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ROUGH STATISTICAL CONVERGENCE OF DIFFERENCE DOUBLE SEQUENCES IN NORMED LINEAR SPACES

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Abstract. In this paper, we introduce rough statistical convergence of difference double sequences in normed linear spaces as an extension of rough convergence. We define the set of rough statistical limit points of a difference double sequence and analyze the results with proofs.

1. Introduction

Fast [1] initiated statistical convergence for a real sequence. Mursaleen and Edely [2] examined the statistical convergence via double sequences. The other studies of this concept can be examined by [3, 4, 5]. In the wake of the study of ideal convergence defined by Kostyrko et al. [6], there has been comprehensive research to discover applications and summability studies of the classical theories. Kostyrko et al. [7] studied the idea of $\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit points. Notable results on this topic can be found in [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24].

The idea of rough convergence was first introduced by Phu [25] in finite-dimensional normed spaces. In another paper [26] related to this subject, Phu defined the rough continuity of linear operators and showed that every linear operator $f : X \to Y$ is r -continuous at every point $x \in X$ under the assumption $dimY < \infty$ and $r > 0$, where X and Y are normed spaces. In [27], Phu extended the results given in [25] to infinite-dimensional normed spaces. Aytar [29] studied the rough statistical convergence. Also, Aytar [30] studied that the rough limit set and the core of a real sequence. Pal et al. [31] generalized the idea of rough convergence into rough statistical convergence and rough ideal convergence. Recently, rough convergence of double sequences has been introduced by Malik and Maity [32] and investigated some basic properties of this type of convergence and also studied relation between rough convergence and Pringsheim convergence for double sequences. In [33] rough statistical convergence of double sequences in finite dimensional normed

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linear spaces was studied and investigated some basic properties of this type of convergence rough statistical convergence of double sequences. Recently, Dündar and Çakan [34] studied rough ideal convergence. Also Dündar and Çakan [35] investigated the rough convergence of double sequence. Then after Dündar [36] studied rough ideal convergence of double sequence. Savaş et al. [37] introduced rough $\mathcal{I}$-statistical convergence as an extension of rough convergence. Rough convergence, rough statistical convergence and $\Delta\mathcal{I}$-convergence for difference sequences and for double difference sequences have been studied. For details, see [38, 39, 40, 41, 42, 43, 44, 45].

The idea of rough statistical convergence has developed a new perspective for non-convergent sequences. Applying this new perspective to difference sequences, which are known with their own properties, will produce very interesting results.

2. Definitions and notations

In this section, we recall some definitions and notations, which form the base for the present study.

During the paper, let r be a nonnegative real number and $\mathbb{R}^n$ denotes the real n-dimensional space with the norm $\|.\|$. Consider a sequence $x = (x_k) \subset X = \mathbb{R}^n$.

The sequence $x = (x_k)$ is said to be r-convergent to x_*, denoted by $x_k \xrightarrow{r} x_*$ provided that

$$\forall \varepsilon > 0 \ \exists i_\varepsilon \in \mathbb{N} : \quad k \geq i_\varepsilon \Rightarrow \|x_k - x_*\| < r + \varepsilon.$$

The set

$$\mathrm{LIM}^r x := \{x_* \in \mathbb{R}^n : x_k \xrightarrow{r} x_*\}$$

is called the r-limit set of the sequence $x = (x_k)$. A sequence $x = (x_i)$ is said to be r-convergent if $\mathrm{LIM}^r x \neq \emptyset$. In this case, r is called the convergence degree of the sequence $x = (x_k)$. For $r = 0$, we get the ordinary convergence. There are several reasons for this interest (see [25]).

A double sequence $x = (x_{mn})_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ of real numbers is said to be bounded if there exists a positive real number M such that $|x_{mn}| < M$, for all $m, n \in \mathbb{N}$. That is

$$\|x\|_\infty = \sup_{m,n} |x_{mn}| < \infty.$$

A double sequence $x = (x_{mn})$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense (shortly, p-convergent to $L \in \mathbb{R}$), if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$, whenever $m, n > N_\varepsilon$. In this case, we write

$$\lim_{m,n\to\infty} x_{mn} = L.$$

We recall that a subset K of $\mathbb{N} \times \mathbb{N}$ is said to have natural density $d(K)$ if

$$d(K) = \lim_{m,n \to \infty} \frac{K(m,n)}{m.n},$$

where $K(m,n) = |\{(j,k) \in \mathbb{N} \times \mathbb{N} : j \leq m, k \leq n\}|$.

Throughout the paper we consider a sequence $x = (x_{mn})$ such that $(x_{mn}) \in \mathbb{R}^n$.

Let $x = (x_{mn})$ be a double sequence in a normed space $(X, \|.\|)$ and r be a non negative real number. x is said to be r-statistically convergent to ξ, denoted by $x \overset{r-st_2}{\longrightarrow} \xi$, if for $\varepsilon > 0$ we have $d(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - \xi\| \geq r + \varepsilon\}$. In this case, ξ is called the r-statistical limit of x.

A double sequence $x = (x_{mn})$ is said to be rough convergent (r-convergent) to x_* with the roughness degree r, denoted by $x_{mn} \overset{r}{\longrightarrow} x_*$ provided that

$$\forall \varepsilon > 0 \;\; \exists k_\varepsilon \in \mathbb{N} : m,n \geq k_\varepsilon \Rightarrow \|x_{mn} - x_*\| < r + \varepsilon,$$

or equivalently, if

$$\limsup \|x_{mn} - x_*\| \leq r.$$

A double sequence (Δx_{kl}) is said to be bounded if there exists a positive real number K such that $\|\Delta x_{kl}\| < K$ for all $(k,l) \in \mathbb{N} \times \mathbb{N}$.

A double sequence (Δx_{kl}) is said to be statistically bounded if there exists a positive real number K

$$d\left(\{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl}\| \geq K\}\right) = 0.$$

A point $c \in X$ is said to be a statistical cluster point of a double sequence (Δx_{kl}) if for any $\varepsilon > 0$, the set

$$d\left(\{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - c\| < \varepsilon\}\right) \neq 0.$$

We use the notation $\Gamma^2 (\Delta x_{kl})$ to denote the set of all statistical cluster points of (Δx_{kl}).

3. Main results

In this section, we define the concept of rough statistical convergence for difference double sequences in $(\mathbb{R}^n, \|.\|)$ space, where $\mathbb{R}^n$ is real n-dimensional normed space and we prove some important theorems.

Definition 3.1. *Let (Δx_{kl}) be a double sequence in a normed linear space $(\mathbb{R}^n, \|.\|)$ and r be a nonnegative real number. Then, (Δx_{kl}) is said to be rough statistically convergent to x_* or r-statistically convergent to x_* if for each $\varepsilon > 0$, the set*

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - x_*\| \geq r + \varepsilon\}$$

has natural density zero or equivalently

$$st_2 - \limsup \|\Delta x_{kl} - x_*\| \leq r.$$

In this case x_* is called the r-st_2-limit point of a difference double sequence (Δx_{kl}) and we denote it by $\Delta x_{kl} \overset{r-st_2}{\longrightarrow} x_*$. Here r is called roughness degree. If we take $r = 0$, we obtain the notion statistical convergence of double sequence.

In general the r-st_2-limit point of a difference double sequence may not be unique for the roughness degree $r > 0$. We define the set of all r-st_2-limit points of a difference double sequence (Δx_{kl}) with

$$st_2\text{-LIM}^r(\Delta x_{kl}) = \left\{ x_* \in \mathbb{R}^n : \Delta x_{kl} \overset{r-st_2}{\longrightarrow} x_* \right\}.$$

The following example gives us an example of a difference double sequence which is not statistically convergent but r-statistically convergent.

Example 3.2. *Let the difference sequence (Δy_{kl}) be a statistically convergent to y_* and can not be measured exactly. Additionally, let (Δx_{kl}) be a sequence that provides the property $\|\Delta x_{kl} - \Delta y_{kl}\| \leq r$ (a.a.k, l). Then, the sets*

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta y_{kl} - x_*\| \geq \varepsilon\}$$

and

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - \Delta y_{kl}\| \geq r\},$$

have natural density zero for any $\varepsilon > 0$. According to these informations we can not say that (Δx_{kl}) is statistically convergent. But we know that

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - y_*\| \geq r + \varepsilon\} \subseteq \{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta y_{kl} - y_*\| \geq \varepsilon\}$$

and this relation gives us that the natural density of the set on the left will be zero. So, the sequence (Δx_{kl}) is r-statistically convergent.

For the set of all r-st_2-limit points of (Δx_{kl}), if $st_2\text{-LIM}^r(\Delta x_{kl}) \neq \emptyset$, then

$$st_2\text{-LIM}^r(\Delta x_{kl}) = [st_2\text{-}\limsup \Delta x_{kl} - r,\ st_2\text{-}\liminf \Delta x_{kl} + r].$$

On the other hand, we know that if (Δx_{kl}) is unbounded, then the set of r-limit points is empty, i.e., $\text{LIM}^r(\Delta x_{kl}) = \emptyset$. Whereas this sequence might be rough statistically convergent. The following example explains this situation.

Example 3.3. *Let*

$$(\Delta x_{kl}) = \begin{cases} kl, & \text{if } k, l \text{ are squares,} \\ (-1)^{k+l}, & \text{otherwise,} \end{cases}$$

So, we get

$$st_2\text{-}LIM^r(\Delta x_{kl}) = \begin{cases} \emptyset, & \text{if } r < 1, \\ [1-r, r-1], & \text{otherwise.} \end{cases}$$

and $LIM^r(\Delta x_{kl}) = \emptyset$ for all $r \geq 0$.

Corollary 3.4. *st_2-LIM^r $(\Delta x_{kl}) \neq \emptyset$ does not imply LIM^r $(\Delta x_{kl}) \neq \emptyset$, but LIM^r $(\Delta x_{kl}) \neq \emptyset$ implies st_2-LIM^r $(\Delta x_{kl}) \neq \emptyset$. Therefore,*

$$LIM^r(\Delta x_{kl}) \subseteq st_2\text{-}LIM^r(\Delta x_{kl})$$

and

$$\operatorname{diam}(LIM^r(\Delta x_{kl})) \subseteq \operatorname{diam}(st_2\text{-}LIM^r(\Delta x_{kl}))\,.$$

Theorem 3.5. *For any difference double sequence (Δx_{kl}),*

$$\operatorname{diam}(st_2\text{-}LIM^r(\Delta x_{kl})) \leq 2r.$$

In general diam(st_2-LIM^r (Δx_{kl})) has no smaller bound.

Proof. Assume that diam(st_2-LIM^r $(\Delta x_{kl})) > 2r$. Then, there exist $y, z \in st_2$-LIM^r (Δx_{kl}) such that

$$d := \|y - z\| > 2r.$$

Now, we select $\varepsilon > 0$ so that $\varepsilon < \frac{d}{2} - r$. Define A_1 and A_2 sets such that

$$A_1 := \{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - y\| \geq r + \varepsilon\}$$

and

$$A_2 := \{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - z\| \geq r + \varepsilon\}\,.$$

Since $y, z \in st_2$-LIM^r (Δx_{kl}), for every $\varepsilon > 0$, we have $d(A_1) = 0$, $d(A_2) = 0$ and from the properties of natural density, $d(A_1^c \cap A_2^c) = 1$. So

$$\begin{aligned} \|y - z\| &\leq \|\Delta x_{kl} - y\| + \|\Delta x_{kl} - z\| \\ &< 2(r+\varepsilon) < 2r + 2\left(\tfrac{d}{2} - r\right) = d = \|y - z\|\,, \end{aligned}$$

for all $(k,l) \in A_1^c \cap A_2^c$. This is a contradiction. Thus,

$$\operatorname{diam}(st_2\text{-LIM}^r(\Delta x_{kl})) \leq 2r.$$

Now, let's show that there is generally no smaller bound. For this, we show that st_2-LIM^r $(\Delta x_{kl}) = \overline{B}_r(x_*)$. We know that diam$\left(\overline{B}_r(x_*)\right) = 2r$ for

$$\overline{B}_r(x_*) := \{y \in X : \|x_* - y\| \leq r\}\,.$$

Choose a difference double sequence (Δx_{kl}) such that st_2-$\lim^r \Delta x_{kl} = x_*$. For each $\varepsilon > 0$ we get

$$d(\{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - x_*\| \geq \varepsilon\}) = 0.$$

Then

$$\|\Delta x_{kl} - y\| \leq \|\Delta x_{kl} - x_*\| + \|x_* - y\| \leq \|\Delta x_{kl} - x_*\| + r,$$

for each $y \in \overline{B}_r(x_*)$. In this case,

$$\|\Delta x_{kl} - y\| < r + \varepsilon,$$

for each $(k,l) \in \{(k,l) \in \|\Delta x_{kl} - x_*\| < \varepsilon\}$. At the same time, we know that

$$d(\{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - x_*\| < \varepsilon\}) = 1$$

and so, $y \in st_2$-$\lim^r \Delta x_{kl}$. Then, we obtain $\overline{B}_r(x_*) = st_2$-LIM^r (Δx_{kl}). □

Theorem 3.6. *For a bounded double sequence (Δx_{kl}), there is a non-negative real number r such that $st_2\text{−}LIM^r(\Delta x_{kl}) \neq \emptyset$.*

The question of whether the converse of the above theorem is also valid is a question that can immediately come to mind. The answer is no. But if the sequence is statistically bounded, the converse is valid. The theorem that gives this case is below.

Theorem 3.7. *A double sequence (Δx_{kl}) is statistically bounded iff there exists a non-negative real number r such that st_2-$LIM^r(\Delta x_{kl}) \neq \emptyset$.*

Proof. First, let's show that st_2-$\mathrm{LIM}^r(\Delta x_{kl}) \neq \emptyset$, when (Δx_{kl}) is statistically bounded. From the definition of statistically boundedness, there exists a positive real number K such that $d(A) = 0$, where

$$A = \{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl}\| \geq K\}.$$

Let

$$r' := \sup\{\|\Delta x_{kl}\| : (k,l) \in A^c\}.$$

Then, st_2-$\mathrm{LIM}^{r'}(\Delta x_{kl})$ contains the origin of $\mathbb{R}^n$ and st_2-$\mathrm{LIM}^{r'}(\Delta x_{kl}) \neq \emptyset$.

Conversely, let st_2-$\mathrm{LIM}^{r'}(\Delta x_{kl}) \neq \emptyset$ for some $r \geq 0$. Let $x_* \in st_2$-$\mathrm{LIM}^{r'}(\Delta x_{kl})$. In that case,

$$d(\{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - x_*\| \geq r + \varepsilon\}) = 0,$$

for each $\varepsilon > 0$. Therefore, we can say that almost all (Δx_{kl}) are contained in some ball with any radius greater than r and (Δx_{kl}) is statistically bounded. □

In rough convergence, we know that when $\left(\Delta x_{k_p l_q}\right)$ is a subset of (Δx_{kl})

$$\mathrm{LIM}^r(\Delta x_{kl}) \subseteq \mathrm{LIM}^r\left(\Delta x_{k_p l_q}\right).$$

In the case of rough statistical convergence, the subsequence must be non-thin to satisfy this condition.

Now let $\{k_p\}_{p\in\mathbb{N}}$ and $\{l_q\}_{q\in\mathbb{N}}$ be two strictly increasing sequences of natural numbers. If (Δx_{kl}) is a double sequence, then we define $\left(\Delta x_{k_p l_q}\right)$ as a subsequence of (Δx_{kl}).

Definition 3.8. *$\left(\Delta x_{k_p l_q}\right)$ is a non-thin subsequence of (Δx_{kl}) provided that the set M does not have natural density zero where $M = \{(k_p, l_q), p, q \in \mathbb{N}\}$, i.e., $d(\{(k_p, l_q), p, q \in \mathbb{N}\}) = 1$.*

If $\left(\Delta x_{k_p l_q}\right)$ is a subsequence of (Δx_{kl}), then $LIM^r(\Delta x_{kl}) \subset LIM^r\left(\Delta x_{k_p l_q}\right)$. But this result is not true for statistical convergence. To show this we consider the following example.

Example 3.9. *Consider the double sequence (Δx_{kl}) in $\mathbb{R}^n$ defined by*

$$(\Delta x_{kl}) = \begin{cases} kl, & \text{if } k,\, l \text{ both are squares} \\ 0, & \text{otherwise,} \end{cases}$$

Then, $\left(\Delta x_{k_pl_q}\right)$ *is a subsequence of* (Δx_{kl}) *for all* $r>0$, st_2*-LIM*$^r(\Delta x_{kl})=[-r,r]$ *but* st_2*-LIM*$^r\left(\Delta x_{k_pl_q}\right)=\emptyset$.

Theorem 3.10. *If* $\left(\Delta x_{k_pl_q}\right)$ *is a non-thin subsequence of* (Δx_{kl})*, then*

$$st_2\text{-}LIM^r(\Delta x_{kl})\subset st_2\text{-}LIM^r\left(\Delta x_{k_pl_q}\right).$$

Proof. Let $x_*\in st_2$-LIM$^r(\Delta x_{kl})$. Then, for any $\varepsilon>0$, $d(A(\varepsilon))=0$, where

$$A(\varepsilon)=\{(k,l)\in\mathbb{N}\times\mathbb{N}:\|\Delta x_{kl}-x_*\|\ge r+\varepsilon\}.$$

Then, $d(A^c(\varepsilon))=1$.

Since $\left(\Delta x_{k_pl_q}\right)$ is a non-thin subsequence of (Δx_{kl}), so $d(K)=1$, where

$$K=\{(k_p,l_q):p,q\in\mathbb{N}\}.$$

Then, $d(A^c(\varepsilon)\cap K)=1$.

Let

$$A'(\varepsilon)=\left\{(p,q)\in\mathbb{N}\times\mathbb{N}:\left\|\Delta x_{k_pl_q}-x_*\right\|\ge r+\varepsilon\right\}.$$

Now

$$\left\{(p,q)\in\mathbb{N}\times\mathbb{N}:\left\|\Delta x_{k_pl_q}-x_*\right\|<r+\varepsilon\right\}\supset A^c(\varepsilon)\cap K.$$

Therefore, $d\left((A'(\varepsilon))^c\right)=1$ and so $d(A'(\varepsilon))=0$, which implies $x_*\in st_2$-LIM$^r\left(\Delta x_{k_pl_q}\right)$. □

Theorem 3.11. *For all* $r\ge 0$*, the r-statistical limit set* st_2*-LIM*$^r(\Delta x_{kl})$ *of a double sequence* (Δx_{kl}) *is closed.*

Proof. For this proof, we use one of the well-known theorems of functional analysis. According to this theorem, "For a convergent sequence $\Delta y_{kl}\to y_*$, when $\Delta y\in st_2$-LIM$^r(\Delta x_{kl})$ (at the same time $y_*\in st_2$-LIM$^r(\Delta x_{kl})$), then st_2-LIM$^r(\Delta x_{kl})$ is closed". If st_2-LIM$^r(\Delta x_{kl})=\emptyset$, then the proof is trivial. Assume that st_2-LIM$^r(\Delta x_{kl})\ne\emptyset$. Then, we have a sequence $(\Delta y_{kl})\subseteq st_2$-LIM$^r(\Delta x_{kl})$ such that $\Delta y_{kl}\to y_*$. From the definition of convergence, for each $\varepsilon>0$, there exists $k_{\frac{\varepsilon}{2}},l_{\frac{\varepsilon}{2}}\in\mathbb{N}$ such that $\|\Delta y_{kl}-y_*\|<\frac{\varepsilon}{2}$ for all $k>k_{\frac{\varepsilon}{2}}$, $l>l_{\frac{\varepsilon}{2}}$. Choose a $(k_0,l_0)\in\mathbb{N}\times\mathbb{N}$ such that $k_0>k_{\frac{\varepsilon}{2}}$, $l_0>l_{\frac{\varepsilon}{2}}$. Then, $\|\Delta y_{k_0l_0}-y_*\|<\frac{\varepsilon}{2}$.

On the other hand, since $(\Delta y_{kl})\subseteq st_2$-LIM$^r(\Delta x_{kl})$, we have $y_{k_0l_0}\in st_2$-LIM$^r(\Delta x_{kl})$, i.e.,

$$d\left(\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\|\Delta x_{kl}-y_{k_0l_0}\|\ge r+\frac{\varepsilon}{2}\right\}\right)=0.$$

Now, we need to show following inclusion

$$\begin{aligned}&\{(k,l)\in\mathbb{N}\times\mathbb{N}:\|\Delta x_{kl}-y_*\|<r+\varepsilon\}\\ &\supseteq\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\|\Delta x_{kl}-y_{k_0l_0}\|<r+\tfrac{\varepsilon}{2}\right\}.\end{aligned}$$

Let

$$(p,r)\in\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\|\Delta x_{kl}-y_{k_0l_0}\|<r+\frac{\varepsilon}{2}\right\}.$$

Then, we have

$$\|\Delta x_{pr}-y_{k_0l_0}\|<r+\frac{\varepsilon}{2}$$

and hence

$$\|\Delta x_{pr} - y_*\| \leq \|\Delta x_{pr} - y_{k_0 l_0}\| + \|y_{k_0 l_0} - y_*\| < r + \varepsilon.$$

It means

$$(p, r) \in \{(k, l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - y_{k_0 l_0}\| < r + \varepsilon\}.$$

Hence, st_2-LIM$^r(\Delta x_{kl})$ of a sequence (Δx_{kl}) is a closed set. □

Theorem 3.12. *The set st_2-LIM$^r(\Delta x_{kl})$ of a double sequence (Δx_{kl}) is convex.*

Proof. Let $y_0, y_1 \in st_2$-LIM$^r(\Delta x_{kl})$. Let $\varepsilon > 0$ be given. Then, $d(A_0(\varepsilon)) = 0$, $d(A_1(\varepsilon)) = 0$, where

$$A_0(\varepsilon) := \{(k, l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - y_0\| \geq r + \varepsilon\},$$

and

$$A_1(\varepsilon) := \{(k, l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - y_1\| \geq r + \varepsilon\}.$$

We know that $d(A_0^c(\varepsilon) \cap A_1^c(\varepsilon)) = 1$ from the assumption.

Let $\lambda \in [0, 1]$ and

$$C(\varepsilon) = \{(k, l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - [(1 - \lambda) y_0 + \lambda y_1]\| \geq r + \varepsilon\}.$$

Now $(k, l) \notin A_0(\varepsilon) \cup A_1(\varepsilon) \Rightarrow \|\Delta x_{kl} - y_0\| < r + \varepsilon$ and $\|\Delta x_{kl} - y_1\| < r + \varepsilon$

$\Rightarrow \|\Delta x_{kl} - [(1 - \lambda) y_0 + \lambda y_1]\| \leq (1 - \lambda)\|\Delta x_{kl} - y_0\| + \lambda \|\Delta x_{kl} - y_1\| < r + \varepsilon$

$\Rightarrow (k, l) \notin C(\varepsilon)$.

Contra-positively, $(k, l) \in C(\varepsilon)$ implies $(k, l) \in A_0(\varepsilon) \cup A_1(\varepsilon)$. Thus, $C(\varepsilon) \subset A_0(\varepsilon) \cup A_1(\varepsilon)$ and $d(C(\varepsilon)) = 0$. Therefore, $[(1 - \lambda) y_0 + \lambda y_1] \in st_2$-LIM$^r(\Delta x_{kl})$. Hence, the set st_2-LIM$^r(\Delta x_{kl})$ is convex. □

Theorem 3.13. *Let $r > 0$. Then, a double sequence (Δx_{kl}) is rough statistically convergent to x_* iff there exists a difference double sequence (Δy_{kl}) such that st_2-LIM$^r(\Delta y_{kl}) = x_*$ and $\|\Delta x_{kl} - \Delta y_{kl}\| \leq r$, for all $(k, l) \in \mathbb{N} \times \mathbb{N}$.*

Proof. Necessity: Let $\Delta x \overset{r-st_2}{\longrightarrow} x_*$. From the definition

$$st_2\text{-}\limsup \|\Delta x_{kl} - x_*\| \leq r.$$

Now we define a double sequence (Δy_{kl}) by

$$\Delta y_{kl} := \begin{cases} x_*, & \text{if } \|\Delta x_{kl} - x_*\| \leq r, \\ \Delta x_{kl} + r\frac{x_* - \Delta x_{kl}}{\|x_* - \Delta x_{kl}\|}, & \text{otherwise.} \end{cases}$$

Then, it is easy to see that

$$\|\Delta y_{kl} - x_*\| = \begin{cases} 0, & \text{if } \|\Delta x_{kl} - x_*\| \leq r, \\ \\ \|\Delta x_{kl} - x_*\| - r, & \text{otherwise,} \end{cases}$$

and $\|\Delta x_{kl} - \Delta y_{kl}\| \leq r$, for all $(k, l) \in \mathbb{N} \times \mathbb{N}$.

Conversely, let there exists a double sequence (Δy_{kl}) such that $st_2\text{-LIM}^r(\Delta y_{kl}) = x_*$ and $\|\Delta x_{kl} - \Delta y_{kl}\| \leq r$, for all $(k,l) \in \mathbb{N} \times \mathbb{N}$. From the definition of statistical convergence, for each $\varepsilon > 0$, we get

$$d\left(\{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta y_{kl} - x_*\| \geq \varepsilon\}\right) = 0.$$

We know that

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta y_{kl} - x_*\| \geq \varepsilon\} \supseteq \{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - x_*\| \geq r + \varepsilon\}.$$

and we have

$$d\left(\{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - x_*\| \geq r + \varepsilon\}\right) = 0.$$

Hence, (Δx_{kl}) is rough statistically convergent to x_*. □

In order to prove the next theorem, we will need the following lemma, which is related to statistical cluster points.

Lemma 3.14. *Let $\Gamma^2_{(\Delta x_{kl})}$ be the set of all statistical cluster points of (Δx_{kl}) and c be an arbitrary element of this set. For all $x_* \in st_2\text{-LIM}^r(\Delta x_{kl})$ we have $\|x_* - c\| \leq r$.*

Proof. Let's accept the contrary of the lemma and find the contradiction. Assume that there exist a point $c \in \Gamma^2_{(\Delta x_{kl})}$ and $x_* \in st_2\text{-LIM}^r(\Delta x_{kl})$ such that $\|x_* - c\| > r$. Define $\varepsilon = \frac{\|x_* - c\| - r}{3}$. In that case,

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - x_*\| \geq r + \varepsilon\} \supseteq \{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - c\| < \varepsilon\}.$$

From the fact that $c \in \Gamma^2_{(\Delta x_{kl})}$, we know that the natural density of the set

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - c\| < \varepsilon\}$$

is not zero. So, by using the inclusion above, we obtain

$$d\left(\{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - x_*\| \geq r + \varepsilon\}\right) \neq 0$$

and this completes the proof. □

Theorem 3.15. *A double sequence (Δx_{kl}) is rough statistically convergent to x_* iff $st_2\text{-LIM}^r(\Delta x_{kl}) = \overline{B}_r(x_*)$.*

Proof. In Theorem 3.5, we proved the necessity part. So, we need to prove if $st_2\text{-LIM}^r(\Delta x_{kl}) = \overline{B}_r(x_*)$, then $\Delta x \overset{r-st_2}{\longrightarrow} x_*$. We know that if the statistical cluster point of a statistically bounded sequence is unique, then the sequence is statistically convergent to this point.

In that case, if $st_2\text{-LIM}^r(\Delta x_{kl}) = \overline{B}_r(x_*) \neq \emptyset$, then (Δx_{kl}) is statistically bounded. Let (Δx_{kl}) sequence has two different statistical cluster points, such as x_* and x'_*. Then, the point

$$\overline{x}_* := x_* + \frac{r}{\|x_* - x'_*\|}(x_* - x'_*),$$

satisfies

$$\|\overline{x}_* - x'_*\| = \left(\frac{r}{\|x_* - x'_*\|} + 1\right)\|x_* - x'_*\| = r + \|x_* - x'_*\| > r.$$

From the previous lemma, $\overline{x}_* \notin st_2\text{-LIM}^r(\Delta x_{kl})$ but this contradicts the fact that $\|\overline{x}_* - x_*\| = r$ and $st_2\text{-LIM}^r(\Delta x_{kl}) = \overline{B}_r(x_*)$. This means that x_* is the unique statistical cluster point of (Δx_{kl}). Therefore, (Δx_{kl}) is statistically convergent to x_*. □

Corollary 3.16. *If $(X, \|.\|)$ is a strictly convex space and (Δx_{kl}) is a double sequence in X. Also if there exist $y_1, y_2 \in st_2\text{–}LIM^r(\Delta x_{kl})$ such that $\|y_1 - y_2\| = 2r$, then, this sequence is rough statistically convergent to $\frac{y_1+y_2}{2}$.*

The proof is straightforward and so is omitted.

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