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FINITE RANK LITTLE HANKEL OPERATORS ON THE DIRICHLET SPACE OF THE POLYDISK

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Abstract. We characterize pluriharmonic symbols of the finite rank little Hankel operators on the Dirichlet space of the unit polydisk.

1. Introduction

Let D be the unit disk in the complex plane $\mathbb{C}$. For a fixed integer $n \geq 1$, the unit polydisk D^n is the cartesian product of n copies of D and V is the Lebesgue volume measure on D^n normalized so that $V(D^n) = 1$. The Sobolev space $\mathscr{S}$ is the completion of the space of all smooth functions f on D^n for which

$$\|f\| = \left\{ \left| \int_{D^n} f\, dV \right|^2 + \int_{D^n} \left(|\mathcal{R}f|^2 + |\widetilde{\mathcal{R}}f|^2 \right) dV \right\}^{1/2} < \infty$$

where

$$\mathcal{R}f(z) = \sum_{i=1}^{n} z_i \frac{\partial f}{\partial z_i}(z), \quad \widetilde{\mathcal{R}}f(z) = \sum_{i=1}^{n} \overline{z_i} \frac{\partial f}{\partial \overline{z_i}}(z)$$

for $z = (z_1, \dots, z_n) \in D^n$. Then the Sobolev space $\mathscr{S}$ is a Hilbert space with respect to the inner product

$$\langle f, g \rangle = \int_{D^n} f\, dV \overline{\int_{D^n} g\, dV} + \int_{D^n} \{ \mathcal{R}f \overline{\mathcal{R}g} + \widetilde{\mathcal{R}}f \overline{\widetilde{\mathcal{R}}g} \}\, dV.$$

The Dirichlet space $\mathscr{D}$ is then a closed subspace of $\mathscr{S}$ consisting of all holomorphic functions on D^n. Put

$$\mathscr{L}^{1,\infty} = \left\{ \varphi \in \mathscr{S} \ : \varphi, \frac{\partial \varphi}{\partial z_j}, \frac{\partial \varphi}{\partial \bar{z}_j} \in L^\infty, j = 1, \dots, n \right\}$$

where the derivatives are taken in the sense of distribution and $L^p = L^p(D^n, V)$ denotes the usual Lebesgue space on D^n. For $\varphi, \psi \in \mathscr{L}^{1,\infty}$, we note that $\mathcal{R}\varphi, \widetilde{\mathcal{R}}\varphi \in L^\infty$ and $\varphi\psi \in \mathscr{L}^{1,\infty}$.

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Let Q be the Hilbert space orthogonal projection from $\mathscr{S}$ onto $\mathscr{D}$. Given a function $u \in \mathscr{L}^{1,\infty}$, the *little Hankel operator* $h_u : \mathscr{D} \to \mathscr{D}$ with symbol u is a bounded linear operator on $\mathscr{D}$ defined by

$$h_u f = Q(u\hat{f})$$

for functions $f \in \mathscr{D}$. Here, for a function g on D^n, we let $\hat{g}$ be the function on D^n defined by $\hat{g}(z) = g(\bar{z})$ for $z \in D^n$.

In this paper, we study the problem of when the little Hankel operators have finite ranks on the Dirichlet space of the polydisk. The corresponding problem has been well studied on various function spaces. On the Hardy space of the unit disk, a well known Kronecker's theorem says that the little Hankel operator with symbol u has finite rank if and only if $p\hat{u}$ is analytic for some polynomial p; see [4] for example. Also, on the Bergman space of bounded symmetric domains, holomorphic symbols of finite rank little Hankel operators have been characterized in [2]. Recently, the corresponding problem on the Dirichlet space of the unit ball has been also studied in [3].

We in this paper consider pluriharmonic symbols and characterize finite rank little Hankel operators with pluriharmonic symbol on the Dirichlet space of the unit polydisk. Recall that a twice continuously differentiable function u on D^n is said to be pluriharmonic if the one-variable function $\lambda \mapsto u(a + \lambda b)$, defined for $\lambda \in \mathbb{C}$ such that $a + \lambda b \in D^n$, is harmonic for each $a \in D^n$ and $b \in \mathbb{C}^n$. Recall that u is pluriharmonic on D^n if and only if $u = f + \overline{g}$ for some holomorphic functions f and g on D^n.

In order to introduce the our main result, we need some notations. Given a point $a = (a_1, \cdots, a_n) \in D^n$, we let

$$\varphi_a(z) = (\varphi_{a_1}(z_1), \cdots, \varphi_{a_n}(z_n))$$

for $z = (z_1, \ldots, z_n) \in D^n$ where each φ_{a_i} is the usual Möbius automorphism of D defined by

$$\varphi_{a_i}(z_i) = \frac{a_i - z_i}{1 - \overline{a_i} z_i}, \qquad z_i \in D.$$

Then, it is well known that φ_a is an automorphism on D^n and $\varphi_a(0) = a$. Also, we let k_a be the normalized Bergman kernel defined by

$$k_a(z) = \prod_{j=1}^{n} \frac{1 - |a_j|^2}{(1 - \overline{a_j} z_j)^2}$$

for $z = (z_1, \ldots, z_n) \in D^n$.

The following is the our main theorem.

Main Theorem. *Let $u \in \mathscr{L}^{1,\infty}$ be a pluriharmonic symbol. Then h_u has finite rank on $\mathscr{D}$ if and only if there exist finite points $a_1, \cdots, a_N \in D^n$ and polynomials $p_1, \cdots, p_N$ for which*

$$\mathcal{R}u = \sum_{j=1}^{N} p_j \circ \varphi_{a_j} k_{a_j}.$$

In Section 2, we collect some basic facts on the projection Q which will be useful in our proofs. We will prove our main theorem in Section 3.

2. Preliminaries

For any multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ where each α_k is a nonnegative integer, we will write $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$. We will also write $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ for $z = (z_1, \cdots, z_n) \in D^n$.

Let A^2 be the well known Bergman space consisting of all holomorphic functions in L^2. Note that $\mathscr{D} \subset A^2$ and moreover $||f||_2 \leq \|f\|$ holds for all $f \in \mathscr{D}$. Here and in what follows, we write

$$\langle \varphi, \psi \rangle_2 = \int_{D^n} \varphi \bar{\psi}\, dV$$

and $||\varphi||_2 = \sqrt{\langle \varphi, \varphi \rangle_2}$ for functions $\varphi, \psi \in L^2$. Let P denote the well known Bergman projection which is the orthogonal projection from L^2 onto A^2. Note that P can be written as

$$P\varphi(z) = \langle \varphi, K_z \rangle_2, \qquad z \in D^n$$

for $\varphi \in L^2$ where K_z is the well known Bergman kernel on A^2 given by

$$K_z(w) = \sum_{|\alpha| \geq 0} (1 + \alpha_1)(1 + \alpha_2) \cdots (1 + \alpha_n) \overline{z}^\alpha w^\alpha$$

for all $z, w \in D^n$. Thus P has the following explicit formula

$$P\varphi(z) = \sum_{|\alpha| \geq 0} (1 + \alpha_1) \cdots (1 + \alpha_n) z^\alpha \int_{D^n} \overline{w}^\alpha \varphi\, dV \tag{1}$$

for all $z \in D^n$ and $\varphi \in L^2$.

Also, each point evaluation is a bounded linear functional on $\mathscr{D}$ and hence, for each $z \in D^n$, there exists $R_z \in \mathscr{D}$ which has the reproducing property on $\mathscr{D}$:

$$f(z) = \langle f, R_z \rangle$$

for every $f \in \mathscr{D}$. Since the set $\{z^\alpha : |\alpha| \geq 0\}$ spans a dense subset of $\mathscr{D}$ and

$$||z^\alpha||^2 = \frac{|\alpha|^2}{(\alpha_1 + 1) \cdots (\alpha_n + 1)}$$

for each multi-index α with $|\alpha| > 0$, we can check that R_z can be given by

$$R_z(w) = 1 + \sum_{|\alpha| > 0} \frac{(1 + \alpha_1) \cdots (1 + \alpha_n)}{|\alpha|^2} \overline{z^\alpha} w^\alpha \tag{2}$$

for $z, w \in D^n$. Since $R_z(0) = 1$ for all $z \in D^n$, it follows from (2) that

$$
\begin{aligned}
Q\psi(z) &= \langle \psi, R_z \rangle \\
&= \int_{D^n} \psi \, dV + \langle \mathcal{R}\psi, \mathcal{R}R_z \rangle_2 \\
&= \int_{D^n} \psi \, dV + \sum_{|\alpha|>0} \frac{(1+\alpha_1)\cdots(1+\alpha_n)}{|\alpha|} z^\alpha \int_{D^n} \overline{w^\alpha} \mathcal{R}\psi \, dV
\end{aligned} \tag{3}
$$

for all $z \in D^n$. Note $\mathcal{R}z^\alpha = |\alpha| z^\alpha$ for all multi-index α. Thus, by (1), we have

$$
\begin{aligned}
\mathcal{R}(Q\psi)(z) &= \sum_{|\alpha|>0} (1+\alpha_1)\cdots(1+\alpha_n) z^\alpha \int_{D^n} \overline{w^\alpha} \mathcal{R}\psi \, dV \\
&= P(\mathcal{R}\psi)(z) - P(\mathcal{R}\psi)(0), \qquad z \in D^n
\end{aligned} \tag{4}
$$

for functions $\psi \in \mathscr{S}$. For each $a \in D^n$, we put

$$E_a(z) := \sum_{|\alpha|>0} \frac{(1+\alpha_1)\cdots(1+\alpha_n)}{|\alpha|} \overline{a^\alpha} z^\alpha, \qquad z \in D^n$$

for notational simplicity. Then, we can easily see that $\mathcal{R}E_a = K_a - 1$ and

$$\sup_{z \in B} |E_a(z)| \le \prod_{j=1}^{n} \left(\frac{1}{1 - |a_j|} \right)^2 \tag{5}$$

for each $a \in D^n$. Also, we note from (3) that

$$Q\psi(a) = \int_{D^n} \psi \, dV + \int_{D^n} (\mathcal{R}\psi) \overline{E_a} \, dV, \qquad a \in D^n$$

for functions $\psi \in \mathscr{S}$.

3. Proof of the main theorem

Given a Hilbert space K with an inner product $(\ ,\ )$ and $a, b \in K$, we let $a \otimes b$ be the rank one operator on K defined by

$$[a \otimes b]x = (x, b)a$$

for $x \in K$. Note that each finite rank operator S on K can be written as the form

$$S = \sum_{j=1}^{N} a_j \otimes b_j$$

for some sets $\{a_1, \cdots, a_N\}$ and $\{b_1, \cdots, b_N\}$ of linearly independent in K.

We first need the following lemma which is taken from Lemma 2.4 of [1].

Lemma 3.1. *Let $\{x_j\}_{j=1}^N$ be a linearly independent set in $\mathscr{D}$. Then there exist points $z_1, \dots, z_N \in D^n$ such that the matrix*

$$\begin{pmatrix} x_1(z_1) & \dots & x_1(z_N) \\ \vdots & & \vdots \\ x_N(z_1) & \dots & x_N(z_N) \end{pmatrix}$$

is invertible.

In the following, we let $\mathcal{R}^2\psi = \mathcal{R}(\mathcal{R}\psi)$ as usual.

Proposition 3.2. *Let $u \in \mathscr{L}^{1,\infty}$ and $x_j, y_j \in \mathscr{D}$ for $j = 1, \cdots, N$. Suppose $\{x_j\}_{j=1}^N$ is a linearly independent set and*

$$h_u = \sum_{j=1}^N x_j \otimes y_j$$

holds on $\mathscr{D}$. Then $\mathcal{R}^2 y_j \in A^2$ for all $j = 1, \cdots, N$.

Proof. Since $E_a(0) = 0$, $\mathcal{R}E_a = K_a - 1$ and $\mathcal{R}y_j(0) = 0$, we first note

$$\langle E_a, y_j\rangle = \langle K_a - 1, \mathcal{R}y_j\rangle_2 = \overline{\mathcal{R}y_j(a)} - \overline{\mathcal{R}y_j(0)} = \overline{\mathcal{R}y_j(a)}$$

for each j and $a \in D^n$. Hence

$$Q[u\hat{E}_a](z) = h_u E_a(z) = \sum_{j=1}^N \langle E_a, y_j\rangle x_j(z) = \sum_{j=1}^N \overline{\mathcal{R}y_j(a)} x_j(z)$$

for every $z, a \in D^n$. Thus, letting $G_z(a) = \overline{Q[u\hat{E}_a](z)}$, we have

$$\begin{pmatrix} G_{z_1}(a) \\ \vdots \\ G_{z_N}(a) \end{pmatrix} = \begin{pmatrix} \overline{x_1(z_1)} & \dots & \overline{x_N(z_1)} \\ \vdots & & \vdots \\ \overline{x_1(z_N)} & \dots & \overline{x_N(z_N)} \end{pmatrix} \begin{pmatrix} \mathcal{R}y_1(a) \\ \vdots \\ \mathcal{R}y_N(a) \end{pmatrix}$$

for all $a \in D^n$ and $z_1, \dots, z_N \in D^n$. Now, since functions $x_1, \dots, x_N$ are linearly independent by the assumption, Lemma 3.1 tells that the $N \times N$ matrix in the above displayed equation is invertible for some points $z_1, \dots, z_N \in D^n$. Thus, each $\mathcal{R}y_j$ is a linear combination of functions $G_{z_1}, \cdots, G_{z_N}$. So, in order to show $\mathcal{R}^2 y_j \in A^2$ for each j, it suffices to prove $\mathcal{R}G_z \in A^2$ for every $z \in D^n$. To prove this, fix a point $z \in D^n$. Note from (3) that

$$\begin{aligned} G_z(a) &= \overline{Q[u\hat{E}_a](z)} \\ &= \int_{D^n} \overline{u(w) E_a(\bar{w})}\, dV(w) + \int_{D^n} \overline{\mathcal{R}u(w) E_a(\bar{w})} E_z(w)\, dV(w) \end{aligned} \tag{6}$$

and

$$\begin{aligned}\mathcal{R}_a\overline{E_a(\bar{w})} &= \mathcal{R}_a \sum_{|\alpha|>0} \frac{(1+\alpha_1)\cdots(1+\alpha_n)}{|\alpha|} a^\alpha w^\alpha \\ &= \sum_{|\alpha|>0} (1+\alpha_1)\cdots(1+\alpha_n) a^\alpha w^\alpha \\ &= \overline{K_a(\bar{w})} - 1\end{aligned}$$

for every $a, w \in D^n$. Here, the notation $\mathcal{R}_a$ means the radial derivative $\mathcal{R}$ with respect to the variable a. Since $\int_{D^n} \psi \, dV = P\psi(0)$ for every $\psi \in L^2$, we also note

$$\begin{aligned}&\int_{D^n} \overline{u(w)} \left[\overline{K_a(\bar{w})} - 1\right] dV(w) \\ &\qquad = \int_{D^n} \overline{u(\bar{w})} \left[\overline{K_a(w)} - 1\right] dV(w) \\ &\qquad = P(\bar{\hat{u}})(a) - P(\bar{\hat{u}})(0)\end{aligned}$$

and

$$\begin{aligned}&\int_{D^n} \overline{\mathcal{R}u(w)} \left[\overline{K_a(\bar{w})} - 1\right] E_z(w) \, dV(w) \\ &\qquad = \int_{D^n} \overline{\mathcal{R}u(\bar{w})} \left[\overline{K_a(w)} - 1\right] E_z(\bar{w}) \, dV(w) \\ &\qquad = P(\overline{\widehat{\mathcal{R}u}}\widehat{E_z})(a) - P(\overline{\widehat{\mathcal{R}u}}\widehat{E_z})(0).\end{aligned}$$

It follows from (6) that

$$\mathcal{R}G_z = P(\bar{\hat{u}}) - P(\bar{\hat{u}})(0) + P(\overline{\widehat{\mathcal{R}u}}\widehat{E_z}) - P(\overline{\widehat{\mathcal{R}u}}\widehat{E_z})(0).$$

Note $\bar{\hat{u}}$ and $\overline{\widehat{\mathcal{R}u}}\widehat{E_z}$ are bounded on D^n by (5). Thus $\mathcal{R}G_z$ is the Bergman projection of a bounded function and hence $\mathcal{R}G_z \in A^2$ as desired. The proof is compete. $\square$

Given a holomorphic function g on D^n with $g(0) = 0$ and the power series expansion $g(z) = \sum_{|\alpha|>0} b_\alpha z^\alpha$, we let $\mathcal{R}^{-1}g$ be the holomorphic function on D^n defined by

$$\mathcal{R}^{-1}g(z) = \sum_{|\alpha|>0} \frac{b_\alpha}{|\alpha|} z^\alpha, \qquad z \in D^n.$$

Then, we note $\mathcal{R}^{-1}g(0) = 0$ and $\mathcal{R}(\mathcal{R}^{-1}g) = g$. Also, we let $(\mathcal{R}^{-1})^2 g = \mathcal{R}^{-1}(\mathcal{R}^{-1}g)$ as usual.

Using the exactly same argument as in the proof of Proposition 13 of [3], we have the following property which will be useful in our proofs.

Proposition 3.3. *The following statements hold.*

(a) $\langle \mathcal{R}f, \mathcal{R}g\rangle_2 = \langle f, \mathcal{R}^2 g\rangle_2$ *for every* $f, g \in \mathscr{D}$ *with* $\mathcal{R}^2 g \in A^2$.

(b) $\langle \mathcal{R}^{-1}f, \mathcal{R}^{-1}g\rangle = \langle f, (\mathcal{R}^{-1})^2 g\rangle$ *for every* $f \in \mathscr{D}$ *and* $g \in A^2$ *with* $f(0) = g(0) = 0$.

(c) $\langle f, g\rangle_2 = \langle f, (\mathcal{R}^{-1})^2[g - g(0)] + g(0)\rangle$ *for every* $f \in \mathscr{D}$ *and* $g \in A^2$.

In our characterization of finite rank little Hankel operators on the Dirichlet space, we will use a known result for little Hankel operators on the Bergman space. So we need to introduce little Hankel operators acting on the Bergman space of the polydisk. Given $u \in L^\infty$, the little Hankel operator b_u is a bounded operator defined on A^2 by

$$b_u f = P(u\hat{f})$$

for functions $f \in A^2$.

The following connection between finite rank little Hankel operators on the Bergman space and Dirichlet space will be very useful.

Proposition 3.4. *Let* $u \in \mathscr{L}^{1,\infty}$ *be holomorphic. Then* h_u *has finite rank on* $\mathscr{D}$ *if and only if* $b_{\mathcal{R}u}$ *has finite rank on* A^2.

Proof. First suppose h_u has finite rank on $\mathscr{D}$. Then there exist sets $\{x_j\}_{j=1}^N$ and $\{y_j\}_{j=1}^N$ linearly independent in $\mathscr{D}$ for which

$$h_u = \sum_{j=1}^N x_j \otimes y_j$$

on $\mathscr{D}$. Let p be any polynomial. Then we have

$$Q[u\widehat{p}] = h_u p = \sum_{j=1}^N [x_j \otimes y_j]p = \sum_{j=1}^N \langle p, y_j\rangle x_j. \tag{7}$$

Noting

$$\int_{D^n} (\mathcal{R}u)\widehat{p}\, dV = \int_{D^n} (\widehat{\mathcal{R}u})p\, dV = \langle p, \overline{\widehat{\mathcal{R}u}}\rangle_2 = \langle p, P(\overline{\widehat{\mathcal{R}u}})\rangle_2$$

and taking $\mathcal{R}$ to both sides of (7), we obtain by (4) and Proposition 3.3(a)

$$\begin{aligned}
b_{\mathcal{R}u}p = P[(\mathcal{R}u)\widehat{p}] &= P[(\mathcal{R}u)\widehat{p}](0) + \sum_{j=1}^N \langle p, y_j\rangle \mathcal{R}x_j \\
&= \int_{D^n} (\mathcal{R}u)\widehat{p}\, dV + \sum_{j=1}^N \left[p(0)\overline{y_j(0)} + \langle \mathcal{R}p, \mathcal{R}y_j\rangle_2\right] \mathcal{R}x_j \\
&= \langle p, \rho\rangle_2 + \sum_{j=1}^N \left[\langle p, y_j(0)\rangle_2 + \langle p, \mathcal{R}^2 y_j\rangle_2\right] \mathcal{R}x_j \\
&= [1 \otimes \rho]p + \sum_{j=1}^N \left[\mathcal{R}x_j \otimes y_j(0)\right] p + \sum_{j=1}^N [\mathcal{R}x_j \otimes \mathcal{R}^2 y_j]p
\end{aligned}$$

where $\rho = P(\widehat{\overline{\mathcal{R}u}})$. It follows that

$$b_{\mathcal{R}u} = 1 \otimes \rho + \sum_{j=1}^{N} \mathcal{R}x_j \otimes y_j(0) + \sum_{j=1}^{N} \mathcal{R}x_j \otimes \mathcal{R}^2 y_j.$$

Note $\rho, \mathcal{R}x_j \in A^2$ for each j. Also, $\mathcal{R}^2 y_j \in A^2$ for each j by Proposition 3.2. Hence $b_{\mathcal{R}u}$ has finite rank on A^2.

Now suppose $b_{\mathcal{R}u}$ has finite rank on A^2. Then there exist linearly independent sets $\{x_j\}_{j=1}^N$ and $\{y_j\}_{j=1}^N$ in A^2 for which

$$b_{\mathcal{R}u} = \sum_{j=1}^{N} x_j \otimes y_j$$

on A^2 and then

$$P[(\mathcal{R}u)\hat{f}] = \sum_{j=1}^{N} \langle f, y_j \rangle_2 x_j$$

for every $f \in A^2$. Note $\mathscr{D} \subset A^2$. Thus, we have by (4)

$$\begin{aligned} \mathcal{R}Q(u\hat{f}) &= P[(\mathcal{R}u)\hat{f}] - P[(\mathcal{R}u)\hat{f}](0) \\ &= \sum_{j=1}^{N} \langle f, y_j \rangle_2 x_j - P[(\mathcal{R}u)\hat{f}](0) \end{aligned} \tag{8}$$

for every $f \in \mathscr{D}$. Taking $z = 0$ in (8), we have

$$\sum_{j=1}^{N} \langle f, y_j \rangle_2 x_j(0) = P[(\mathcal{R}u)\hat{f}](0)$$

and then by (8) again

$$\mathcal{R}Q(u\hat{f}) = \sum_{j=1}^{N} \langle f, y_j \rangle_2 [x_j - x_j(0)]$$

for every $f \in \mathscr{D}$. Taking $\mathcal{R}^{-1}$ to both sides, we obtain from Proposition 3.3(c)

$$\begin{aligned} h_u f = Q[u\hat{f}] &= \sum_{j=1}^{N} \langle f, y_j \rangle_2 \mathcal{R}^{-1}[x_j - x_j(0)] \\ &= \sum_{j=1}^{N} \langle f, (\mathcal{R}^{-1})^2 [y_j - y_j(0)] + y_j(0) \rangle X_j \\ &= \sum_{j=1}^{N} [X_j \otimes Y_j] f, \qquad f \in \mathscr{D} \end{aligned} \tag{9}$$

where

$$X_j = \mathcal{R}^{-1}[x_j - x_j(0)] \qquad \text{and} \qquad Y_j = (\mathcal{R}^{-1})^2 [y_j - y_j(0)] + y_j(0).$$

Note each X_j belongs to $\mathscr{D}$. Also, since $\mathcal{R}Y_j = \mathcal{R}^{-1}[y_j - y_j(0)] \in \mathscr{D} \subset A^2$, we see that each Y_j belongs to $\mathscr{D}$. Hence, (9) shows that h_u has finite rank on $\mathscr{D}$, as desired. The proof is complete. $\square$

The following characterization for finite rank little Hankel operators on the Bergman space is taken from Theorem 4.2 of [2].

Lemma 3.5. *Let $u \in L^\infty$ be a holomorphic function on D^n. Then b_u has finite rank on A^2 if and only if there exist points $a_1, \cdots, a_N \in D^n$ and polynomials $p_1, \cdots, p_N$ for which*

$$u = \sum_{j=1}^{N} p_j \circ \varphi_{a_j} k_{a_j}.$$

We are now ready to prove the our main theorem.

Proof of the main theorem. Let $u = \varphi + \bar{\psi}$ where φ, ψ are holomorphic on D^n. By a simple application of the mean value property, we note from (3)

$$Q(\bar{\psi}\hat{f}) = \int_{D^n} \bar{\psi}(w) f(\bar{w})\, dV(w) = \overline{\psi(0)} f(0) = [1 \otimes \psi(0)] f$$

for every $f \in \mathscr{D}$. Thus $h_{\bar{\psi}} = 1 \otimes \psi(0)$ and $h_{\bar{\psi}}$ has always finite rank on $\mathscr{D}$. Hence h_u has finite rank if and only if h_φ has finite rank on $\mathscr{D}$, which is in turn equivalent to that $b_{\mathcal{R}\varphi}$ has finite rank on A^2 by Proposition 3.4. Since $\mathcal{R}u = \mathcal{R}\varphi$, the result follows from Lemma 3.5. The proof is complete. $\square$

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