\gamma+\delta_A(a)\cdot m\cdot\gamma$, for all $a\in A$ and $m\in M$. Thus, δ_M is a pseudo generalized (δ_A,δ_B)-derivation with Jordan coefficient γ. The converse is clear. □

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