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DOUBLE INTEGRALS INVOLVING PRODUCT OF TWO GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract. In this paper two interesting double integrals involving product of two generalized hypergeometric functions have been evaluated in terms of gamma function. The results are derived with the help of known integrals involving hypergeometric functions recorded in the paper of Rathie *et al.* [6]. We also give several very interesting special cases.

1. Introduction and results required

The generalized hypergeometric function ${}_pF_q$ with p numerator and q denominator parameters is defined as follows [1, 2, 5, 7]:

$$(1.1)\qquad {}_pF_q\left[\begin{array}{ccc}\alpha_1, & \cdots, & \alpha_p\\ \beta_1, & \cdots, & \beta_q\end{array};x\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{x^n}{n!},$$

where $(\alpha)_n$ is the well-known Pochhammer symbol defined (for $\alpha\in\mathbb{C}$) by

$$(1.2)\qquad (\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\qquad(\alpha\in\mathbb{C}\backslash\mathbb{Z}_0^-)$$
$$=\begin{cases}\alpha(\alpha+1)\cdots(\alpha+n-1), & n\in\mathbb{N}\\ 1, & n=0\end{cases}$$

and $\Gamma(\alpha)$ is the familiar gamma function. Here an empty product is to be interpreted as unity, and we assume that the variable x, the numerator parameters $\alpha_1,\cdots,\alpha_p$ and the denominator parameters $\beta_1,\cdots,\beta_q$ take on complex values, provided that no zeros appear in denominator of (1.1), that is,

$$\beta_j\in\mathbb{C}\backslash\mathbb{Z}_0^-\,;\ j=1,\cdots,q.$$

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Here we let $\mathbb{C}$, $\mathbb{Z}$ and $\mathbb{N}$ be the sets of complex numbers, integers and positive integers, respectively and moreover, let

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \quad \text{and} \quad \mathbb{Z}_0^- := \mathbb{Z} \backslash \mathbb{N}.$$

See [1, 2, 5, 7] for more information on ${}_pF_q$ including its convergence, absolute convergence, various special and limiting cases .

It is interesting to mention here that whenever the generalized hypergeometric function ${}_pF_q$ and its important special case ${}_2F_1$ with some specified argument such as 1 or $\frac{1}{2}$ or -1 can be summed to be expressed in terms of gamma functions, the results may be important from the application point of view. Thus the classical summation theorems such as those of Gauss, Gauss's second, Kummer and Bailey for the series ${}_2F_1$; Watson, Dixon, Whipple and Saalschütz for the series ${}_3F_2$ and others play an important role in the theory and application. Applications of the above mentioned classical summation theorems are well-known now.

For our purpose, we would like to mention the following classical Watson's summation theorem [2] :

$$(1.3) \qquad {}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c \end{matrix} ; 1\right] = \frac{\Gamma(\frac{1}{2})\,\Gamma(c+\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}b+\frac{1}{2})}$$

provided $\mathrm{Re}(2c-a-b) > -1$.

From (1.3), it is not difficult to evaluate the following two double integrals involving hypergeometric function [6] :

$$(1.4) \qquad \int_0^1 \int_0^1 y^c(1-x)^{c-1}(1-y)^{c-1}\,(1-xy)^{1-2c} \times {}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+1) \end{matrix} ; \frac{y(1-x)}{1-xy}\right] dxdy = \frac{\Gamma(\frac{1}{2})\,\Gamma^2(c)\Gamma(c+\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})}{\Gamma(2c)\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}b+\frac{1}{2})} = \Omega,$$

provided $\mathrm{Re}(c) > 0$ and $\mathrm{Re}(2c-a-b) > -1$, and

$$(1.5) \qquad \int_0^1 \int_0^1 y^c(1-x)^{c-1}(1-y)^{c-1}\,(1-xy)^{1-2c} \times {}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+1), \end{matrix} ; \frac{1-y}{1-xy}\right] dxdy = \Omega,$$

provided $\mathrm{Re}(c) > 0$ and $\mathrm{Re}(2c - a - b) > -1$. The value Ω is the same as given in (1.4).

These results have been obtained by the following well-known result of Edwards [3] :

$$\int_0^1 \int_0^1 y^{\alpha}(1-x)^{\alpha-1}(1-y)^{\beta-1}\,(1-xy)^{1-\alpha-\beta}dxdy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{1.6}$$

provided $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$.

In this paper we evaluate two interesting double integrals involving product of two generalized hypergeometric functions in terms of gamma function. The integrals are evaluated with the help of known integrals (1.4) and (1.5). Several very interesting special cases have also been given.

2. Main integral formulas

In this section we evaluate two double integrals involving product of two generalized hypergeometric functions asserted in the following theorem.

Theorem 2.1. *For $Re(c) > 0$ and $Re(2c-a-b) > -1$, the following results hold.*

$$\begin{aligned}
&\int_0^1 \int_0^1 y^{c}(1-x)^{c-1}(1-y)^{c-1}\,(1-xy)^{1-2c} \\
&\qquad \times\ {}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+1) \end{matrix}\,;\frac{y(1-x)}{1-xy}\right] \\
&\qquad \times\ {}_2F_2\left[\begin{matrix} c-\frac{1}{2}a+\frac{1}{2}, \quad c-\frac{1}{2}b+\frac{1}{2} \\ c, \quad c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2} \end{matrix}\,;\frac{4y(1-x)(1-y)}{(1-xy)^2}\right] dxdy \\
&= \frac{e\sqrt{\pi}\,\Gamma^2(c)\Gamma(c+\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})}{\Gamma(2c)\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}b+\frac{1}{2})} \\
&= \Omega_1 = e\,\Omega
\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}
&\int_0^1 \int_0^1 y^{c}(1-x)^{c-1}(1-y)^{c-1}\,(1-xy)^{1-2c} \\
&\qquad \times\ {}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+1) \end{matrix}\,;\frac{1-y}{1-xy}\right] \\
&\qquad \times\ {}_2F_2\left[\begin{matrix} c-\frac{1}{2}a+\frac{1}{2}, \quad c-\frac{1}{2}b+\frac{1}{2} \\ c, \quad c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2} \end{matrix}\,;\frac{4y(1-x)(1-y)}{(1-xy)^2}\right] dxdy \\
&= \Omega_1.
\end{aligned} \tag{2.2}$$

The value of Ω_1 *is the same as given in* (2.1).

Proof. In order to evaluate the integral (2.1), we proceed as follows. Denoting the left-hand side of (2.1) by I, we have

$$I = \int_0^1 \int_0^1 y^c (1-x)^{c-1} (1-y)^{c-1} (1-xy)^{1-2c} \times {}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+1) \end{matrix} ; \frac{y(1-x)}{1-xy}\right] \times {}_2F_2\left[\begin{matrix} c-\frac{1}{2}a+\frac{1}{2}, \quad c-\frac{1}{2}b+\frac{1}{2} \\ c, \quad c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2} \end{matrix} ; \frac{4y(1-x)(1-y)}{(1-xy)^2}\right] dxdy.$$

Express ${}_2F_2$ as a series, interchanging the order of integration and summation, which is easily seen to be justified due to the uniform convergence of the series involved in the process, we have

$$I = \sum_{n=0}^{\infty} \frac{(c-\frac{1}{2}a+\frac{1}{2})_n \, (c-\frac{1}{2}b+\frac{1}{2})_n \, 2^{2n}}{(c)_n \, (c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})_n \, n!} \times \int_0^1 \int_0^1 y^{c+n} (1-x)^{c+n-1} (1-y)^{c+n-1} (1-xy)^{1-2c-2n} \times {}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+1) \end{matrix} ; \frac{y(1-x)}{1-xy}\right] dxdy.$$

Evaluating the double integral with the help of the result (1.4) and simplifying it, we have

$$I = \frac{\Gamma(\frac{1}{2})\Gamma^2(c)\Gamma(c+\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})}{\Gamma(2c)\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}b+\frac{1}{2})} \times \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Finally, observing that $\sum_{n=0}^{\infty} \frac{1}{n!} = e$, we easily arrive at the right-hand side of the result (2.1). This completes the proof of the result (2.1) asserted in the theorem.

In the same manner, we can establish the result (2.2) with the help of the known result (1.5).

□

3. Special cases

In this section we mention a few very interesting special cases of our main integrals (2.1) and (2.2) in the following forms.

(a) In (2.1), if we let $b = -2n$ and replace a by $a + 2n$, where $n \in \mathbb{N}_0$, then we get the following result.

$$\begin{aligned}
(3.1) \qquad & \int_0^1 \int_0^1 y^c(1-x)^{c-1}(1-y)^{c-1}\,(1-xy)^{1-2c} \\
& \quad \times {}_2F_1\left[\begin{matrix} -2n,\ a+2n \\ \frac{1}{2}(a+1) \end{matrix};\frac{y(1-x)}{1-xy}\right] \\
& \quad \times {}_2F_2\left[\begin{matrix} c+n+\frac{1}{2},\ c-\frac{1}{2}a+\frac{1}{2}-n \\ c,\quad c-\frac{1}{2}a+\frac{1}{2} \end{matrix};\frac{4y(1-x)(1-y)}{(1-xy)^2}\right] dxdy \\
& = \frac{e\,\Gamma^2(c)}{\Gamma(2c)}\,\frac{(\frac{1}{2})_n\,(\frac{1}{2}+\frac{1}{2}a-c)_n}{(c+\frac{1}{2})_n\,(\frac{1}{2}a+\frac{1}{2})_n} \\
& = \Omega_2.
\end{aligned}$$

(b) In (2.1), if we let $b=-2n-1$ and replace a by $a+2n+1$, where $n\in\mathbb{N}_0$, then we get the following interesting result.

$$\begin{aligned}
(3.2) \qquad & \int_0^1 \int_0^1 y^c(1-x)^{c-1}(1-y)^{c-1}\,(1-xy)^{1-2c} \\
& \quad \times\ {}_2F_1\left[\begin{matrix} -2n-1,\ a+2n+1 \\ \frac{1}{2}(a+1) \end{matrix};\frac{y(1-x)}{1-xy}\right] \\
& \quad \times {}_2F_2\left[\begin{matrix} c+n+1,\ c-\frac{1}{2}a+\frac{1}{2}-n \\ c,\quad c-\frac{1}{2}a+\frac{1}{2} \end{matrix};\frac{4y(1-x)(1-y)}{(1-xy)^2}\right] dxdy \\
& = 0.
\end{aligned}$$

(c) In (2.1), if we take $a=b=\frac{1}{2}$ and making use of the known result [4, p. 473, Equ.(75)]

$$(3.3) \qquad {}_2F_1\left[\begin{matrix} \frac{1}{2},\ \frac{1}{2} \\ 1 \end{matrix};x\right] = \frac{2}{\pi}K(\sqrt{x}),$$

where $K(k)$ is the complete elliptic function of the first kind defined by

$$(3.4) \qquad K(k) = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1-k^2 sin^2 t}},$$

then we get the following interesting result.

$$\int_0^1\int_0^1 y^c(1-x)^{c-1}(1-y)^{c-1}\,(1-xy)^{1-2c}K\left(\sqrt{\frac{y(1-x)}{1-xy}}\right) \tag{3.5}$$
$$\times\, {}_2F_2\left[\begin{matrix} c+\frac{1}{4},\ c+\frac{1}{4} \\ c,\quad c \end{matrix};\frac{4y(1-x)(1-y)}{(1-xy)^2}\right]dxdy$$
$$=\frac{e\,\pi^{3/2}}{2}\,\frac{\Gamma^3(c)\Gamma(c+\frac{1}{2})}{\Gamma(2c)\,\Gamma^2(\frac{3}{4})\,\Gamma^2(c+\frac{1}{4})}$$
$$=\Omega_3,$$

provided $\mathrm{Re}(c)>0$.

(d) In (2.1), if we take $a=b=1$ and making use of the known result [4, p. 476, Equ.(147)]

$$ {}_2F_1\left[\begin{matrix}1,\ 1\\ \frac{3}{2}\end{matrix};x\right]=\frac{\sin^{-1}(\sqrt{x})}{\sqrt{x(1-x)}}, \tag{3.6}$$

then we get the following interesting result.

$$\int_0^1\int_0^1 y^{c-\frac{1}{2}}(1-x)^{c-\frac{3}{2}}(1-y)^{c-\frac{3}{2}}\,(1-xy)^{-2c}\sin^{-1}\left(\sqrt{\frac{y(1-x)}{1-xy}}\right) \tag{3.7}$$
$$\times\, {}_1F_1\left[\begin{matrix} c \\ c-\frac{1}{2} \end{matrix};\frac{4y(1-x)(1-y)}{(1-xy)^2}\right]dxdy$$
$$=\frac{e\,\pi}{2}\,\frac{\Gamma(c-\frac{1}{2})\Gamma(c+\frac{1}{2})}{\Gamma(2c)}$$
$$=\Omega_4,$$

provided $\mathrm{Re}(c)>\frac{1}{2}$.

(e) In (2.1), if we take $b=-a$ and making use of the known result [4, p. 459, Equ.(83)]

$$ {}_2F_1\left[\begin{matrix}a,\ -a\\ \frac{1}{2}\end{matrix};x\right]=\cos(2a\sin^{-1}(\sqrt{x})), \tag{3.8}$$

then we get the following interesting result.

$$\begin{aligned}(3.9)\quad &\int_0^1\int_0^1 y^c(1-x)^{c-1}(1-y)^{c-1}\,(1-xy)^{1-2c}\cos\left(2a\sin^{-1}\sqrt{\frac{y(1-x)}{1-xy}}\right)\\ &\quad\times {}_2F_2\left[\begin{matrix} c-\frac{1}{2}a+\frac{1}{2},\ c+\frac{1}{2}a+\frac{1}{2}\\ c,\quad c+\frac{1}{2}\end{matrix};\frac{4y(1-x)(1-y)}{(1-xy)^2}\right]dxdy\\ &=\frac{e\,\pi\,\Gamma^2(c)\Gamma^2(c+\frac{1}{2})}{\Gamma(2c)\Gamma(\frac{1}{2}-\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c+\frac{1}{2}a+\frac{1}{2})}\\ &=\Omega_5.\end{aligned}$$

(f) In (2.2), if we let $b=-2n$ and replace a by $a+2n$, where $n\in\mathbb{N}_0$, then we get the following result.

$$\begin{aligned}&\int_0^1\int_0^1 y^c(1-x)^{c-1}(1-y)^{c-1}\,(1-xy)^{1-2c}\\ (3.10)\quad &\quad\times {}_2F_1\left[\begin{matrix}-2n,\ a+2n\\ \frac{1}{2}(a+1)\end{matrix};\frac{1-y}{1-xy}\right]\\ &\quad\times {}_2F_2\left[\begin{matrix} c+n+\frac{1}{2},\ c-\frac{1}{2}a+\frac{1}{2}-n\\ c,\quad c-\frac{1}{2}a+\frac{1}{2}\end{matrix};\frac{4y(1-x)(1-y)}{(1-xy)^2}\right]dxdy\\ &=\Omega_2,\end{aligned}$$

where Ω_2 is the same as defined in (3.1).

(g) In (2.2), if we let $b=-2n-1$ and replace a by $a+2n+1$, where $n\in\mathbb{N}_0$, then we get the following interesting result.

$$\begin{aligned}&\int_0^1\int_0^1 y^c(1-x)^{c-1}(1-y)^{c-1}\,(1-xy)^{1-2c}\\ (3.11)\quad &\quad\times {}_2F_1\left[\begin{matrix}-2n-1,\ a+2n+1\\ \frac{1}{2}(a+1)\end{matrix};\frac{1-y}{1-xy}\right]\\ &\quad\times {}_2F_2\left[\begin{matrix} c+n+1,\ c-\frac{1}{2}a+\frac{1}{2}-n\\ c,\quad c-\frac{1}{2}a+\frac{1}{2}\end{matrix};\frac{4y(1-x)(1-y)}{(1-xy)^2}\right]dxdy\\ &=0.\end{aligned}$$

(h) In (2.2), if we take $a=b=\frac{1}{2}$ and making use of the known result (3.3), then we get the following interesting result.

$$\begin{aligned}(3.12)\quad &\int_0^1\int_0^1 y^c(1-x)^{c-1}(1-y)^{c-1}\,(1-xy)^{1-2c}K\left(\sqrt{\frac{1-y}{1-xy}}\right)\\ &\quad\times {}_2F_2\left[\begin{matrix} c+\frac{1}{4},\ c+\frac{1}{4}\\ c,\quad c\end{matrix};\frac{4y(1-x)(1-y)}{(1-xy)^2}\right]dxdy\\ &=\Omega_3,\end{aligned}$$

where Ω_3 is the same as defined in (3.5).

(i) In (2.2), if we take $a = b = 1$ and making use of the known result (3.6), then we get the following interesting result.

$$\begin{aligned}(3.13)\qquad &\int_0^1\int_0^1 y^{c-\frac{1}{2}}(1-x)^{c-\frac{3}{2}}(1-y)^{c-\frac{3}{2}}\,(1-xy)^{-2c}\sin^{-1}\left(\sqrt{\frac{1-y}{1-xy}}\right)\\ &\qquad\times {}_1F_1\left[\begin{matrix} c \\ c-\frac{1}{2}\end{matrix}\,;\frac{4y(1-x)(1-y)}{(1-xy)^2}\right]dxdy\\ &= \Omega_4,\end{aligned}$$

where Ω_4 is the same as defined in (3.7).

(j) In (2.2), if we take $b = -a$ and making use of the known result (3.8) then we get the following interesting result.

$$\begin{aligned}(3.14)\qquad &\int_0^1\int_0^1 y^{c}(1-x)^{c-1}(1-y)^{c-1}\,(1-xy)^{1-2c}\cos\left(2a\sin^{-1}\sqrt{\frac{1-y}{1-xy}}\right)\\ &\qquad\times {}_2F_2\left[\begin{matrix} c-\frac{1}{2}a+\frac{1}{2},\ c+\frac{1}{2}a+\frac{1}{2} \\ c,\quad c+\frac{1}{2}\end{matrix}\,;\frac{4y(1-x)(1-y)}{(1-xy)^2}\right]dxdy\\ &= \Omega_5,\end{aligned}$$

where Ω_5 is the same as defined in (3.9).

Similarly, other results can be obtained.

4. Concluding Remark

In this paper we have evaluated two interesting double integrals involving product of two generalized hypergeometric functions in terms of gamma function. It should be remarked here that whenever an integral is evaluated in terms of gamma function, the result may be useful in application point of view. Thus our results established in this paper may be potentially useful.

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