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# SOFT CONGRUENCE RELATIONS OVER SEMIRINGS

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**Abstract.** In this paper, we generalize the notion of soft congruence relations from rings to semirings. We construct some examples in order to show that these relations exist over semirings. Some properties of these relations are investigated.

## 1. Introduction

In order to solve different problems in different branches of science certain methods in classical mathematics were not always working because of the fact that different uncertainties occured for the mentioned problems. In order to avoid those uncertainties different people worked on different theories [2], [7], [12], [14], [17], [25], [26], [27], [31], [32]. However, all of the mentioned theories have their own difficulties and to deal with these difficulties, Molodtsov [24] initiated the notion of soft sets. In nowadays different people work on soft set theory and it progresses quickly. Maji et al. [21] deeply mentioned the applications of soft set theory to different decision making problems. Chen et al. [9] defined soft set parametrization reduction in a new way and then compared it to the related notion of attributes reduction in rough set theory. Maji et al. [22] defined and discussed some operations on soft sets while Ali et al. [4] defined restricted union, restricted intersection, extended intersection, and restricted difference of soft sets and then discussed some of their properties. In [28], the authors discussed the fundamental properties of intersection, restricted union, restricted difference, and restricted union on soft sets, while in [3], the authors made a comparison among soft sets, fuzzy sets and rough sets. The authors of [3] then defined the notion of soft groups and explored some elegant properties. Furthermore, Jun [16] initiated the notion of soft BCK-algebra as well the notion of BCI-algebra and explored some useful properties. We also have seen that Feng et al. [11] have already discussed the structure of a soft semiring, while in [18] the authors introduced the notion of $(M,\ N)$-soft intersection near semirings as well the notion of $(M,\ N)$-$\alpha$-inclusion and explored them

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by different properties. In [1], Acar et al. defined soft ring and investigated some properties. Liu et al. [19], [20] have discussed the well-known three isomorphism theorems and fuzzy isomorphism theorems of soft rings. Zhan and Jun [33] discussed BL-algebras based on fuzzy sets. In the same years authors of [23] introduced the concept of soft mappings. Furthermore, the author of [5] generalized the notion of binary relations and defined soft binary relations as well as soft equivalence relations while Ali et al. [6] discussed algebraic structures of soft sets associated with some new operations. In the paper [29], the authors extended soft binary relations from sets to soft congruence relations over rings.

## 2. Preliminaries

In this section, we recall several concepts and definitions that will be used throughout the paper. For these concepts and definitions, we refer the reader to [6], [8], [11], [13], [24], [30]. We start with the definitions given below which are taken from [8], [13].

**Definition 2.1.** *Let $\boldsymbol{L} \neq \emptyset$ be a set equipped with the binary operations of additions "+" and multiplication "$\cdot$". If*
*(i) $(\boldsymbol{L}, +)$ is a commutative semigroup,*
*(ii) $(\boldsymbol{L}, \cdot)$ is a semigroup,*
*(iii) $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$, $\forall\ x, y, z \in \boldsymbol{L}$, hold, then $\boldsymbol{L}$ is said to be a semiring.*

For theoretical understanding of semirings, we give some examples. These examples have been taken from the source [13].

**Example 2.2.** *(i) All rings are semirings.*
*(ii) If $\boldsymbol{L} = [0, 1]$, then $\boldsymbol{L}$ is a semiring by taking "+" = max and "$\cdot$" = min or "+" = min and "$\cdot$" = max or by taking "+" = max and "$\cdot$" = usual multiplication of real numbers.*
*(iii) Let $\boldsymbol{W}$ be the set of whole numbers, then $\boldsymbol{W}$ is a semiring by taking "+" = ordinary addition and "$\cdot$" = ordinary multiplication of numbers.*
*(iv) Let $\boldsymbol{R}^+$ be the set of non-negative real numbers, then $\boldsymbol{R}^+$ is a semiring by taking "+" = usual addition and "$\cdot$" = usual multiplication of numbers.*

Moreover, we note that:

If a semiring $(\boldsymbol{L}, +, \cdot)$ has a multiplicative identity 1, then the semiring $(\boldsymbol{L}, +, \cdot)$ is called a semiring with identity. The semiring $(\boldsymbol{L}, +, \cdot)$ is said to be commutative if "$\cdot$" is commutative in $\boldsymbol{L}$.

Further, we discuss substructures of semirings, that is, subsemirings as well as ideals. We start with the following definition of a subsemiring which is taken from [13].

**Definition 2.3.** *A subset $\boldsymbol{H} \neq \emptyset$ of a semiring $(\boldsymbol{L}, +, \cdot)$ is said to be a subsemiring of $(\boldsymbol{L}, +, \cdot)$ if the subset $\boldsymbol{H}$ is itself a semiring under the operations of $\boldsymbol{L}$.*

It should be noted that every semiring $(\boldsymbol{L}, +, \cdot)$ has at least two subsemirings, i.e. $\{0\}$ and $\boldsymbol{L}$ itself called the trivial subsemirings; all other subsemirings are called non-trivial subsemirings.

The following result gives us equivalent conditions for subsemirings. The following result is taken from [13].

**Theorem 2.4.** *A subset $\boldsymbol{I} \neq \emptyset$ of a semiring $(\boldsymbol{L}, +, \cdot)$ is said to be a subsemiring if and only if*
*(i) $a + b \in \boldsymbol{I}$ for all $a, b \in \boldsymbol{I}$,*
*(ii) $a \cdot b \in \boldsymbol{I}$ for all $a, b \in \boldsymbol{I}$,*
*(iii) $0 \in \boldsymbol{I}$.*

Let us define left (right) ideals in a semiring.

**Definition 2.5.** *Let $\boldsymbol{H} \neq \emptyset$ be a subset of a semiring $(\boldsymbol{L}, +, \cdot)$, then $\boldsymbol{H}$ is called a left (right) ideal of $\boldsymbol{L}$ if the following conditions are satisfied.*
*(i) $a + b \in \boldsymbol{H}$ for all $a, b \in \boldsymbol{H}$.*
*(ii) $x \cdot a \in \boldsymbol{H}$ ($a \cdot x \in \boldsymbol{H}$) for all $a \in \boldsymbol{H}$ and $x \in \boldsymbol{L}$.*
*$\boldsymbol{H}$ is called two sided ideal or simply an ideal of $\boldsymbol{L}$ if it is both left as well as right ideal of $\boldsymbol{L}$. Note that if f $(\boldsymbol{L}, +, \cdot)$ is commutative, then each left or right ideal is a two-sided ideal of $(\boldsymbol{L}, +, \cdot)$.*

In the following, we define right compatible, left compatible and compatible relations over semirings.

**Definition 2.6.** *Let $(\boldsymbol{S}, +, \cdot)$ be a semiring. A relation $\rho$ on $\boldsymbol{S}$ is said to be left compatible if $\forall\ a, b, x \in \boldsymbol{S} \ni (a, b) \in \rho$ implies that $(x + a, x + b)$ and $(x \cdot a, x \cdot b) \in \rho$. The relation $\rho$ is said to be a right compatible if $\forall\ a, b, y \in \boldsymbol{S}$ such that $(a, b) \in \rho$ implies that $(a + y, b + y)$ and $(a \cdot y, a \cdot y) \in \rho$. It is called compatible if $\forall\ a, b, m, n \in \boldsymbol{S}$ such that $(a, b)$ and $(m, n) \in \rho$ implies that$(a + m, b + n)$ and $(a \cdot m, b \cdot n) \in \rho$.*
*Moreover, we note that:*
*A left (right) compatible equivalence relation is called a left (right) congruence relation. A compatible equivalence relation is called a congruence relation.*

Let us present some properties. The following result provides us equivalent conditions for congruence relations over semirings. The result is true in case of semigroups and here we prove it for semirings. The idea of this result has come from the book [15].

**Proposition 2.7.** *A relation $\rho$ on a semiring $\boldsymbol{S}$ is a congruence relation if and only if it is both right and left congruence relation.*

*Proof.* Let $\rho$ be a congruence relation over $\boldsymbol{S}$. If $a,\ b,\ x \in \boldsymbol{S} \ni (a,\ b) \in \rho$ then $(x + a,\ x + b)$ and $(x \cdot a,\ x \cdot b) \in \rho$ since $(x,\ x) \in \rho$. This shows that $\rho$ is a left congruence relation. Similarly we can show that $\rho$ is a right congruence relation.

Conversely, let us suppose that $\rho$ is both right as well as left congruence relation. If $s,\ t,\ m,\ n \in \boldsymbol{S}$ such that $(a,\ b) \in \rho$ as well as $(m,\ n) \in \rho$. This implies that $(a + m,\ b + m)$ and $(a \cdot m,\ b \cdot m) \in \rho$ since $\rho$ is a right compatible relation and $(b + m,\ b + n) \in \rho$ and $(b \cdot m,\ b \cdot n) \in \rho$ since $\rho$ is a left compatible relation. This implies that $(a + m,\ b + n)$, $(a \cdot m,\ b \cdot n) \in \rho$, since $\rho$ is transitive. So it follows that $\rho$ is a congruence relation. □

Further, we have the following result which will be used later. This result is true in case of rings and here we prove it for semirings. The idea of this result has come from [29].

**Lemma 2.8.** *Let us suppose that $\eta$ is a congruence over a semiring $\boldsymbol{S}$. Then $\boldsymbol{I} = 0\eta$ is an ideal of $\boldsymbol{S}$. Conversely, let us suppose that $\boldsymbol{I}$ is an ideal of $\boldsymbol{S}$ and define a relation $\eta$ on $\boldsymbol{S}$ by the rule that $(x,\ y) \in \eta \iff x + i = y + j$ for some $i,\ j \in \boldsymbol{I}$. Then $\eta$ is congruence relation.*

*Proof.* Let us suppose $\eta$ is a congruence relation over $\boldsymbol{S}$. We show that $\boldsymbol{I} = 0\eta$ is an ideal of $\boldsymbol{S}$. Hence $\boldsymbol{I} = 0\eta = \{y \in \boldsymbol{S}\colon (0,\ y) \in \eta\}$. As $(0,\ 0) \in \eta \Rightarrow 0 \in \boldsymbol{I} = 0\eta \Rightarrow \boldsymbol{I} = 0\eta$ is non-empty. Now assume that $y_1,\ y_2 \in \boldsymbol{I} = 0\eta \Rightarrow (0,\ y_1)$, $(0,\ y_2) \in \eta \Rightarrow (0 + 0,\ y_1 + y_2) \in \eta \Rightarrow\ y_1 + y_2 \in \boldsymbol{I} = 0\eta$.

Now let $r \in \boldsymbol{S}$ and $y \in \boldsymbol{I} = 0\eta \Rightarrow r \in \boldsymbol{S}$ and $(0,\ y) \in \eta \Rightarrow (r0,\ ry)$ and $(0r,\ yr) \in \eta \Rightarrow (0,\ ry),\ (0,\ yr) \in \eta \Rightarrow ry,\ yr \in \boldsymbol{I} = 0\eta$. It follows that $\boldsymbol{I} = 0\eta$ is an ideal.

Conversely, let us suppose that $\boldsymbol{I}$ is an ideal of $\boldsymbol{S}$ and let us define $(x,\ y) \in \eta \iff x + i = y + j$ for some $i,\ j \in \boldsymbol{I}$. We show that $\eta$ is a congruence relation. As $x + i = x + i,\ \forall\ x \in \boldsymbol{S}$, so it follows that $(x,\ x) \in \eta\ \forall\ x \in \boldsymbol{S}$, thus $\eta$ is reflexive. Now let $(x,\ y) \in \eta \Rightarrow x + i = y + j$ for some $i,\ j \in \boldsymbol{I}$ then we have $y + j = x + i$ for some $i,\ j \in \boldsymbol{I} \Rightarrow (y,\ x) \in \eta$. It follows that $\eta$ is symmetric.

Now let $(x,\ y),\ (y,\ z) \in \eta \Rightarrow x + i = y + j$ for some $i,\ j \in \boldsymbol{I}$ and $y + k = z + l$ for some $k,\ l \in \boldsymbol{I}$. Now $x + i + k = y + j + k = y + k + j = z + l + j \Rightarrow x + i' = z + j'$ where $i' = i + k \in \boldsymbol{I}$ and $j' = l + j \in \boldsymbol{I} \Rightarrow (x,\ z) \in \eta$. Thus $\eta$ is transitive. It follows that $\eta$ is an equivalence relation.

*Compatibility with respect to addition*: Let $(x,\ y),\ (u,\ v) \in \eta \Rightarrow x + i = y + j$ and $u + k = v + l$ for some $i,\ j,\ k,\ l \in \boldsymbol{I}$.

Now we have

$x + i + u + k = y + j + v + l \Rightarrow x + u + i + k = y + v + j + l \Rightarrow x + u + i' = y + v + j'$, where $i' = i + k \in \boldsymbol{I}$ and $j' = j + l \in \boldsymbol{I}$. Thus it follows that $(x + u,\ y + v) \in \eta$.

*Left Compatibility with respect to multiplication*: Let $(x, y) \in \eta$ and $r \in \boldsymbol{S}$ $\Rightarrow x + i = y + j \Rightarrow r(x + i) = r(y + j) \Rightarrow rx + ri = ry + rj \Rightarrow rx + i' = ry + j'$ where $i' = ri \in \boldsymbol{I}$ and $j' = rj \in \boldsymbol{I} \Rightarrow (rx, ry) \in \eta$.

*Right Compatibility with respect to multiplication:* Let $(x, y) \in \eta$ and $r \in \boldsymbol{S}$ $\Rightarrow x + i = y + j \Rightarrow r(x + i) = r(y + j) \Rightarrow rx + ri = ry + rj \Rightarrow rx + i' = ry + j'$ where $i' = ri \in \boldsymbol{I}$ and $j' = rj \in \boldsymbol{I} \Rightarrow (rx, ry) \in \eta$.

It follows that $\eta$ is compatible with respect to multiplication. Thus $\eta$ is a congruence relation □

We are now going to define homomorphisms of semirings which are those maps which preserve binary operations. We start with the given definition which has been taken from [13].

**Definition 2.9.** *Let us suppose $(\boldsymbol{S}, +, \cdot)$ and $(\boldsymbol{T}, \star, \diamond)$ are two semirings. A mapping $f : \boldsymbol{S} \to \boldsymbol{T}$ is called a homomorphism if the following conditions are satisfied:*

*(i) $f(x + y) = f(x) \star f(y)\ \forall\ x, y \in \boldsymbol{S}$;*

*(ii) $f(x \cdot y) = f(x) \diamond f(y)\ \forall\ x, y \in \boldsymbol{S}$;*

*(iii) $f(0) = 0$.*

Note that monomorphism, epimorphism, isomorphism, endomorphism and automorphism can be defined in the usual way. To understand the above concept, we give an example.

**Example 2.10.** *Let us suppose $\boldsymbol{W}$ is the set of whole numbers. Then $(\boldsymbol{W}, +, \cdot)$ and $(2\boldsymbol{W}, +, \cdot)$ are semirings. Now define $f : \boldsymbol{W} \to 2\boldsymbol{W}$ by the rule $f(n) = 2n,\ \forall\ n \in \boldsymbol{W}$. Then it can be easily verified that $f$ is a homomorphism.*

Let us state and prove some properties. The following result is true in case of rings. Here we prove it for semirings and the idea of this result has come from the paper [29]

**Proposition 2.11.** *Let us assume that $f : \boldsymbol{S} \to \boldsymbol{S}'$ is an epimorphism and let $\eta$ be a congruence relation over $\boldsymbol{S}$. Let*

*$f(\eta) = \{(f(r), f(s)) \in \boldsymbol{S}' \times \boldsymbol{S}' : (r, s) \in \eta\}$; then $f(\eta)$is a congruence relation over $\boldsymbol{S}'$.*

*Proof.* As $(r, r) \in \eta\ \forall\ r \in \boldsymbol{S}$, so it follows that $(f(r), f(s)) \in f(\eta))\ \forall\ f(r) \in Imf = \boldsymbol{S}'$. Thus $f(\eta)$ is reflexive. Now let $r', s' \in \boldsymbol{S}'$ such that $(r', s') \in f(\eta)$. As $f$ is onto so $\exists\ r, s \in \boldsymbol{S}$ such that $r' = f(r)$ and $s' = f(s)$. Thus $(f(r), f(s)) \in f(\eta)$ and this implies $(r, s) \in \eta$ and this implies $(s, r) \in \eta$, as $\eta$ is symmetric. It follows that $(f(s), f(r)) \in f(\eta)$. Thus $f(\eta)$ is symmetric.

Now $r', s', t' \in \boldsymbol{S}'$ such that $(r', s'), (s', t') \in f(\eta)$. As $f$ is onto so $\exists\ r, s, t \in \boldsymbol{S} \ni r' = f(r), s' = f(s)$ and $t' = f(t)$. Thus $(f(r), f(s)), (f(s), f(t)) \in f(\eta)$. It follows that $(r, s), (s, t) \in \eta$ and this implies $(r, t) \in \eta$, as $\eta$ a transitive. Thus, $(f(r), f(s)) \in f(\eta)$. It follows that $f(\eta)$ is an equivalence relation.

Now let $(r', s'), (u', v') \in f(\eta)$ for some $r', s', u', v' \in \boldsymbol{S}'$. As $f$ is onto so $\exists$ $r, s, u, v \in \boldsymbol{S}$ such that $r' = f(r), s' = f(s), u' = f(u)$ and $v' = f(v)$. Then we have $(f(r), f(s)), (f(u), f(v)) \in f(\eta)$. This implies that $(r, s), (u, v) \in \eta$ and this implies $(r + u, s + v), (r \cdot u, s \cdot v) \in \eta$ as $\eta$ is compatible. Thus it follows that

$(f(r + u), f(s + v)), (f(r \cdot u), f(s \cdot v)) \in f(\eta)$
$\Rightarrow (f(r) \star f(u)), f(s) \star f(v)), (f(r) \diamond f(u)), f(s) \diamond f(v)) \in f(\eta)$
$\Rightarrow (r' \star u', s' \star v'), (r' \diamond u', s' \diamond v') \in f(\eta)$.

It follows that $f(\eta)$ is compatible and so $f(\eta)$ is a congruence relation. □

Further, we have the following definitions which are taken from [[6], [11], [24], [30]]. From here let us assume that $\boldsymbol{U}$ is an initial universe and further suppose that $\boldsymbol{E}$ is a set of parameters. Moreover let $\boldsymbol{P}(\boldsymbol{U})$ represents the power set of $\boldsymbol{U}$ and let $\boldsymbol{L} \neq \emptyset$ and $\boldsymbol{M} \neq \emptyset$ be subsets of $\boldsymbol{E}$.

**Definition 2.12.** *Let $\boldsymbol{A}: \boldsymbol{L} \to \boldsymbol{P}(\boldsymbol{U})$ be a mapping, then the pair $(\boldsymbol{A}, \boldsymbol{L})$ is called a soft set over $\boldsymbol{U}$.*

**Definition 2.13.** *Let $(\boldsymbol{A}, \boldsymbol{L})$ and $(\boldsymbol{B}, \boldsymbol{M})$ be two soft sets over a common universe $\boldsymbol{U}$, then we say that $(\boldsymbol{A}, \boldsymbol{L})$ is a soft subset of $(\boldsymbol{B}, \boldsymbol{M})$ if it satisfies the following:*

*(i) $\boldsymbol{L} \subseteq \boldsymbol{M}$,*
*(ii) $\boldsymbol{A}(\alpha) \subseteq \boldsymbol{B}(\alpha)\ \forall\ \alpha \in \boldsymbol{L}$.*

*Symbolically, we write $(\boldsymbol{A}, \boldsymbol{L}) \tilde{\subset} (\boldsymbol{B}, \boldsymbol{M})$. In this case, $(\boldsymbol{B}, \boldsymbol{M})$ is called to be a soft super set of $(\boldsymbol{A}, \boldsymbol{L})$.*

**Definition 2.14.** *Let us assume that $(\boldsymbol{A}, \boldsymbol{L})$ and $(\boldsymbol{B}, \boldsymbol{M})$ are two soft sets over a common universe $\boldsymbol{U}$, then the* ***AND****-operation of $(\boldsymbol{A}, \boldsymbol{L})$ and $(\boldsymbol{B}, \boldsymbol{M})$ is the soft set $(\boldsymbol{C}, \boldsymbol{N})$ where $\boldsymbol{N} = \boldsymbol{L} \times \boldsymbol{M}$ and for all $(\alpha, \beta) \in \boldsymbol{N}$ we have $\boldsymbol{C}(\alpha, \beta) = \boldsymbol{A}(\alpha) \cap \boldsymbol{B}(\beta)$. Symbolically, $(\boldsymbol{A}, \boldsymbol{L}) \tilde{\wedge} (\boldsymbol{B}, \boldsymbol{M}) = (\boldsymbol{C}, \boldsymbol{N})$.*

**Definition 2.15.** *Let $(\boldsymbol{A}, \boldsymbol{L})$ and $(\boldsymbol{B}, \boldsymbol{M})$ be two soft sets over a common universe $\boldsymbol{U}$, then the* ***OR****-operation of $(\boldsymbol{A}, \boldsymbol{L})$ and $(\boldsymbol{B}, \boldsymbol{M})$ is define to be the soft set $(\boldsymbol{C}, \boldsymbol{N})$ where $\boldsymbol{N} = \boldsymbol{L} \times \boldsymbol{M}$ and for all $(\alpha, \beta) \in \boldsymbol{N}$ we have $\boldsymbol{C}(\alpha, \beta) = \boldsymbol{A}(\alpha) \cup \boldsymbol{B}(\beta)$. In this case, we write $(\boldsymbol{A}, \boldsymbol{L}) \tilde{\vee} (\boldsymbol{B}, \boldsymbol{M}) = (\boldsymbol{C}, \boldsymbol{N})$.*

**Definition 2.16.** *Let us assume that $(\boldsymbol{A}, \boldsymbol{L})$ and $(\boldsymbol{B}, \boldsymbol{M})$ are two soft sets over a common universe $\boldsymbol{U}$, then the extended intersection of $(\boldsymbol{A}, \boldsymbol{L})$ and $(\boldsymbol{B}, \boldsymbol{M})$ is the soft $(\boldsymbol{C}, \boldsymbol{N})$ where $\boldsymbol{N} = \boldsymbol{L} \cup \boldsymbol{M}$ and for all $\alpha \in \boldsymbol{N}$ we have*

$$\boldsymbol{C}(\alpha) = \begin{cases} \boldsymbol{A}(\alpha) & if \alpha \in \boldsymbol{L} \backslash \boldsymbol{M}, \\ \boldsymbol{B}(\alpha) & if\ \alpha \in \boldsymbol{M} \backslash \boldsymbol{L}, \\ \boldsymbol{A}(\alpha) \cap \boldsymbol{B}(\alpha) & if\ \alpha \in \boldsymbol{L} \cap \boldsymbol{M}. \end{cases}$$

*Symbolically, we write $(\boldsymbol{A}, \boldsymbol{L}) \bigcap_{\varepsilon} (\boldsymbol{B}, \boldsymbol{M}) = (\boldsymbol{C}, \boldsymbol{N})$.*

**Definition 2.17.** *Let $(\boldsymbol{A}, \boldsymbol{L})$ and $(\boldsymbol{B}, \boldsymbol{M})$ be two soft sets over a common universe $\boldsymbol{U}$, then the restricted intersection of $(\boldsymbol{A}, \boldsymbol{L})$ and $(\boldsymbol{B}, \boldsymbol{M})$ is the soft set $(\boldsymbol{C}, \boldsymbol{N})$ where $\boldsymbol{N} = \boldsymbol{L} \cap \boldsymbol{M} \neq \emptyset$ and moreover, for all $\alpha \in \boldsymbol{N}$ we have $\boldsymbol{C}(\alpha) = \boldsymbol{A}(\alpha) \cap \boldsymbol{B}(\alpha)$. Symbolically, we write $(\boldsymbol{A}, \boldsymbol{L}) \bigcap_{\Re} (\boldsymbol{B}, \boldsymbol{M}) = (\boldsymbol{C}, \boldsymbol{N})$.*

Note that if $\boldsymbol{L} \cap \boldsymbol{M} = \emptyset$ then $(\boldsymbol{A},\, \boldsymbol{L}) \bigcap_{\Re}(\boldsymbol{B},\, \boldsymbol{M}) = \emptyset_{\emptyset}$, where $\emptyset_{\emptyset}$ is known as the unique soft set over $\boldsymbol{U}$ with an empty parameter set.

**Definition 2.18.** *Let us assume that $(\boldsymbol{A},\, \boldsymbol{L})$ and $(\boldsymbol{B},\, \boldsymbol{M})$ are two soft sets over a common universe $\boldsymbol{U}$, then the extended union of $(\boldsymbol{A},\, \boldsymbol{L})$ and $(\boldsymbol{B},\, \boldsymbol{M})$ is the soft set $(\boldsymbol{C},\, \boldsymbol{N})$ where $\boldsymbol{N} = \boldsymbol{L} \cup \boldsymbol{M}$ and for all $\alpha \in \boldsymbol{N}$ we have*

$$\boldsymbol{C}(\alpha) = \begin{cases} \boldsymbol{A}(\alpha) & if\ \alpha \in \boldsymbol{L} \backslash \boldsymbol{M}, \\ \boldsymbol{B}(\alpha) & if\ \alpha \in \boldsymbol{M} \backslash \boldsymbol{L}, \\ \boldsymbol{A}(\alpha) \cup \boldsymbol{B}(\alpha) & if\ \alpha \in \boldsymbol{L} \cap \boldsymbol{M}. \end{cases}$$

*Symbolically, we write $(\boldsymbol{A},\, \boldsymbol{L}) \bigcup_{\varepsilon}(\boldsymbol{B},\, \boldsymbol{M}) = (\boldsymbol{C},\, \boldsymbol{N})$.*

**Definition 2.19.** *Let us assume that $(\boldsymbol{A},\, \boldsymbol{L})$ and $(\boldsymbol{B},\, \boldsymbol{M})$ are two soft sets over a common universe $\boldsymbol{U}$, then the restricted union of $(\boldsymbol{A},\, \boldsymbol{L})$ and $(\boldsymbol{B},\, \boldsymbol{M})$ is the soft set $(\boldsymbol{C},\, \boldsymbol{N})$ where $\boldsymbol{N} = \boldsymbol{L} \cap \boldsymbol{M} \neq \emptyset$ and for all $\alpha \in \boldsymbol{N}$ we have $\boldsymbol{C}(\alpha) = \boldsymbol{A}(\alpha) \cup \boldsymbol{B}(\alpha)$. Symbolically, we write $(\boldsymbol{A},\, \boldsymbol{L}) \bigcup_{\Re}(\boldsymbol{B},\, \boldsymbol{M}) = (\boldsymbol{C},\, \boldsymbol{N})$.*

It should be noted that if $\boldsymbol{L} \cap \boldsymbol{M} = \emptyset$, then $(\boldsymbol{A},\, \boldsymbol{L}) \bigcup_{\Re}(\boldsymbol{B},\, \boldsymbol{M}) = \emptyset_{\emptyset}$.

In the same way, we may define the **OR**-operation, the **AND**-operation, the restricted intersection, the extended intersection, the extended union, and the restricted union of a collection $\{(\boldsymbol{A}_i,\, \boldsymbol{L}_i) : i \in \boldsymbol{I}\}$ of soft sets over $\boldsymbol{U}$. For details see [30].

## 3. Semirings and Soft Congruence Relations

In this section, we discuss soft congruence relations over semirings and some related properties are investigated. Let $(\boldsymbol{S},\, +,\, \cdot)$ denotes a semiring. We start with the following definitions which has been taken from [[5], [9]].

**Definition 3.1.** *Let us suppose that $(\boldsymbol{A},\, \boldsymbol{L})$ is a soft set over $\boldsymbol{U}$. Then $Supp(\boldsymbol{A},\, \boldsymbol{L}) = \{\beta \in \boldsymbol{L} : \boldsymbol{A}(\beta) \neq \emptyset\}$ is said to be the support of $(\boldsymbol{A},\, \boldsymbol{L})$. We say that a soft set $(\boldsymbol{A},\, \boldsymbol{L})$ is said to be nonnull if $Supp(\boldsymbol{A},\, \boldsymbol{L}) \neq \emptyset$.*

**Definition 3.2.** *Let $(\rho,\, \boldsymbol{L})$ be a soft set over $\boldsymbol{U} \times \boldsymbol{U}$. Then $(\rho,\, \boldsymbol{L})$ is called a soft binary relation over $\boldsymbol{U}$.*

**Definition 3.3.** *Let $(\rho,\, \boldsymbol{L})$ be a soft binary relation over $\boldsymbol{U}$, then $(\rho,\, \boldsymbol{L})$ is said to be a soft equivalence relation over $\boldsymbol{U}$ if $\rho(\beta) \neq \emptyset$ is an equivalence relation on $\boldsymbol{U}$ $\forall\, \beta \in \boldsymbol{L}$.*

Further we have the following definitions.

**Definition 3.4.** *Let $(\rho,\, \boldsymbol{L})$ be a nonnull soft set over $\boldsymbol{S} \times \boldsymbol{S}$. Then $(\rho,\, \boldsymbol{L})$ is called a soft congruence relation over $\boldsymbol{S}$ if $\rho(\alpha)$ $\forall\, \alpha \in Supp(\rho,\, \boldsymbol{L})$ is a congruence relation on $\boldsymbol{S}$.*

*Let $Supp(\rho,\, \boldsymbol{L}) = \emptyset$. Then $(\rho,\, \boldsymbol{L})$ is called a null soft congruence relation over $\boldsymbol{S}$, denoted by ${\emptyset_L}^2$.*

We have the following remark.

**Remark 3.5.** *Let us suppose* $(\rho,\ \boldsymbol{L})$ *is a soft congruence relation over* $\boldsymbol{S}$ *and further assume that the set* $\boldsymbol{L}$ *contains only one element. Then it is easy to see that* $(\rho,\ \boldsymbol{L})$ *is the same as the classical congruence relation. If* $\boldsymbol{L}$ *contains more than one element, then* $(\rho,\ \boldsymbol{L})$ *is a general soft congruence relation. That is, in other words, we can say that the classical congruence relation is a soft congruence relation.*

In order to understand general soft congruence relations, we give two examples.

**Example 3.6.** *Let* $\boldsymbol{W}$ *be the set of whole numbers. Then* $(\boldsymbol{W},\ +,\ \cdot)$ *is a semiring and let us suppose that* $\boldsymbol{L} = \boldsymbol{W}$*. Let map* $\rho : \boldsymbol{L} \rightarrow \boldsymbol{P}(\boldsymbol{W} \times \boldsymbol{W})$ *be defined by*

$\rho(\alpha) = \{(a,\ b) \in \boldsymbol{W} \times \boldsymbol{W}\colon \alpha a = \alpha b\}$ *for all* $\alpha \in \boldsymbol{L}$*.*

*Then* $\rho(\alpha)$ *is a congruence relation on* $(\boldsymbol{W},\ +,\ \cdot)$*. Therefore* $(\rho,\ \boldsymbol{L})$ *is a soft congruence relation over* $\boldsymbol{W}$*.*

**Example 3.7.** *Let* $\boldsymbol{L} = \boldsymbol{S}^1$ *be a semiring with identity and let* $\rho : \boldsymbol{L} \rightarrow \boldsymbol{P}(\boldsymbol{S}^1 \times \boldsymbol{S}^1)$ *be a function which is defined by the rule* $\rho(\alpha) = \{(a,\ b) \in \boldsymbol{S}^1 \times \boldsymbol{S}^1 : a + i = b + j\}$ *for some* $i,\ j \in \langle\alpha\rangle$ *where* $\langle\alpha\rangle$ *is the principle ideal generated by* $\alpha$*. It is simply to verify that* $\rho(\alpha)$ *is a congruence relation on* $\boldsymbol{S}$*. Thus* $(\rho,\ \boldsymbol{L})$ *is a soft congruence over* $\boldsymbol{S}$*.*

Let us state and prove some properties.

**Theorem 3.8.** *Let us suppose* $(\rho,\ \boldsymbol{M})$ *and* $(\sigma,\ \boldsymbol{N})$ *are soft congruence relations over a semiring* $\boldsymbol{S}$*. Then the following statements hold:*

*(i) If* $(\rho,\ \boldsymbol{M})\ \tilde{\wedge}\ (\sigma,\ \boldsymbol{N})$ *is nonnull, then it is a soft congruence relation over the semiring* $\boldsymbol{S}$*.*

*(ii)* $(\rho,\ \boldsymbol{M}) \bigcap_{\varepsilon} (\sigma,\ \boldsymbol{N})$ *is a soft congruence relation over the semiring* $\boldsymbol{S}$*.*

*(iii) If* $(\rho,\ \boldsymbol{M}) \bigcap_{\Re} (\sigma,\ \boldsymbol{N})$ *is nonnull, then it is a soft congruence relation over the semiring* $\boldsymbol{S}$*.*

*Proof.* (i) According to Definition 2.14, let us suppose that $(\rho,\ \boldsymbol{M})\ \tilde{\wedge}\ (\sigma,\ \boldsymbol{N}) = (\eta,\ \boldsymbol{L})$ where $\boldsymbol{L} = \boldsymbol{M} \times \boldsymbol{N}$ and $\eta(\alpha,\ \beta) = \rho(\alpha) \cap \sigma\ (\beta)$, $\forall\ (\alpha,\ \beta) \in \boldsymbol{L}$. Then by the hypothesis, there exists $(\alpha,\ \beta) \in \mathrm{Supp}(\eta,\ \boldsymbol{L})$, and so $\eta(\alpha,\ \beta) = \rho(\alpha) \cap \sigma(\beta) \neq \emptyset$. It is given that the non-empty sets $\rho(\alpha)$ and $\sigma(\beta)$ are both congruence relations on the semiring $\boldsymbol{S}$, therefore $\eta(\alpha,\ \beta)$ is a congruence relation on the semiring $\boldsymbol{S}$**,** $\forall\ (\alpha,\ \beta) \in \mathrm{Supp}(\eta,\ \boldsymbol{L})$. Therefore $(\rho,\ \boldsymbol{M})\ \tilde{\wedge}\ (\sigma,\ \boldsymbol{N}) = (\eta,\ \boldsymbol{L})$ is a soft congruence relation over the semiring $\boldsymbol{S}$.

(ii) According to Definition 2.16, let us suppose that $(\rho,\ \boldsymbol{M}) \bigcap_{\varepsilon} (\sigma,\ \boldsymbol{N}) = (\eta,\ \boldsymbol{L})$ where $\boldsymbol{L} = \boldsymbol{M} \cup \boldsymbol{N}$ and $\forall\ \alpha \in \boldsymbol{L}$,

$$\eta(\alpha) = \begin{cases} \rho\,(\alpha) & if\ \alpha \in \boldsymbol{M} \backslash \boldsymbol{N}, \\ \sigma\,(\beta) & if\ \alpha \in \boldsymbol{N} \backslash \boldsymbol{M}, \\ \rho\,(\alpha) \cap \sigma\,(\beta) & if\ \alpha \in \boldsymbol{M} \cap \boldsymbol{N}. \end{cases}$$

Let us suppose $\alpha \in \mathrm{Supp}(\eta,\ \boldsymbol{L})$. Let $\alpha \in \boldsymbol{M}\backslash\boldsymbol{N}$, then $\eta(\alpha) = \rho(\alpha) \neq \emptyset$ is a congruence relation on the semiring $\boldsymbol{S}$. Now if $\alpha \in \boldsymbol{N}\backslash\boldsymbol{M}$, then $\eta(\alpha) = \sigma(\alpha)$ $\neq \emptyset$ is a congruence relation on the semiring $\boldsymbol{S}$. Moreover if $\alpha \in \boldsymbol{M} \cap \boldsymbol{N}$, then $\eta(\alpha) = \rho(\alpha) \cap \sigma(\beta) \neq \emptyset$. It is given that the sets $\rho(\alpha) \neq \emptyset$ and $\sigma(\alpha) \neq \emptyset$ are both congruence relations on the semiring $\boldsymbol{S}$ and so their intersection as well. Therefore $\eta(\alpha)$ is a congruence relation on the semiring $\boldsymbol{S}$, $\forall\ \alpha \in \mathrm{Supp}(\eta,\ \boldsymbol{L})$ and therefore $(\rho,\ \boldsymbol{M}) \bigcap_{\varepsilon}(\sigma,\ \boldsymbol{N}) = (\eta,\ \boldsymbol{L})$ is a soft congruence relation over the semiring $\boldsymbol{S}$.

(iii) According to Definition 2.17, let $(\rho,\ \boldsymbol{M}) \bigcap_{\Re}(\sigma,\ \boldsymbol{N}) = (\eta,\ \boldsymbol{L})$ where $\boldsymbol{L}$ $= \boldsymbol{M} \cap \boldsymbol{N} \neq \emptyset$ and $\eta(\alpha) = \rho(\alpha) \cap \sigma(\alpha),\ \forall\ \alpha \in \boldsymbol{L}$. By hypothesis it is given that $(\eta,\ \boldsymbol{L})$ is a nonnull soft set on $\boldsymbol{S}$, so there exists $\alpha \in \mathrm{Supp}(\eta,\ \boldsymbol{L})$, and so $\eta(\alpha) = \rho(\alpha) \cap \sigma(\alpha) \neq \emptyset$. As the non-empty sets $\rho(\alpha)$ and $\sigma(\alpha)$ are both congruence relation on the semiring $\boldsymbol{S}$, therefore $\eta\ (\alpha)$ is a congruence relation on the semiring $\boldsymbol{S}$, $\forall\ \alpha \in \mathrm{Supp}(\eta,\ \boldsymbol{L})$. Thus, it follows that $(\rho,\ \boldsymbol{M}) \bigcap_{\Re}(\sigma,\ \boldsymbol{N})$ $= (\eta,\ \boldsymbol{C})$ is a soft congruence relation over the semiring $\boldsymbol{S}$. □

**Theorem 3.9.** *Let* $(\rho,\ \boldsymbol{M})$ *and* $(\sigma,\ \boldsymbol{N})$ *be soft congruence relations over a semiring* $\boldsymbol{S}$. *Then the following statements hold:*

*(i) Let* $\rho(\alpha) \subseteq \sigma(\beta)$ *or* $\sigma(\beta) \subseteq \rho(\alpha),\ \forall\ (\alpha,\ \beta) \in \boldsymbol{M} \times \boldsymbol{N}$, *then* $(\rho,\ \boldsymbol{M})\ \tilde{\vee}\ (\sigma,\ \boldsymbol{N})$ *is a soft congruence relation over the semiring* $\boldsymbol{S}$.

*(ii) Let* $\rho(\alpha) \subseteq \sigma(\alpha)$ *or* $\rho(\alpha) \subseteq \sigma(\alpha),\ \forall\ \alpha\ \in \boldsymbol{M} \cup \boldsymbol{N}$, *then* $(\rho,\ \boldsymbol{M}) \bigcup_{\varepsilon} (\sigma,\ \boldsymbol{N})$ *is a soft congruence relation over the semiring* $\boldsymbol{S}$.

*(iii) Let* $\rho(\alpha) \subseteq \sigma(\alpha)$ *or* $\sigma(\alpha) \subseteq \rho(\alpha),\ \forall\ \alpha \in \boldsymbol{M} \cap \boldsymbol{N}$ *and* $\boldsymbol{M} \cap \boldsymbol{N} \neq \emptyset$, *then* $(\rho,\ \boldsymbol{M}) \bigcup_{\Re}(\sigma,\ \boldsymbol{N})$ *is a soft congruence relation over the semiring* $\boldsymbol{S}$

*Proof.* (i) According to Definition 2.15, let us suppose that $(\rho,\ \boldsymbol{M})\ \tilde{\vee}\ (\sigma,\ \boldsymbol{N}) = (\eta,\ \boldsymbol{L})$ where $\boldsymbol{L} = \boldsymbol{M} \times \boldsymbol{N}$ and $\eta(\alpha,\ \beta) = \rho(\alpha) \cup \sigma(\beta)\ \forall\ (\alpha,\ \beta) \in \boldsymbol{L}$. If $(\alpha,\ \beta) \in \mathrm{Supp}(\eta,\ \boldsymbol{L})$, then we have $\eta(\alpha,$ $\beta) = \rho(\alpha) \cup \sigma(\beta) \neq \emptyset$. Now given that $\rho(\alpha) \subseteq \sigma(\beta)$ or $\sigma(\beta) \subseteq \rho(\alpha),\ \forall\ (\sigma,\ \beta) \in$ $\boldsymbol{M} \times \boldsymbol{N}$, we have $\eta(\alpha,\ \beta) = \rho(\alpha)$ or $\eta(\alpha,\ \beta) = \sigma(\beta)$. Since the sets $\rho(\alpha) \neq \emptyset$ and $\sigma(\beta) \neq \emptyset$ are both congruence relations on the semiring $\boldsymbol{S}$, then $\eta(\alpha,\ \beta)$ is a congruence relation on the semiring $\boldsymbol{S}$, $\forall\ (\alpha,\ \beta) \in \mathrm{Supp}(\eta,\ \boldsymbol{L})$. Thus it follows that $(\rho,\ \boldsymbol{M})\ \tilde{\vee}\ (\sigma,\ \boldsymbol{N}) = (\eta,\ \boldsymbol{L})$ is a soft congruence relation over the semiring $\boldsymbol{S}$.

(ii) According to Definition 2.18, let us suppose that $(\rho,\ \boldsymbol{M}) \bigcup_{\varepsilon}(\sigma,\ \boldsymbol{N}) = (\eta,\ \boldsymbol{L})$ where $\boldsymbol{L} = \boldsymbol{M} \cup \boldsymbol{N}$ and $\forall\ \alpha \in \boldsymbol{L}$.

$$\eta(\alpha) = \begin{cases} \rho\,(\alpha) & if\ \alpha \in \boldsymbol{M}\backslash\boldsymbol{N}, \\ \sigma\,(\alpha) & if\ \alpha \in \boldsymbol{N}\backslash\boldsymbol{M}, \\ \rho\,(\alpha) \cup \sigma\,(\alpha) & if\ \alpha \in \boldsymbol{M} \cap \boldsymbol{N}. \end{cases}$$

Let us suppose $\alpha \in \mathrm{Supp}(\eta,\ \boldsymbol{L})$. First let $\alpha \in \boldsymbol{M} \setminus \boldsymbol{N}$, then $\eta(\alpha) = \rho(\alpha) \neq$ $\emptyset$ is congruence relation on the semiring $\boldsymbol{S}$. Secondly let $\alpha \in \boldsymbol{N}\backslash\boldsymbol{M}$, then $\eta(\alpha)$ $= \sigma(\alpha) \neq \emptyset$ is a congruence relation on the semiring $\boldsymbol{S}$. Thirdly let us assume that $\alpha \in \boldsymbol{M} \cap \boldsymbol{N}$, then $\eta\ (\alpha) = \rho(\alpha) \cup \sigma(\alpha) \neq \emptyset$. Now given that $\rho(\alpha) \subseteq \sigma(\alpha)$ or $\sigma(\alpha) \subseteq \rho(\alpha)$, for all $\alpha \in \boldsymbol{M} \cup \boldsymbol{N}$, we have $\eta\ (\alpha)$ is a congruence relation on the semiring $\boldsymbol{S}\ \forall\ \alpha \in \mathrm{Supp}(\eta,\ \boldsymbol{L})$. Consequently $(\rho,\ \boldsymbol{M}) \bigcup_{\varepsilon}(\sigma,\ \boldsymbol{N}) = (\eta,\ \boldsymbol{L})$ is a soft congruence relation over the semiring $\boldsymbol{S}$.

(iii) By Definition 2.19, let us suppose that $(\rho,\ \boldsymbol{M}) \bigcup_{\Re}(\sigma,\ \boldsymbol{N}) = (\eta,\ \boldsymbol{L})$ where $\boldsymbol{L} = \boldsymbol{M} \cap \boldsymbol{N} \neq \emptyset$ and $\eta\ (\alpha) = \rho(\alpha) \cup \sigma(\alpha)\ \forall\ \alpha \in \boldsymbol{L}$. Let us assume that $\alpha \in \mathrm{Supp}(\eta,\ \boldsymbol{L})$; then $\eta(\alpha) = \rho(\alpha) \cup \sigma(\alpha) \neq \emptyset$. Now $\rho(\alpha) \subseteq \sigma(\alpha)$ or $\sigma(\alpha) \subseteq \rho(\alpha)$, $\forall\ \alpha \in \boldsymbol{M} \cap \boldsymbol{N}$, we have either $\eta(\alpha) = \rho(\alpha)$ or $\eta(\alpha) = \sigma(\alpha)$. As the sets $\rho(\alpha) \neq \emptyset$ and $\sigma(\alpha) \neq \emptyset$ are both congruence relations on the semiring $\boldsymbol{S}$, therefore $\eta\ (\alpha)$ is a congruence relation on the semiring $\boldsymbol{S}$, $\forall\ \alpha \in \mathrm{Supp}(\eta,\ \boldsymbol{L})$. It follows that $(\rho,\ \boldsymbol{M}) \bigcup_{\Re}(\sigma,\ \boldsymbol{N}) = (\eta,\ \boldsymbol{L})$ is a soft congruence relation over the semiring $\boldsymbol{S}$. □

We now give the following definitions.

**Definition 3.10.** *Let us assume that $(\boldsymbol{F},\ \boldsymbol{M})$ is a nonnull soft set over a semiring $\boldsymbol{S}$. Then $(\boldsymbol{F},\ \boldsymbol{A})$ is said to be a soft semiring over $\boldsymbol{S}$ if $\boldsymbol{F}(\alpha)$ is a subsemiring of $\boldsymbol{S}$ for all $\alpha \in$ Supp$(\boldsymbol{F},\ \boldsymbol{M})$. Symbolically we write $(\boldsymbol{F},\ \boldsymbol{M}) \widetilde{<} \boldsymbol{S}$.*

**Definition 3.11.** *Let us suppose that $(\boldsymbol{F},\ \boldsymbol{M})$ is a nonnull soft set over a semiring $\boldsymbol{S}$. Then $(\boldsymbol{F},\ \boldsymbol{M})$ is said to be an idealistic soft semiring over $\boldsymbol{S}$ if $\boldsymbol{F}(\alpha)$ is an ideal of $\boldsymbol{S}$, $\forall\ \alpha \in$ Supp$(\boldsymbol{F},\ \boldsymbol{M})$.*

**Definition 3.12.** *Let $(\rho,\ \boldsymbol{M})$ be a soft congruence relation over a semiring $\boldsymbol{S}$. Let $a \in \boldsymbol{S}$. We define a function $a\ \rho : \boldsymbol{M} \to \boldsymbol{P}(\boldsymbol{S})$ by $(a\rho)(\alpha) = a\rho(\alpha)$, $\forall\ \alpha \in \boldsymbol{M}$. Here $a\rho(\alpha)$ is the congruence class of $a$ with respect to $\rho(\alpha)$. Moreover we say that $a(\rho,\ \boldsymbol{M})$ is a soft congruence class of $a$ with respect to $(\rho,\ \boldsymbol{M})$.*

**Definition 3.13.** *Let us assume that $(\rho,\ \boldsymbol{M})$ is a soft congruence relation over a semiring $(\boldsymbol{S},\ +,\ \cdot)$ and let $\boldsymbol{S}\ /(\rho,\ \boldsymbol{M}) = \{a\rho(\alpha) :\ a \in \boldsymbol{S},\ \alpha \in \boldsymbol{M}\}$ be an initial universe set. Let us define the function $\rho^*$: $\boldsymbol{M} \to \boldsymbol{P}(\boldsymbol{S}/\ (\rho,\ \boldsymbol{M}))$ by the rule $\rho^*(\ \alpha) = \{a\ \rho(\alpha) :\ a \in \boldsymbol{S}\}$ for all $\alpha \in \boldsymbol{M}$. We say that $(\rho^*,\ \boldsymbol{M})$ is a soft quotient set of the semiring $\boldsymbol{S}$.*

Define $a\rho(\alpha)\ +^* b\rho(\alpha) = (a + b)\rho(\alpha)$ and $(a\rho(\alpha))\ \cdot^*\ (b\rho(\alpha)) = (a \cdot b)\ \rho(\alpha)$, $\forall\ a,\ b \in \boldsymbol{S}$ and $\forall\ \alpha \in \boldsymbol{M}$.

Then $(\rho^*(\ \alpha),\ +^*,\ \cdot^*)$ for all $\alpha \in\ \boldsymbol{M}$ is a semiring which is called quotient semiring of $(\boldsymbol{S},\ +,\ \cdot)$ with respect to $\rho(\alpha)$. We say that $(\rho^*,\ \boldsymbol{M})$ is a soft quotient semiring of $\boldsymbol{S}$ with respect to the soft congruence relation $(\rho,\ \boldsymbol{M})$.

To understand the above concept, we give an example.

**Example 3.14.** *Let us suppose $\boldsymbol{S} = \{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9\}$ and define "$+$" and "$\cdot$" in the following tables:*

| + | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 7 |
| 2 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 7 | 8 |
| 3 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 7 | 8 | 9 |
| 4 | 4 | 5 | 6 | 7 | 8 | 9 | 7 | 8 | 9 | 7 |
| 5 | 5 | 6 | 7 | 8 | 9 | 7 | 8 | 9 | 7 | 8 |
| 6 | 6 | 7 | 7 | 9 | 7 | 8 | 9 | 7 | 8 | 9 |
| 7 | 7 | 8 | 8 | 7 | 8 | 9 | 7 | 8 | 9 | 7 |
| 8 | 8 | 9 | 7 | 8 | 9 | 7 | 8 | 9 | 7 | 8 |
| 9 | 9 | 7 | 8 | 9 | 7 | 8 | 9 | 7 | 8 | 9 |

| $\cdot$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2 | 0 | 2 | 4 | 6 | 8 | 7 | 9 | 8 | 7 | 9 |
| 3 | 0 | 3 | 6 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |
| 4 | 0 | 4 | 8 | 9 | 7 | 8 | 9 | 7 | 8 | 9 |
| 5 | 0 | 5 | 7 | 9 | 8 | 7 | 9 | 8 | 7 | 9 |
| 6 | 0 | 6 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |
| 7 | 0 | 7 | 8 | 9 | 7 | 8 | 9 | 7 | 8 | 9 |
| 8 | 0 | 8 | 7 | 8 | 8 | 7 | 9 | 8 | 7 | 9 |
| 9 | 0 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 8 | 9 |

*Then according to* [10], $(\boldsymbol{S}, +, \cdot)$ *is a semiring. Let* $\boldsymbol{M} = \{1, 2\}$ *and further assume that* $\rho : \boldsymbol{M} \to \boldsymbol{P}(\boldsymbol{S} \times \boldsymbol{S})$ *is a function given as follows:*

$\rho(\alpha) = \{(a, b) : \alpha a = \alpha b\}, \forall\ \alpha \in \boldsymbol{M}$.

*Then*

$\rho(1) = \{(a, b) : a = b\}$,

*i.e.*

$\rho(1) = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8), (9, 9)\}$.

$\rho(2) = \{(a, b) : 2a = 2b\}$,

*i.e.*

$\rho(2) = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8), (9, 9), (4, 7), (5, 8), (6, 9), (7, 4), (8, 5), (9, 6)\}$.

*are congruence relations on the semiring* $\boldsymbol{S}$*. Therefore* $(\rho, \boldsymbol{M})$ *is soft congruence relation over the semiring* $\boldsymbol{S}$*. Also let* $(\rho^*, \boldsymbol{M})$ *be a soft quotient semiring of* $\boldsymbol{S}$ *with respect to* $(\rho, \boldsymbol{M})$*. Now*

$0(\rho, \boldsymbol{M}) = \{0\rho(1), 0\rho(2)\} \Rightarrow 0(\rho, \boldsymbol{M}) = \{\{0\}\}$.

$1(\rho, \boldsymbol{M}) = \{1\rho(1), 1\rho(2)\} \Rightarrow 1(\rho, \boldsymbol{M}) = \{\{1\}\}$.

$2(\rho, \boldsymbol{M}) = \{2\rho(1), 2\rho(2)\} \Rightarrow 2(\rho, \boldsymbol{M}) = \{\{2\}\}$.

$3(\rho, \boldsymbol{M}) = \{3\rho(1), 3\rho(2)\} \Rightarrow 3(\rho, \boldsymbol{M}) = \{\{3\}\}$.

*$4(\rho,\, \boldsymbol{M}) = \{4\rho(2),\, 4\rho(2)\} \Rightarrow 4(\rho,\, \boldsymbol{M}) = \{\{4\},\, \{4,\, 7\}\}$.*
*$5(\rho,\, \boldsymbol{M}) = \{5\rho(1),\, 5\rho(2)\} \Rightarrow 5(\rho,\, \boldsymbol{M}) = \{\{5\},\, \{5,\, 8\}\}$.*
*$6(\rho,\, \boldsymbol{M}) = \{6\rho(1),\, 6\rho(2)\} \Rightarrow 6(\rho,\, \boldsymbol{M}) = \{\{6\},\, \{6,\, 9\}\}$.*
*$7(\rho,\, \boldsymbol{M}) = \{7\rho(1),\, 7\rho(2)\} \Rightarrow 7(\rho,\, \boldsymbol{M}) = \{\{7\},\, \{7,\, 4\}\}$.*
*$8(\rho,\, \boldsymbol{M}) = \{8\rho(1),\, 8\rho(2)\} \Rightarrow 8(\rho,\, \boldsymbol{M}) = \{\{8\},\, \{8,\, 5\}\}$.*
*$9(\rho,\, \boldsymbol{M}) = \{9\rho(2),\, 9\rho(2)\} \Rightarrow 9(\rho,\, \boldsymbol{M}) = \{\{9\},\, \{9,\, 6\}\}$.*
*Soft quotient semiring $(\rho^*,\, \boldsymbol{M})$ is given below:*
*$\rho^*(1) = \{0\rho(1),\, 1\rho(1),\, 2\rho(1),\, 3\rho(1),\, 4\rho(1),\, 5\rho(1),\, 6\rho(1),\, 7\rho(1),\, 8\rho(1),\, 9\rho(1)\}$,*
*$\rho^*(2) = \{0\rho(2),\, 1\rho(2),\, 2\rho(2),\, 3\rho(2),\, 4\rho(2),\, 5\rho(2),\, 6\rho(2),\, 7\rho(2),\, 8\rho(2),\, 9\rho(2)\}$.*

The following theorem shows a relation between soft congruence relations and idealistic soft semirings over semirings.

**Theorem 3.15.** *(i) Let us assume that $\boldsymbol{S}$ is a semiring and suppose that $(\rho,\, \boldsymbol{M})$ is a soft congruence relation over the semiring $\boldsymbol{S}$. If $(\boldsymbol{F},\, \boldsymbol{M})$ is a nonnull soft set and $(\boldsymbol{F},\, \boldsymbol{M}) = 0(\rho,\, \boldsymbol{M})$, then $(\boldsymbol{F},\, \boldsymbol{M})$ is an idealistic soft semiring over $\boldsymbol{S}$.*

*(ii) Let us suppose that $(\boldsymbol{F},\, \boldsymbol{M})$ is an idealistic soft semiring over a semiring $\boldsymbol{S}$ and let us define a function $\rho : \boldsymbol{M} \to \boldsymbol{P}(\boldsymbol{S} \times \boldsymbol{S})$ by*
*$\rho(\alpha) = \{(a,\, b) \in \boldsymbol{S} \times \boldsymbol{S} : a + i = b + j$ for some $i,\, j \in \boldsymbol{F}(\alpha)\},\ \forall\ \alpha \in \boldsymbol{M}$.*
*Then $(\rho,\, \boldsymbol{M})$ is a soft congruence relation over the semiring $\boldsymbol{S}$.*

*Proof.* (i) According to Definition 3.4, $\rho(\alpha)\ \forall\ \alpha \in$ Supp $(\rho,\, \boldsymbol{M})$ is a congruence relation on the semiring $\boldsymbol{S}$. By Lemma 2.8, we have $\boldsymbol{F}(\alpha) = 0\rho(\alpha)$ is an ideal of $\boldsymbol{S}$. Since $(\boldsymbol{F},\, \boldsymbol{M}) = 0(\rho,\, \boldsymbol{M})$ is a nonnull soft set so it follows that $(\boldsymbol{F},\, \boldsymbol{M})$ is an idealistic soft semiring over $\boldsymbol{S}$ by Definition 3.11.

(ii) According to Definition 3.11, we know that, $\forall\ \alpha \in$ Supp $(\boldsymbol{F},\, \boldsymbol{M})$, $\boldsymbol{F}(\alpha)$ is an ideal of the semiring $\boldsymbol{S}$. By Lemma 2.8, we have $\forall\ \alpha \in$ Supp $(\rho,\, \boldsymbol{M})$, $\rho(\alpha)$ is a congruence relation on the semiring $\boldsymbol{S}$. Therefore, $(\rho,\, \boldsymbol{M})$ is a soft congruence relation over the semiring $\boldsymbol{S}$. □

## 4. Soft Congruence Relations and Homomorphisms

In this section, we define trivial and whole soft congruence relations. We also give some examples. We then show that every homomorphism defines a soft congruence relation.

**Definition 4.1.** *A soft congruence relation $(\rho,\, \boldsymbol{L})$ over a semiring $\boldsymbol{S}$ is called trivial if $\rho(\alpha) = \{(a,\, a) : a \in \boldsymbol{S}\},\ \forall\ \alpha \in \boldsymbol{L}$. It is denoted by the symbol $\boldsymbol{I}^2{}_L$.*

*A soft congruence relation $(\rho,\, \boldsymbol{L})$ over $\boldsymbol{S}$ is called whole if $\rho(\alpha) = \{(a,\, b) : a,\, b \in \boldsymbol{S}\},\ \forall\ \alpha \in \boldsymbol{L}$. It is denoted by the symbol $\boldsymbol{S}^2{}_L$.*

Let us give an example.

**Example 4.2.** *Let $\boldsymbol{S} = \boldsymbol{W}/\langle 4\rangle$ which is a semiring and further assume that $(\rho,\, \boldsymbol{L}_i)_{i=1,2}$ are nonnull soft sets over $\boldsymbol{S} \times \boldsymbol{S}$. Define*

*$\rho(\alpha) = \{(a, b) \in \boldsymbol{S} \times \boldsymbol{S} : \alpha\ a = \alpha\ b\}$ for all $\alpha \in \boldsymbol{L}_i$.*

*If $\boldsymbol{L}_1 = \{1\}$ then it is easy to see that $(\rho, \boldsymbol{L}_1)$ is a soft congruence over the semiring $\boldsymbol{S}$ and $\rho(\alpha) = \{(a, a)\colon a \in \boldsymbol{S}\}\ \forall\ \alpha \in \boldsymbol{L}_1$. Thus $(\rho, \boldsymbol{L}_1)$ is the trivial soft congruence over the semiring $\boldsymbol{S}$.*

*If $\boldsymbol{L}_2 = \{0\}$ then it is obvious that $(\rho, \boldsymbol{L}_2)$ is a soft congruence over the semiring $\boldsymbol{S}$ and $\rho(0) = \{(a, b)\colon a, b \in \boldsymbol{S}\}$. Thus $(\rho, \boldsymbol{L}_2)$ is the whole soft congruence over the semiring $\boldsymbol{S}$.*

Before going further, we have the following remark.

**Remark 4.3.** *Let $\boldsymbol{S}$ and $\boldsymbol{S}'$ be two semirings and furthermore, assume that $g : \boldsymbol{S} \to \boldsymbol{S}'$ is a mapping. Let $(\boldsymbol{L}, \boldsymbol{B})$ be a soft set over the semiring $\boldsymbol{S}$. Then we may define a soft set $(g(\boldsymbol{L}),\boldsymbol{B})$ over the semiring $\boldsymbol{S}'$ where $g(\boldsymbol{L}) : \boldsymbol{B} \to \rho(\boldsymbol{S}')$ is defined by $g(\boldsymbol{L})(\alpha) = g(\boldsymbol{L}(\alpha))$, for all $\alpha \in \boldsymbol{B}$.*

*Here, we note that $Supp(g(\boldsymbol{L}), \boldsymbol{B}) = Supp(\boldsymbol{L}, \boldsymbol{B})$.*

**Proposition 4.4.** *Let $\boldsymbol{S}$ and $\boldsymbol{S}'$ be two semirings and $f : \boldsymbol{S} \to \boldsymbol{S}'$ be an epimorphism. If $(\rho, \boldsymbol{L})$ is a soft congruence relation over the semiring $\boldsymbol{S}$, then $(f(\rho), \boldsymbol{L})$ is a soft congruence relation over $\boldsymbol{S}'$, where*

*$f(\rho)(\alpha) = \{(f(a), f(b)) \in \boldsymbol{S}' \times \boldsymbol{S}' : (a, b) \in \rho(\alpha)\}, \forall\ \alpha \in \boldsymbol{L}$.*

*Proof.* Here $(f(\rho), \boldsymbol{L})$ is a nonnull soft set because $(\rho, \boldsymbol{L})$ is a nonnull soft set. Let $\alpha \in$ Supp $(f(\rho), \boldsymbol{L})$ then we have $f(\rho)(\alpha) = f(\rho(\alpha)) \neq \emptyset$. As the set $\rho(\alpha) \neq \emptyset$ is a congruence relation on the semiring $\boldsymbol{S}$, then by Proposition 2.11, $f(\rho(\alpha))$ is a congruence relation on the semiring $\boldsymbol{S}'$. Thus it follows that $\forall\ \alpha \in$ Supp $(f(\rho), \boldsymbol{L})$, $f(\rho)(\alpha)$ is a congruence on the semiring $\boldsymbol{S}'$. In other words, $(f(\rho), \boldsymbol{L})$ is a soft congruence relation over the semiring $\boldsymbol{S}'$. □

**Proposition 4.5.** *Let us assume that $\boldsymbol{S}$ and $\boldsymbol{S}'$ are two semirings and let $f : \boldsymbol{S} \to \boldsymbol{S}'$ be an epimorphism. Let $(\rho, \boldsymbol{L})$ be a soft congruence relation over the semiring $\boldsymbol{S}$, where*

*$f(\rho)(\alpha) = \{(f(a), f(b)) \in \boldsymbol{S}' \times \boldsymbol{S}' : (a, b) \in \rho(\alpha)\}, \forall\ \alpha \in \boldsymbol{L}$. The following statements hold:*

*(i) If $(\rho, \boldsymbol{L})$ is trivial, then $(f(\rho), \boldsymbol{L})$ will be the trivial soft congruence relation over the semiring $\boldsymbol{S}'$.*

*(ii) If $(\rho, \boldsymbol{L})$ is whole, then $(f(\rho), \boldsymbol{L})$ will be the whole soft congruence relation over the semiring $\boldsymbol{S}'$.*

*Proof.* (i) According to Definition 4.1, $\rho(\alpha) = \{(a, a)\colon a \in \boldsymbol{S}\}, \forall\ \alpha \in \boldsymbol{L}$. As $f$ is an epimorphism, then it follows that

$f(\rho)(\alpha) = f(\rho(\alpha)) = \{(a', a')\colon a' = f(a), a \in \boldsymbol{S}\}, \forall\ \alpha \in \boldsymbol{L}$.

Thus by Proposition 4.4, $(f(\rho), \boldsymbol{L})$ is the trivial soft congruence relation over the semiring $\boldsymbol{S}'$

(ii) According to Definition 4.1, $\rho(\alpha) = \{(a, b) : a, b \in \boldsymbol{S}\}\ \forall\ \alpha \in \boldsymbol{L}$. As $f$ is an epimorphism, then we have that

$f(\rho)(\alpha) = \{(a', b') : a' = f(a), b' = f(b), (a, b) \in \rho(\alpha), a, b \in \boldsymbol{S}\}\ \forall\ \alpha \in \boldsymbol{L}$.

By using Proposition 4.4, $(f(\rho),\ \boldsymbol{L})$ is the whole soft congruence relation over the semiring $\boldsymbol{S}'$. $\square$

**Proposition 4.6.** *Let $(\boldsymbol{S},\ +,\ \cdot)$ and $(\boldsymbol{S}',\ \star,\ \diamond)$ be semirings and $f:\ \boldsymbol{S}\to\boldsymbol{S}'$ be a homomorphism. The following statements hold:*

*(i) Let $\rho$: $\boldsymbol{L}\to\boldsymbol{P}(\boldsymbol{S}\times\ \boldsymbol{S})$ be defined by*

*$\rho(\alpha)=\{(a,\ b):\ a,\ b\in\boldsymbol{S},\ f(a)=f(b)\}\ \forall\ \alpha\in\boldsymbol{L}$. Then $(\rho,\ \boldsymbol{L})$ will be a soft congruence relation over the semiring $\boldsymbol{S}$.*

*(ii) Let $\boldsymbol{F}:\boldsymbol{L}\to\boldsymbol{P}(\boldsymbol{S})$ be defined by*

*$\boldsymbol{F}(\alpha)=\mathrm{Ker}\, f=\{a\in\boldsymbol{S}:f(a)=0\}\ \forall\ \alpha\in\boldsymbol{L}$. Then $(\boldsymbol{F},\ \boldsymbol{L})$ will be an idealistic soft semiring over the semiring $\boldsymbol{S}$.*

*Proof.* (i) It is enough to show that $\rho(\alpha)$ for all $\alpha\in\mathrm{Supp}(\rho,\ \boldsymbol{L})$ is a congruence relation.

*Reflexive*: As $f(a)=f(a),\ \forall\ a\in\boldsymbol{S}\Rightarrow(a,\ a)\in\rho(\alpha),\ \forall\ a\in\boldsymbol{S}\Rightarrow\rho(\alpha)$ is reflexive.

*Symmetric*: Let $(a,\ b)\in\rho(\alpha)\Rightarrow f(a)=f(b)\Rightarrow f(b)=f(a)\Rightarrow(b,\ a)\in\rho(\alpha)\Rightarrow\rho(\alpha)$ is symmetric.

*Transitive*: Let $(a,\ b),\ (b,\ c)\in\rho(\alpha)\Rightarrow f(a)=f(b)$ and $f(b)=f(c)\Rightarrow f(a)=f(c)\Rightarrow(a,\ c)\in\rho(\alpha)\Rightarrow\rho(\alpha)$ is transitive.

It follows $\rho(\alpha)$ is an equivalence relation.

Now let $(a,\ b),\ (u,\ v)\in\rho(\alpha)\Rightarrow f(a)=f(b)$ and $f(u)=f(v)$. Now

$f\ (a+u)=f(a)\star f(u)=f(b)\star f(v)=f(b+v)\Rightarrow(a+u,\ b+v)\in\rho(\alpha)$.

Similarly $f\ (a\cdot u)=f(a)\diamond f(u)=f(b)\diamond f(v)=f(b\cdot v)\Rightarrow(a\cdot u,\ b\cdot v)\in\rho(\alpha)$.

It follows that $\rho(\alpha)$ is compatible. Thus $\rho(\alpha)$ is a congruence relation.

(ii) It is enough to show that $\boldsymbol{F}(\alpha)$ for all $\alpha\in\mathrm{Supp}\ (\rho,\ \boldsymbol{L})$ is an ideal. As $f(0)=0$, so it follows that $0\in\boldsymbol{F}(\alpha)$. Thus $\boldsymbol{F}(\alpha)$ is non-empty. Now let $a,\ b\in\boldsymbol{F}(\alpha)\Rightarrow f\ (a)=f(b)=0$.

Now $f\ (a+b)=f(a)\star f(b)=0\star 0=0\Rightarrow a+b\in\boldsymbol{F}(\alpha)$.

Furthermore if $x\in\boldsymbol{S}$, then

$f\ (x\cdot a)=f(x)\diamond f(a)=f(x)\diamond 0=0\Rightarrow x\cdot a\in\boldsymbol{F}(\alpha)$.

Similarly

$f\ (a\cdot x)=f(a)\diamond f(x)=0\diamond f(x)=0\Rightarrow a\cdot x\in\boldsymbol{F}(\alpha)$.

Thus $\boldsymbol{F}(\alpha)$ is an ideal.Therefore $(\boldsymbol{F},\ \boldsymbol{L})$ is an idealistic soft semiring over the semiring $\boldsymbol{S}$. $\square$

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