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WEIGHTED PROJECTIVE LINES WITH WEIGHT PERMUTATION

Lina Han and Xintian Wang

Abstract. Let $\mathbb{X}$ be a weighted projective line defined over the algebraic closure $k = \overline{\mathbb{F}}_q$ of the finite field $\mathbb{F}_q$ and σ be a weight permutation of $\mathbb{X}$. By folding the category coh-$\mathbb{X}$ of coherent sheaves on $\mathbb{X}$ in terms of the Frobenius twist functor induced by σ, we obtain an $\mathbb{F}_q$-category, denoted by coh-$(\mathbb{X}, \sigma; q)$. We then prove that coh-$(\mathbb{X}, \sigma; q)$ is derived equivalent to the valued canonical algebra associated with $(\mathbb{X}, \sigma)$.

1. Introduction

The notions of weighted projective lines $\mathbb{X}$ and categories coh-$\mathbb{X}$ of their coherent sheaves were first introduced by Geigle and Lenzing in [9]. It turns out that weighted projective lines are important from various points of view and are closely related to many branches of mathematics, such as representation theory of finite dimensional algebras, singularity theory, invariant theory, and function theory, etc.

The key observation of Geigle and Lenzing was that each weighted projective line $\mathbb{X}$ admits a tilting sheaf T such that the endomorphism ring Λ is a canonical algebra in the sense of Ringel [14]. Thus, by applying a result of Happel [11], the category of finite dimensional Λ-modules is derived equivalent to the category coh-$\mathbb{X}$. Because coh-$\mathbb{X}$ is a hereditary abelian category, the structure of this derived category is closely related to that of coh-$\mathbb{X}$.

In [5], the authors introduced a Frobenius morphism F on an algebra A over the algebraic closure $k = \overline{\mathbb{F}}_q$ of the finite field $\mathbb{F}_q$ and constructed naturally Frobenius twist functors on both the category mod-A of finite dimensional A-modules and its bounded derived category $D^b(\text{mod-}A)$. They further proved that the module category mod-A^F for the F-fixed point algebra A^F is equivalent to the subcategory $(\text{mod-}A)^F$ of F-stable A-modules. This equivalence is

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further lifted to the derived category level in [6]. In particular, by applying the construction above to canonical algebras with Frobenius morphisms, we obtain valued canonical $\mathbb{F}_q$-algebras; see the general case studied by Ringel [14] and a special case in [3].

In the present paper we deal with a weighted projective line $\mathbb{X}$ over $k = \overline{\mathbb{F}}_q$ together with weight permutation σ. On the one hand, σ induces a Frobenius twist functor on the category coh-$\mathbb{X}$ which gives an $\mathbb{F}_q$-category coh-$(\mathbb{X}, \sigma; q)$ consisting of F-stable objects in coh-$\mathbb{X}$. On the other hand, σ defines a Frobenius morphism on the canonical algebra Λ associated with $\mathbb{X}$ which induces a Frobenius twist functor on mod-Λ. We then prove that the liftings of these two functors on their derived categories D^b(coh-$\mathbb{X}$) and D^b(mod-Λ) are compatible. As an application, we obtain that the bounded derived categories of coh-$(\mathbb{X}, \sigma; q)$ and mod-Λ^F are equivalent.

We organize the paper as follows. Section 2 recalls the notion of Frobenius morphism on an algebra A over $k = \overline{\mathbb{F}}_q$ and gives a brief introduction on the category of coherent sheaves over a weighted projective line $\mathbb{X}$. In Section 3, we consider a weighted projective line $\mathbb{X}$ together with weight permutation σ and define the corresponding $\mathbb{F}_q$-category coh-$(\mathbb{X}, \sigma; q)$. In Section 4, we prove that the Frobenius twist functors on D^b(coh-$\mathbb{X}$) and D^b(mod-Λ) are compatible. As a result, the category coh-$(\mathbb{X}, \sigma; q)$ is derived equivalent to the category of finite dimensional modules over the fixed point algebra Λ^F.

Throughout this paper we always assume that $\mathbb{F}_q$ is a finite field with q elements, $k = \overline{\mathbb{F}}_q$ is the algebraic closure of $\mathbb{F}_q$. For an algebra A over a field, by mod-A we denote the category of all finite dimensional left A-modules.

2. Preliminaries

In this section we recall some basic facts on Frobenius twist functors induced by Frobenius morphisms on k-algebras from [5, 7], and those on the categories of coherent sheaves over weighted projective lines from [9].

We first recall the notion of a Frobenius morphism on a k-algebra. Let V be a k-vector space. An $\mathbb{F}_q$-linear isomorphism $F : V \to V$ is called a Frobenius map on V if

(1) $F(\lambda v) = \lambda^q F(v)$ for all $v \in V$ and $\lambda \in k$,
(2) For any $v \in V, F^t(v) = v$ for some $t \geq 1$.

Let A be a k-algebra with identity 1. A map $F : A \to A$ is called a Frobenius morphism on A if it is a Frobenius map on the underlying k-vector space A which additionally satisfies

$$F(ab) = F(a)F(b) \text{ for all } a, b \in A.$$

Then the F-fixed point algebra $A^F = \{a \in A \mid F(a) = a\}$ is clearly a finite dimensional $\mathbb{F}_q$-algebra.

Suppose that $\mathfrak{f} : k \to k$ is the field automorphism given by $\mathfrak{f}(\lambda) = \lambda^q$. For each A-module M, let $M^{[1]}$ be the new A-module obtained from M by base change via $\mathfrak{f}$:

$$M^{[1]} = M \otimes_{\mathfrak{f}} k,$$

and the action of A on $M^{[1]}$ is given as

$$a \cdot (m \otimes 1) = (F_A^{-1}(a)m) \otimes 1, \ \forall\, a \in A, \ m \in M. \tag{1}$$

Thus, for $m \in M$ and $\lambda \in k$, we have $\lambda m \otimes 1 = m \otimes \lambda^q$. In other words, let $m^{(1)} := m \otimes 1$,

$$\begin{aligned} (m+n)^{(1)} &= m^{(1)} + n^{(1)}, \\ (\lambda m)^{(1)} &= \lambda^q m^{(1)}, \\ a \cdot (m^{(1)}) &= (F_A^{-1}(a)m)^{(1)}. \end{aligned}$$

We call $M^{[1]}$ the Frobenius twist of M. If $f : M \to N$ is an A-module homomorphism, then the map $f^{[1]} := f \otimes 1 : M^{[1]} \to N^{[1]}$ is again an A-module homomorphism.

Proposition 2.1 ([7, Prop. 2.9]). *The correspondence $M \mapsto M^{[1]}$ induces a category equivalence* $(\ )^{[1]}$: mod-$A \to$ mod-A, *called the Frobenius* (*twist*) *functor on* mod-A.

Inductively, define the s-fold Frobenius twist $M^{[s]} := (M^{[s-1]})^{[1]}$ of M and $f^{[s]} := (f^{[s-1]})^{[1]}$ for $s \geq 1$, where $M^{[0]} = M$ and $f^{[0]} = f$ by convention.

Let M be an A-module and F_M be a Frobenius map on M. Define $M^{[F_M]}$ to be the A-module with $M^{[F_M]} = M$ as a vector space and the A-module structure given by

$$a * m = F_M(F_A^{-1}(a)F_M^{-1}(m)). \tag{2}$$

We call $M^{[F_M]}$ the F_M-twist of M. If $f : M \to N$ is an A-module homomorphism, then the map $f^{[F]} = F_N \circ f \circ F_M^{-1} : M^{[F_M]} \to N^{[F_N]}$ is again an A-module homomorphism.

Inductively, define the s-fold F_M-twist $M^{[F_M]^s} := (M^{[F_M]^{s-1}})^{[F_M]}$ of M and $f^{[F]^s} := (f^{[F]^{s-1}})^{[F]} = F_N^s \circ f \circ F_M^{-s}$ for $s \geq 1$, where $M^{[F_M]^0} = M$ and $f^{[F]^0} = f$ by convention.

Lemma 2.2 ([7, Lem. 2.11]). *Let M be an A-module and F_M a Frobenius morphism on M. The F_M-twist $M^{[F_M]}$ and the Frobenius twist $M^{[1]}$ are isomorphic as A-modules.*

Proof. Let $\varpi_M : M \to M^{[1]}$ be the $\mathbb{F}_q$-linear isomorphism given by $\varpi_M(m) = m \otimes 1$, $\varphi_M = \varpi_M \circ F_M^{-1} : M^{[F_M]} \to M^{[1]}$ is a k-linear isomorphism and

$$\begin{aligned} \varphi_M(a * m) &= \varpi_M F_M^{-1}(F_M F_A^{-1}(a) F_M^{-1}(m)) = (F_A^{-1}(a)F_M^{-1}(m))^{(1)} \\ &= a \cdot ((F_M^{-1}(m))^{(1)}) = a \cdot (\varphi_M(m)). \end{aligned}$$

So φ_M is an A-module isomorphism. $\square$

An A-module M is called Frobenius periodic (or F-periodic) if $M \cong M^{[r]}$ for some $r \geq 1$. The minimal positive integer r with this property is called the F-period of M, denoted by $p_F(M)$. If $p_F(M) = 1$, we call M is Frobenius stable (or F-stable).

Let $(\text{mod-}A)^F$ be the category as follows:

Objects: M such that $\phi_M : M^{[1]} \xrightarrow{\sim} M$ in mod-A,

Morphisms: $\mathrm{Hom}_{(\text{mod-}A)^F}(M, N) = \{f \in \mathrm{Hom}_{\text{mod-}A}(M, N) \mid \phi_N \circ f^{[1]} = f \circ \phi_M\}$.

Theorem 2.3 ([6, Thm. 2.10])**.** *The category* $(\text{mod-}A)^F$ *defined above is equivalent to the category* mod-A^F *of finite dimensional left* A^F*-modules.*

From now on, we assume that A is a finite dimensional k-algebra. By [6], the Frobenius twist functor on mod-A can be lifted to the derived category level. Let $C(\text{mod-}A)$ denote the category of (chain) complexes of mod-A. For each complex $\mathcal{M} \in C(\text{mod-}A)$,

$$\mathcal{M} = (M^i, d^i) = \cdots \longrightarrow M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \xrightarrow{d^{i+1}} \cdots,$$

where $d^2 = 0$. Applying the Frobenius functor to each M^i, we obtain a new chain complex

$$\begin{aligned}\mathcal{M}^{[1]} &= ((M^i)^{[1]}, (\, d^i)^{[1]}) \\ &= \cdots \longrightarrow (M^{i-1})^{[1]} \xrightarrow{(d^{i-1})^{[1]}} (M^i)^{[1]} \xrightarrow{(d^i)^{[1]}} (M^{i+1})^{[1]} \xrightarrow{(d^{i+1})^{[1]}} \cdots .\end{aligned}$$

This will be called the Frobenius twist of $\mathcal{M}$. Further, each complex morphism $f = (f^i) : \mathcal{M} \to \mathcal{N}$ induces a morphism $f^{[1]} = ((f^i)^{[1]}) : \mathcal{M}^{[1]} \to \mathcal{N}^{[1]}$. Thus, the Frobenius functor on mod-A induces a functor

$$(\ )^{[1]} = (\ )^{[1]}_{C(\text{mod-}A)} : C(\text{mod-}A) \longrightarrow C(\text{mod-}A),$$

which we still call the Frobenius (twist) functor (on complexes).

A morphism $f : \mathcal{M} \to \mathcal{N}$ is homotopic to zero if and only if so is $f^{[1]}$. Thus, the Frobenius functor $(\ )^{[1]}$ on $C(\text{mod-}A)$ induces a functor

$$(\ )^{[1]} = (\ )^{[1]}_{H(\text{mod-}A)} : H(\text{mod-}A) \longrightarrow H(\text{mod-}A),$$

which is an equivalence of triangulated categories.

A morphism $f : \mathcal{M} \to \mathcal{N}$ is a quasi-isomorphism if and only if so is $f^{[1]}$. Thus, the Frobenius functor $(\ )^{[1]}$ on $H(\text{mod-}A)$ induces a functor

$$(\ )^{[1]} = (\ )^{[1]}_{D^b(\text{mod-}A)} : D^b(\text{mod-}A) \longrightarrow D^b(\text{mod-}A),$$

which is again an equivalence of triangulated categories. If

$$\xi \in \mathrm{Hom}_{D^b(\text{mod-}A)}(\mathcal{M}, \mathcal{N})$$

denotes the equivalence class $\overline{(\mathcal{L}, s, f)}$ of the triple $(\mathcal{L}, s, f)$, where $\mathcal{L}$ is an object in $D^b(\text{mod-}A)$, $s \in \mathrm{Hom}_{H(\text{mod-}A)}(\mathcal{L}, \mathcal{M})$ is quasi-isomorphism, and $f \in \mathrm{Hom}_{H(\text{mod-}A)}(\mathcal{L}, \mathcal{N})$, then $\xi^{[1]}$ is the equivalence class of $(\mathcal{L}^{[1]}, s^{[1]}, f^{[1]})$.

Let $(D^b(\text{mod-}A))^F$ denote the category with

Objects: M such that $\phi_M : M^{[1]} \xrightarrow{\sim} M$ in $D^b(\text{mod-}A)$,

Morphisms: $\mathrm{Hom}_{(D^b(\text{mod-}A))^F}(M, N)$

$$= \{\xi \in \mathrm{Hom}_{D^b(\text{mod-}A)}(M, N) \mid \phi_N \circ \xi^{[1]} = \xi \circ \phi_M\}.$$

Theorem 2.4 ([6, Thm. 5.4])**.** *The category $(D^b(\text{mod-}A))^F$ defined above is equivalent to the category $D^b(\text{mod-}A^F)$.*

We now briefly introduce the category coh-$\mathbb{X}$ of coherent sheaves on a weighted projective line $\mathbb{X}$, and its basic properties. Let $\mathbb{P}^1_k$ be the projective line over k and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ be a collection of distinct closed points of $\mathbb{P}^1_k$. Without loss of generality, we may assume $\lambda_1 = \infty, \lambda_2 = 0$. Further, let $\boldsymbol{p} = (p_1, p_2, \ldots, p_t)$ be a weight sequence, that is, a sequence of positive integers, and let $\mathbb{L} = \mathbb{L}(p)$ denote the rank 1 abelian group

$$\mathbb{L} = \mathbb{L}(\boldsymbol{p}) = \langle \vec{x}_1, \ldots, \vec{x}_t \mid p_1\vec{x}_1 = \cdots = p_t\vec{x}_t \rangle.$$

The element $\vec{c} := p_1\vec{x}_1$ is called the canonical element of $\mathbb{L}$, and, moreover, each element $\vec{x} \in \mathbb{L}$ can be uniquely written in the normal form $\vec{x} = \sum_{i=1}^t l_i\vec{x}_i + l\vec{c}$ with $0 \le l_i \le p_i - 1$ and $l \in \mathbb{Z}$.

Set

$$S = S(\boldsymbol{p}, \boldsymbol{\lambda}) = k[X_1, X_2, \ldots, X_t]/(X_i^{p_i} - (X_2^{p_2} - \lambda_i X_1^{p_1}))_{i=3}^t := k[x_1, x_2, \ldots, x_t].$$

Then S becomes an $\mathbb{L}$-graded commutative algebra by setting $\deg x_i = \vec{x}_i$, $i = 1, \ldots, t$, called the $\mathbb{L}$-graded k-algebra associated with $(\boldsymbol{p}, \boldsymbol{\lambda})$. The weighted projective line $\mathbb{X} = (\mathbb{P}^1_k, \boldsymbol{p}, \boldsymbol{\lambda})$ associated with the pair $(\boldsymbol{p}, \boldsymbol{\lambda})$ is defined to be

$$\mathrm{Spec}^{\mathbb{L}} S, \text{ the spectrum of } \mathbb{L}\text{-graded homogeneous ideals of } S.$$

By an $\mathbb{L}$-graded version of the Serre construction [15], the category of coherent sheaves on $\mathbb{X}$ is defined to be the quotient category

$$\text{coh-}\mathbb{X} = \text{mod}^{\mathbb{L}}\text{-}S/\text{mod}_0^{\mathbb{L}}\text{-}S,$$

where $\text{mod}^{\mathbb{L}}\text{-}S$ is the category of finitely generated $\mathbb{L}$-graded S-modules and $\text{mod}_0^{\mathbb{L}}\text{-}S$ is the Serre subcategory of finite length $\mathbb{L}$-graded S-modules.

Note that $\mathbb{L}$ acts on $\text{mod}^{\mathbb{L}}\text{-}S$ by grading shift: each $\vec{x} \in \mathbb{L}$ defines a functor

$$\vec{x} : \text{mod}^{\mathbb{L}}\text{-}S \longrightarrow \text{mod}^{\mathbb{L}}\text{-}S, \ M \longmapsto M(\vec{x}),$$

where $M(\vec{x})$ is the S-module M with the new grading $M(\vec{x})_{\vec{y}} = M_{\vec{x}+\vec{y}}$. The free module S gives the structure sheaf $\mathcal{O}$, and each line bundle L on $\mathbb{X}$ has the form $L = \mathcal{O}(\vec{x})$ for some uniquely determined $\vec{x} \in \mathbb{L}$.

Theorem 2.5 ([9])**.** *The category coh-$\mathbb{X}$ is k-linear and abelian with the following properties*:

(1) coh-$\mathbb{X}$ *is connected and Noetherian, that is, ascending chains of subobjects are stationary.*
(2) coh-$\mathbb{X}$ *is Ext-finite and hereditary.*

(3) coh-$\mathbb{X}$ *has a Serre duality in the form*

$$D\mathrm{Ext}^{\,1}_{\text{coh-}\mathbb{X}}(X,Y)\cong \mathrm{Hom}_{\text{coh-}\mathbb{X}}(Y,X(\vec{\omega})),$$

where $D=\mathrm{Hom}_k(-,k)$ *and* $\vec{\omega}:=(t-2)\vec{c}-\sum_{i=1}^{t}\vec{x}_i$ *(called the dualizing element of* $\mathbb{L}$*). In particular, Auslander–Reiten translation* $\tau:\text{coh-}\mathbb{X}\to\text{coh-}\mathbb{X}$ *is the grading shift* $M\mapsto M(\vec{\omega})$*.*

(4) coh-$\mathbb{X}$ *has a splitting torsion pair* $(\text{coh}_0\text{-}\mathbb{X},\text{vect-}\mathbb{X})$*, where* coh_0-$\mathbb{X}$ *and* vect-$\mathbb{X}$ *are full subcategories of torsion sheaves and vector bundles, respectively.*

(5) *For any* $\vec{x},\vec{y}\in\mathbb{L}$*,* $\mathrm{Hom}_{\text{coh-}\mathbb{X}}(\mathcal{O}(\vec{x}),\mathcal{O}(\vec{y}))\cong S_{\vec{y}-\vec{x}}$*.*

The category coh_0-$\mathbb{X}$ decomposes into a coproduct coh_λ-$\mathbb{X}$, where coh_λ-$\mathbb{X}$ denotes the uniserial category of finite length sheaves concentrated at the point λ. If λ is an ordinary point, i.e., $\lambda\in\mathbb{H}_k:=\mathbb{P}^1_k\backslash\{\lambda_1,\ldots,\lambda_t\}$, then there is exactly one simple object in coh_λ-$\mathbb{X}$, while for an exceptional point λ_i, coh_{λ_i}-$\mathbb{X}$ has exactly p_i simple objects (up to isomorphism). The simple finite length sheaf at an ordinary point λ is given as the cokernel of the exact sequence

$$0\longrightarrow\mathcal{O}\xrightarrow{x_2^{p_2}-\lambda x_1^{p_1}}\mathcal{O}(\vec{c})\longrightarrow S_\lambda\longrightarrow 0,$$

while the p_i exceptional simple sheaves concentrated at λ_i arise as the cokernels of exact sequences

$$0\longrightarrow\mathcal{O}(j\vec{x}_i)\xrightarrow{x_i}\mathcal{O}((j+1)\vec{x}_i)\longrightarrow S_{i,j}\longrightarrow 0,\qquad \forall\, j\in\mathbb{Z}/p_i\mathbb{Z}.$$

3. Weighted projective lines with weight permutation

In this section, we consider a weighted projective line $\mathbb{X}$ together with a permutation σ on the weights. We will show that σ induces a Frobenius twist functor on the category coh-$\mathbb{X}$ of coherent sheaves and define a subcategory of coh-$\mathbb{X}$ whose objects are invariant under the Frobenius twist functor. We keep all the notations in the previous section.

Definition. Let $\mathbb{X}$ be the weighted projective line associated to $(\boldsymbol{p},\boldsymbol{\lambda})$, where $\boldsymbol{p}=(p_1,\ldots,p_t)$ and $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_t)$ with $\lambda_1=\infty$ and $\lambda_2=0$. A weight permutation σ for $\mathbb{X}$ is a permutation of $\{1,\ldots,t\}$ such that $p_{\sigma(i)}=p_i$ and

(1) $\lambda_{\sigma(i)}=\lambda_i^q$ for $i\in\{3,\ldots,t\}$ when $t>3$;
(2) either $\lambda_3\in\mathbb{F}_q$ and $\sigma=id$, or $\lambda_3=-1$ and $\sigma=(12)$ when $t=3$;
(3) either $\sigma=id$ or $\sigma=(12)$ when $t=2$.

The triple $(\mathbb{X},\sigma;q)$ is called a weighted projective line with weight permutation σ over $k=\overline{\mathbb{F}}_q$.

Example 3.1. (1) Let $k=\overline{\mathbb{F}}_3$, $\boldsymbol{p}=(2,2,2,2)$, and $\boldsymbol{\lambda}=(\infty,0,\lambda,\lambda^3)$, where $\lambda=\bar{x}\in\mathbb{F}_3[x]/(x^2+1)\cong\mathbb{F}_{3^2}$. Then $\sigma=(34)$ is a weight permutation of $\mathbb{X}$.

(2) Let $\boldsymbol{p}=(2,2,3)$ and $\boldsymbol{\lambda}=(\infty,0,-1)$. Then $\sigma=(12)$ is a weight permutation of $\mathbb{X}$.

Let $(\mathbb{X},\sigma)$ be a weighted projective line $\mathbb{X}$ with weight permutation σ and $S(\boldsymbol{p},\boldsymbol{\lambda})$ be the associated $\mathbb{L}$-graded k-algebra. Then σ induces a Frobenius morphism

$$F = F_S = F_{\mathbb{X},\sigma;q} : S(\boldsymbol{p},\boldsymbol{\lambda}) \longrightarrow S(\boldsymbol{p},\boldsymbol{\lambda})$$
$$\sum_{\mathbf{n}} c_{\mathbf{n}} f_{\mathbf{n}} \longmapsto \sum_{\mathbf{n}} c_{\mathbf{n}}^q \sigma(f_{\mathbf{n}}),$$

where $\mathbf{n} = (n_1,\dots,n_t) \in \mathbb{N}^t$, $c_{\mathbf{n}} \in k$, $f_{\mathbf{n}} = x_1^{n_1}x_2^{n_2}\cdots x_t^{n_t}$, and $\sigma(f_{\mathbf{n}}) = x_{\sigma(1)}^{n_1}x_{\sigma(2)}^{n_2}\cdots x_{\sigma(t)}^{n_t}$.

The weight permutation σ induces a group homomorphism

$$\sigma_{\mathbb{L}} : \mathbb{L} \longrightarrow \mathbb{L},\ \vec{x}_i \longmapsto \vec{x}_{\sigma(i)},\ \forall\, 1 \le i \le t.$$

By (1), each $\mathbb{L}$-graded S-module M gives rise to a new $\mathbb{L}$-graded S-module $M^{[1]} = M \otimes_{\mathfrak{f}} k$ with $\mathbb{L}$-graded structure given by

$$(M^{[1]})_{\vec{x}} = M_{\sigma_{\mathbb{L}}^{-1}(\vec{x})} \otimes_{\mathfrak{f}} k,\ \forall\, \vec{x} \in \mathbb{L}.$$

Indeed, if $a \in S_{\vec{x}}$ and $m^{(1)} \in (M^{[1]})_{\vec{y}} = M_{\sigma_{\mathbb{L}}^{-1}(\vec{y})} \otimes_{\mathfrak{f}} k$, then

$$a \cdot m^{(1)} = (F^{-1}(a)m)^{(1)} \in (M^{[1]})_{\vec{x}+\vec{y}}.$$

We call $M^{[1]}$ the Frobenius twist of $\mathbb{L}$-graded S-module M. If $f : M \to N$ is an $\mathbb{L}$-graded S-module homomorphism, then

$$f^{[1]}((M^{[1]})_{\vec{x}}) = (f\otimes 1)(M_{\sigma_{\mathbb{L}}^{-1}(\vec{x})}\otimes_{\mathfrak{f}} k) = fM_{\sigma_{\mathbb{L}}^{-1}(\vec{x})}\otimes_{\mathfrak{f}} k \subseteq N_{\sigma_{\mathbb{L}}^{-1}(\vec{x})}\otimes_{\mathfrak{f}} k = (N^{[1]})_{\vec{x}}.$$

Hence, $f^{[1]} : M^{[1]} \to N^{[1]}$ is again an $\mathbb{L}$-graded S-module homomorphism.

Proposition 3.2. (1) *The functor* $(\ )^{[1]}_{\mathrm{mod}^{\mathbb{L}}\text{-}S} : \mathrm{mod}^{\mathbb{L}}\text{-}S \to \mathrm{mod}^{\mathbb{L}}\text{-}S$ *is an equivalence. Moreover, its restriction to* $\mathrm{mod}_0^{\mathbb{L}}\text{-}S$ *gives an equivalence* $\mathrm{mod}_0^{\mathbb{L}}\text{-}S \to \mathrm{mod}_0^{\mathbb{L}}\text{-}S$.

(2) *The functor* $(\ )^{[1]}_{\mathrm{mod}^{\mathbb{L}}\text{-}S}$ *compatible with the grading shift, that is, for each* $M \in \mathrm{mod}^{\mathbb{L}}\text{-}S$ *and* $\vec{x} \in \mathbb{L}$,

$$M(\vec{x})^{[1]} = (M^{[1]})(\sigma_{\mathbb{L}}(\vec{x})).$$

Proof. (1) Take $M \in \mathrm{mod}^{\mathbb{L}}\text{-}S$ and consider the k-vector space $M^{[-1]} = M \otimes_{\mathfrak{f}^{-1}} k$, where $\mathfrak{f}^{-1} : k \to k, \lambda \mapsto \sqrt[q]{\lambda}$. Define an S-module structure on $M^{[-1]}$ via

$$a \cdot (m \otimes 1) = (F_S(a)m) \otimes 1,\ \forall\, a \in S,\ m \in M,$$

as well as an $\mathbb{L}$-grading on $M^{[-1]}$ given by

$$(M^{[-1]})_{\vec{x}} = M_{\sigma_{\mathbb{L}}(\vec{x})} \otimes_{\mathfrak{f}} k,\ \forall\, \vec{x} \in \mathbb{L}.$$

Then $M^{[-1]}$ lies in $\mathrm{mod}^{\mathbb{L}}\text{-}S$. Moreover, if $f : M \to N$ is an $\mathbb{L}$-graded S-module homomorphism, then $f^{[-1]} = f \otimes 1$ is again an $\mathbb{L}$-graded S-module homomorphism. Thus, we obtain a functor

$$(\ )^{[-1]}_{\mathrm{mod}^{\mathbb{L}}\text{-}S} : \mathrm{mod}^{\mathbb{L}}\text{-}S \longrightarrow \mathrm{mod}^{\mathbb{L}}\text{-}S.$$

It is easy to check that

$$(\ )^{[1]}_{\mathrm{mod}^{\mathbb{L}}\text{-}S} \circ (\ )^{[-1]}_{\mathrm{mod}^{\mathbb{L}}\text{-}S} \cong \mathrm{id}_{\mathrm{mod}^{\mathbb{L}}\text{-}S} \cong (\ )^{[-1]}_{\mathrm{mod}^{\mathbb{L}}\text{-}S} \circ (\ )^{[1]}_{\mathrm{mod}^{\mathbb{L}}\text{-}S}.$$

Hence, $(\ )^{[1]}_{\mathrm{mod}^{\mathbb{L}}\text{-}S} : \mathrm{mod}^{\mathbb{L}}\text{-}S \to \mathrm{mod}^{\mathbb{L}}\text{-}S$ is an equivalence.

If M is a finite length L-graded S-module, so is $M^{[1]}$. Thus, the restriction of the above functor to $\mathrm{mod}^{\mathbb{L}}_0\text{-}S$

$$(\ )^{[1]}_{\mathrm{mod}^{\mathbb{L}}_0\text{-}S} : \mathrm{mod}^{\mathbb{L}}_0\text{-}S \longrightarrow \mathrm{mod}^{\mathbb{L}}_0\text{-}S$$

is also a category equivalence.

(2) By the definition, for each $\vec{y} \in \mathbb{L}$,

$$\begin{aligned}(M(\vec{x})^{[1]})_{\vec{y}} &= M(\vec{x})_{\sigma_{\mathbb{L}}^{-1}(\vec{y})} \otimes_{\mathfrak{f}} k = M_{\vec{x}+\sigma_{\mathbb{L}}^{-1}(\vec{y})} \otimes_{\mathfrak{f}} k \\ &= (M^{[1]})_{\sigma_{\mathbb{L}}(\vec{x})+\vec{y}} = ((M^{[1]})(\sigma_{\mathbb{L}}(\vec{x})))_{\vec{y}}.\end{aligned}$$

By the definition, the actions of S on the homogeneous spaces $(M(\vec{x})^{[1]})_{\vec{y}}$ and $(M^{[1]})(\sigma_{\mathbb{L}}(\vec{x}))_{\vec{y}}$ coincide. Hence, $M(\vec{x})^{[1]} = (M^{[1]})(\sigma_{\mathbb{L}}(\vec{x}))$. □

The functor $(\ )^{[1]}_{\mathrm{mod}^{\mathbb{L}}\text{-}S}$ is called the Frobenius (twist) functor on $\mathrm{mod}^{\mathbb{L}}\text{-}S$.

Now let M be an $\mathbb{L}$-graded S-module together with a Frobenius map $F_M : M \to M$. As in Section 2, we obtain an S-module $M^{[F_M]}$. Moreover, $M^{[F_M]}$ is also $\mathbb{L}$-graded with

$$(M^{[F_M]})_{\vec{x}} = F_M(M_{\sigma_{\mathbb{L}}^{-1}(\vec{x})}),\ \forall\, \vec{x} \in \mathbb{L}.$$

Indeed, if $a \in S_{\vec{x}}$ and $m \in (M^{[F_M]})_{\vec{y}}$, then by (2),

$$a * m = F_M(F_S^{-1}(a)F_M^{-1}(m)) \in F_M(M_{\sigma_{\mathbb{L}}^{-1}(\vec{x})+\sigma_{\mathbb{L}}^{-1}(\vec{y})}) = (M^{[F_M]})_{\vec{x}+\vec{y}}.$$

We call $M^{[F_M]}$ the F_M-twist of $\mathbb{L}$-graded S-module M. Moreover, if $f : M \to N$ is an $\mathbb{L}$-graded S-module homomorphism and F_M, F_N are Frobenius morphisms, respectively, then

$$\begin{aligned}f^{[F]}((M^{[F_M]})_{\vec{x}}) &= (F_N \circ f \circ F_M^{-1})(F_M(M_{\sigma_{\mathbb{L}}^{-1}(\vec{x})})) \\ &= (F_N \circ f)(M_{\sigma_{\mathbb{L}}^{-1}(\vec{x})}) \subseteq F_N(N_{\sigma_{\mathbb{L}}^{-1}(\vec{x})}) = (N^{[F_N]})_{\vec{x}}.\end{aligned}$$

Hence, $f^{[F]} : M^{[F_M]} \to N^{[F_N]}$ is again an $\mathbb{L}$-graded S-module homomorphism.

Proposition 3.3. *Let M be an $\mathbb{L}$-graded S-module. Then $\varphi_M = \varpi_M \circ F_M^{-1} : M^{[F_M]} \to M^{[1]}$ is an isomorphism of $\mathbb{L}$-graded S-modules.*

Proof. By Lemma 2.2, the homomorphism φ_M is an S-module isomorphism. It suffices to check that φ_M keeps $\mathbb{L}$-gradings. Indeed, for each $m \in (M^{[F_M]})_{\vec{x}}$,

$$\varphi_M(m) = \varpi_M \circ F_M^{-1}(m) \in \varpi_M(M_{\sigma_{\mathbb{L}}^{-1}(\vec{x})}) = (M^{[1]})_{\vec{x}}.$$

□

By [2, Lem. 1.2.4], the category coh-$\mathbb{X}$ can be viewed as the localization of $\mathrm{mod}^{\mathbb{L}}$-$S$ at the class $\mathscr{S}$ of all morphisms in $\mathrm{mod}^{\mathbb{L}}$-$S$ with kernel and cokernel in $\mathrm{mod}_0^{\mathbb{L}}$-$S$. We claim that $f \in \mathscr{S}$ if and only if $f^{[1]} \in \mathscr{S}$. If $f : M \to N \in \mathscr{S}$, then there is an exact sequence

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow M \xrightarrow{f} N \longrightarrow \operatorname{Coker} f \longrightarrow 0.$$

Applying $(\ )^{[1]} : \mathrm{mod}^{\mathbb{L}}\text{-}S \to \mathrm{mod}^{\mathbb{L}}\text{-}S$ to the exact sequence above gives the exact sequence

$$0 \longrightarrow (\operatorname{Ker} f)^{[1]} \longrightarrow M^{[1]} \xrightarrow{f^{[1]}} N^{[1]} \longrightarrow (\operatorname{Coker} f)^{[1]} \longrightarrow 0.$$

Since the functor $(\ )^{[1]}$ is an equivalence, we have

$$\operatorname{Ker}(f^{[1]}) \cong (\operatorname{Ker} f)^{[1]} \in \mathrm{mod}_0^{\mathbb{L}}\text{-}S,\ \operatorname{Coker}(f^{[1]}) \cong (\operatorname{Coker} f)^{[1]} \in \mathrm{mod}_0^{\mathbb{L}}\text{-}S.$$

Hence, $f^{[1]} \in \mathscr{S}$. Similarly, $f^{[1]} \in \mathscr{S}$ implies $f \in \mathscr{S}$. Therefore, the Frobenius functor on $\mathrm{mod}^{\mathbb{L}}$-$S$ induces a functor

$$(\ )^{[1]} = (\ )^{[1]}_{\text{coh-}\mathbb{X}} : \text{coh-}\mathbb{X} \longrightarrow \text{coh-}\mathbb{X},$$

which is again an equivalence.

Proposition 3.4. *Let $(\mathbb{X}, \sigma; q)$ be a weighted projective line with weight permutation σ. Then*

(1) $\mathcal{O}(\vec{x})^{[1]} \cong \mathcal{O}(\sigma_{\mathbb{L}}(\vec{x}))$, $\forall\, \vec{x} \in \mathbb{L}$;
(2) $S_{i,j}^{[1]} \cong S_{\sigma(i),j}$, $\forall\, 1 \le i \le t$, $j \in \mathbb{Z}/p_i\mathbb{Z}$;
(3) *if σ fixes* 1 *and* 2, *then* $(S_\lambda)^{[1]} = S_{\lambda^q}$, *where* $\lambda \in \mathbb{H}_k$; *if σ swaps* 1 *and* 2, *then* $(S_\lambda)^{[1]} = S_{1/\lambda^q}$;
(4) *if $M \in$ coh-$\mathbb{X}$ is indecomposable, then $\tau(M^{[1]}) = (\tau M)^{[1]}$, where τ is the Auslander–Reiten translation in* coh-$\mathbb{X}$.

Proof. (1) The map $\psi_{\vec{x}} : \mathcal{O}(\vec{x})^{[1]} \to \mathcal{O}(\sigma_{\mathbb{L}}(\vec{x}))$ taking $m \otimes 1 \longmapsto F_S(m)$ is an isomorphism of S-modules. Moreover, $\psi_{\vec{x}}((\mathcal{O}(\vec{x})^{[1]})_{\vec{y}}) = \psi_{\vec{x}}(S_{\vec{x}+\sigma_{\mathbb{L}}^{-1}(\vec{y})} \otimes_{\mathfrak{f}} k) \subseteq S_{\sigma_{\mathbb{L}}(\vec{x})+\vec{y}} = \mathcal{O}(\sigma_{\mathbb{L}}(\vec{x}))_{\vec{y}}$. Hence, $\psi_{\vec{x}}$ is an $\mathbb{L}$-graded S-module isomorphism.

(2) For given $1 \le i \le t$, $j \in \mathbb{Z}/p_i\mathbb{Z}$, there are two short exact sequences

$$0 \longrightarrow \mathcal{O}(j\vec{x}_i) \xrightarrow{x_i} \mathcal{O}((j+1)\vec{x}_i) \longrightarrow S_{i,j} \longrightarrow 0 \quad \text{and}$$

$$0 \longrightarrow \mathcal{O}(j\vec{x}_{\sigma(i)}) \xrightarrow{x_{\sigma(i)}} \mathcal{O}((j+1)\vec{x}_{\sigma(i)}) \longrightarrow S_{\sigma(i),j} \longrightarrow 0.$$

Applying the functor $(\ )^{[1]}$ to the first one gives the exact sequence

$$0 \longrightarrow \mathcal{O}(j\vec{x}_i)^{[1]} \xrightarrow{x_i \otimes 1} \mathcal{O}((j+1)\vec{x}_i)^{[1]} \longrightarrow S_{i,j}^{[1]} \longrightarrow 0.$$

Hence, from the commutative diagram

$$\begin{tikzcd} S(j\vec{x}_i)^{[1]} \arrow[r, "x_i\otimes 1"] \arrow[d, "\psi_{j\vec{x}_i}"', "\cong"] & S((j+1)\vec{x}_i)^{[1]} \arrow[d, "\cong"', "\psi_{(j+1)\vec{x}_i}"] \\ S(j\vec{x}_{\sigma(i)}) \arrow[r, "x_{\sigma(i)}"] & S((j+1)\vec{x}_{\sigma(i)}) \end{tikzcd}$$

we obtain that $S_{i,j}^{[1]} \cong S_{\sigma(i),j}$.

(3) Take $\lambda \in \mathbb{H}_k$. If σ fixes 1 and 2, then we have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}^{[1]} \arrow[r, "z_\lambda"] \arrow[d, "\psi_0"', "\cong"] & \mathcal{O}(\vec{c})^{[1]} \arrow[r] \arrow[d, "\cong"', "\psi_{\vec{c}}"] & S_\lambda^{[1]} \arrow[r] \arrow[d, dotted] & 0 \\ 0 \arrow[r] & \mathcal{O} \arrow[r, "x_2^{p_2}-\lambda^q x_1^{p_1}"] & \mathcal{O}(\vec{c}) \arrow[r] & S_{\lambda^q} \arrow[r] & 0, \end{tikzcd}$$

where $z_\lambda = (x_2^{p_2} - \lambda x_1^{p_1}) \otimes 1$. Thus, $(S_\lambda)^{[1]} \cong S_{\lambda^q}$. If σ swaps 1 and 2, then $p := p_1 = p_2$ and the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}^{[1]} \arrow[r, "z_\lambda"] \arrow[d, "\psi_0"', "\cong"] & \mathcal{O}(\vec{c})^{[1]} \arrow[r] \arrow[d, "\cong"', "-\frac{1}{\lambda^q}\psi_{\vec{c}}"] & S_\lambda^{[1]} \arrow[r] \arrow[d, dotted] & 0 \\ 0 \arrow[r] & \mathcal{O} \arrow[r, "x_2^{p}-(\frac{1}{\lambda^q})x_1^{p}"] & \mathcal{O}(\vec{c}) \arrow[r] & S_{1/\lambda^q} \arrow[r] & 0 \end{tikzcd}$$

gives the isomorphism $(S_\lambda)^{[1]} \cong S_{1/\lambda^q}$.

(4) Since $\sigma_{\mathbb{L}}(\vec{\omega}) = \vec{\omega}$, we have by Proposition 3.2(2),

$$\tau(M^{[1]}) = M^{[1]}(\vec{\omega}) = M^{[1]}(\sigma_{\mathbb{L}}(\vec{\omega})) = (M(\vec{\omega}))^{[1]} = (\tau M)^{[1]}. \qquad \square$$

By $(\text{coh-}\mathbb{X})^F$ we denote the category with

Objects: M together with an isomorphism $\phi_M : M^{[1]} \xrightarrow{\sim} M$ in coh-$\mathbb{X}$,

Morphisms: $\operatorname{Hom}_{(\text{coh-}\mathbb{X})^F}(M,N) = \{\zeta \in \operatorname{Hom}_{\text{coh-}\mathbb{X}}(M,N) \mid \phi_N \circ \zeta^{[1]} = \zeta \circ \phi_M\}$.

Since $(\lambda f)^{[1]} = \lambda f^{[1]}$ for each $f : M \to N$ and $\lambda \in \mathbb{F}_q$, it follows that $(\text{coh-}\mathbb{X})^F$ is an $\mathbb{F}_q$-category. In the following, we denote $(\text{coh-}\mathbb{X})^F$ by coh-$(\mathbb{X}, \sigma; q)$, and its objects will be called “coherent sheaves” on $(\mathbb{X}, \sigma; q)$.

Remark 3.5. By an argument similar to that in [5, Thm. 7.4], we infer that coh-$(\mathbb{X}, \sigma; q)$ has Auslander–Reiten sequences, which are obtained by folding those in coh-$\mathbb{X}$.

As in [6], the Frobenius functor $(\ )^{[1]}$: coh-$\mathbb{X} \to$ coh-$\mathbb{X}$ can be lifted to a Frobenius functor $(\ )^{[1]}_{D^b(\text{coh-}\mathbb{X})} : D^b(\text{coh-}\mathbb{X}) \to D^b(\text{coh-}\mathbb{X})$. This is an equivalence of triangulated categories and gives an $\mathbb{F}_q$-category $(D^b(\text{coh-}\mathbb{X}))^F$. Moreover, each object in $D^b(\text{coh-}(\mathbb{X}, \sigma; q))$ can be viewed as an object in $(D^b(\text{coh-}\mathbb{X}))^F$.

Remark 3.6. In [8], the author studies the quotient category $\mathcal{T}^G$ of a triangulated category $\mathcal{T}$ with an action of a group G. It is shown that under certain condition, $\mathcal{T}^G$ is again a triangulated category; see [8, Cor. 6.10]. This implies that $(D^b(\text{coh-}\mathbb{X}))^F$ is a triangulated category.

Theorem 3.7. *The* $\mathbb{F}_q$*-category* $\text{coh-}(\mathbb{X}, \sigma; q)$ *is an abelian category.*

Proof. We first show that $\text{coh-}(\mathbb{X}, \sigma; q)$ is an abelian category. Let

$$M = (M, \phi_M) \text{ and } N = (N, \phi_N)$$

be two objects in $\text{coh-}(\mathbb{X}, \sigma; q)$ and $\zeta \in \operatorname{Hom}_{\text{coh-}(\mathbb{X},\sigma;q)}(M, N)$. It suffices to show that ζ admits a kernel and a cokernel, and the canonical factorization

$$\begin{tikzcd}
\operatorname{Ker}\zeta \arrow[r, "\zeta'"] & M \arrow[r, "\zeta"] \arrow[d] & N \arrow[r, "\zeta''"] & \operatorname{Coker}\zeta \\
& \operatorname{Coker}\zeta' \arrow[r, "\bar{\zeta}"] & \operatorname{Ker}\zeta'' \arrow[u] &
\end{tikzcd}$$

of ζ induces an isomorphism $\bar{\zeta}$. Then we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K^{[1]} \arrow[r] \arrow[d, dotted, "\cong"] & M^{[1]} \arrow[r, "\zeta^{[1]}"] \arrow[d, "\phi_M"', "\cong"] & N^{[1]} \arrow[r] \arrow[d, "\phi_N"', "\cong"] & C^{[1]} \arrow[r] \arrow[d, dotted, "\cong"] & 0 \\
0 \arrow[r] & K \arrow[r] & M \arrow[r, "\zeta"] & N \arrow[r] & C \arrow[r] & 0,
\end{tikzcd}$$

where $K = \operatorname{Ker}\zeta$, $C = \operatorname{Coker}\zeta$. The upper sequence is exact, since $(\ )^{[1]}$ is an equivalence. It implies that $\operatorname{Ker}\zeta$ and $\operatorname{Coker}\zeta$ are both in $\text{coh-}(\mathbb{X}, \sigma; q)$. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & K^{[1]} \arrow[r] \arrow[dd] & M^{[1]} \arrow[rr, "\zeta^{[1]}"] \arrow[dr, "\pi^{[1]}"] \arrow[dd, "\phi_M"] & & N^{[1]} \arrow[rr] \arrow[dd] & & C^{[1]} \arrow[r] \arrow[dd] & 0 \\
& & & M^{[1]}/K^{[1]} \arrow[rr, "g^{[1]}"] \arrow[dd, "\phi_{M/K}"] & & N'^{[1]} \arrow[ul, "\iota^{[1]}"'] \arrow[dd, "\phi_{N'}"] & & \\
0 \arrow[r] & K \arrow[r] & M \arrow[rr, "\zeta"] \arrow[dr, "\pi"'] & & N \arrow[rr] & & C \arrow[r] & 0. \\
& & & M/K \arrow[rr, "g"] & & N' \arrow[ul, "\iota"] & &
\end{tikzcd}$$

We claim that $\phi_{N'} g^{[1]} = g\phi_{M/K}$. Let $g' = \phi_{N'}^{-1} g \phi_{M/K}$, we get $\iota^{[1]} g' \pi^{[1]} = \iota^{[1]} \phi_{N'}^{-1} g \phi_{M/K} \pi^{[1]} = \phi_N^{-1} \iota g \pi \phi_M = \phi_N^{-1} \zeta \phi_M = \zeta^{[1]} = \iota^{[1]} g^{[1]} \pi^{[1]}$. Note that g' and $g^{[1]}$ are isomorphisms, $\iota^{[1]}$ is a monomorphism, and $\pi^{[1]}$ is an epimorphism. Thus $g^{[1]} = g'$, i.e., $\phi_{N'} g^{[1]} = g\phi_{M/K}$. So $\text{coh-}(\mathbb{X}, \sigma; q)$ is abelian. □

It will be seen in the next section that $\text{coh-}(\mathbb{X}, \sigma; q)$ is indeed hereditary and Ext-finite.

4. Valued canonical algebras associated with weight permutations

In this section we point out that a weight permutation σ of a weighted projective line $\mathbb{X}$ induces a Frobenius morphism F on the canonical algebra $\Lambda = \Lambda(\boldsymbol{p}, \boldsymbol{\lambda})$ associated with $\mathbb{X}$. We then prove that the Frobenius twist functors on $D^b(\text{coh-}\mathbb{X})$ and $D^b(\text{mod-}\Lambda)$ are compatible. As a result, the category coh-$(\mathbb{X}, \sigma; q)$ is derived equivalent to the fixed point algebra Λ^F.

Let $\mathbb{X} = (\mathbb{P}^1_k, \boldsymbol{p}, \boldsymbol{\lambda})$ be a weighted projective line with $\boldsymbol{p} = (p_1, \ldots, p_t)$, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_t)$, $\lambda_1 = \infty, \lambda_2 = 0$. The canonical algebra $\Lambda = \Lambda(\boldsymbol{p}, \boldsymbol{\lambda})$ associated with $\mathbb{X}$ is by definition the finite dimensional algebra given by the quiver $Q = Q(\boldsymbol{p}, \boldsymbol{\lambda})$

$$\begin{tikzcd}
& \vec{x}_1 \arrow[ldd, "X_{11}"'] & 2\vec{x}_1 \arrow[l, "X_{12}"'] & \cdots \arrow[l, "X_{13}"'] & (p_1-1)\vec{x}_1 \arrow[l, "X_{1(p_1-1)}"'] & \\
& \vec{x}_2 \arrow[ld, "X_{21}"] & 2\vec{x}_2 \arrow[l, "X_{22}"'] & \cdots \arrow[l, "X_{23}"'] & (p_2-1)\vec{x}_2 \arrow[l, "X_{2(p_2-1)}"'] & \\
\vec{0} & & & & & \vec{c} \arrow[luu, "X_{1p_1}"'] \arrow[lu, "X_{2p_2}"] \arrow[ldd, "X_{tp_t}"'] \\
& \vdots & & & & \\
& \vec{x}_t \arrow[luu, "X_{t1}"] & 2\vec{x}_t \arrow[l, "X_{t2}"'] & \cdots \arrow[l, "X_{t3}"'] & (p_t-1)\vec{x}_t \arrow[l, "X_{t(p_t-1)}"'] &
\end{tikzcd}$$

with the defining relations

$$X_i^{p_i} = X_2^{p_2} - \lambda_i X_1^{p_1},\ i = 3, \ldots, t, \tag{3}$$

where $X_j^{p_j} := X_{j1} \cdots X_{jp_j}$ for $j = 1, \ldots, t$. In other words,

$$\Lambda = kQ/\langle X_i^{p_i} - (X_2^{p_2} - \lambda_i X_1^{p_1}) \mid 3 \leq i \leq t\rangle.$$

Lemma 4.1 ([9, Prop. 4.1])**.** *The object $T = \bigoplus_{0 \leq \vec{x} \leq \vec{c}} \mathcal{O}(\vec{x})$ is the canonical tilting sheaf in* coh-$\mathbb{X}$ *such that its endomorphism algebra* $(\operatorname{End}_{\text{coh-}\mathbb{X}}(T))^{\text{op}}$ *is isomorphic to $\Lambda = \Lambda(\boldsymbol{p}, \boldsymbol{\lambda})$. In particular, there is a derived equivalence $D^b(\text{coh-}\mathbb{X}) \cong D^b(\text{mod-}\Lambda)$.*

Let σ be a permutation of weights of $\mathbb{X}$. On the one hand, by Proposition 3.4, we can identify $T^{[1]}$ with T. Then the correspondence $f \mapsto f^{[1]}$ induces a Frobenius morphism F_T on $(\operatorname{End}_{\text{coh-}\mathbb{X}}(T))^{\text{op}}$.

On the other hand, σ induces an automorphism of Q, still denoted by σ, given by

$$\sigma(\vec{0}) = \vec{0},\ \sigma(\vec{c}) = \vec{c},\ \sigma(k_i\vec{x}_i) = k_i\vec{x}_{\sigma(i)},\ \sigma(X_{ij_i}) = X_{\sigma(i)j_i},$$

where $1 \leq i \leq t$, $1 \leq k_i \leq p_i - 1, 1 \leq j_i \leq p_i$. It is clear that σ permutes the relations in (3), and thus, induces a Frobenius morphism

$$F = F_{\Lambda,\sigma;q} :\ \Lambda \longrightarrow \Lambda,\ \sum_s a_s\overline{p_s} \longmapsto \sum_s a_s^q \overline{\sigma(p_s)}, \tag{4}$$

where the p_s are paths in Q, $\sigma(p_s) = \sigma(\rho_n)\ldots\sigma(\rho_1)$ if $p_s = \rho_n \ldots \rho_1$ with $\rho_i \in Q_1$, and $\overline{p_s}, \overline{\sigma(p_s)}$ denote their residue classes in Λ, respectively. By the construction, we have the following result.

Lemma 4.2. *The Frobenius morphisms F_T and $F_{\Lambda,\sigma;q}$ are compatible with the canonical isomorphism* $(\mathrm{End}_{\text{coh-}\mathbb{X}}(T))^{\mathrm{op}} \cong \Lambda$.

Furthermore, the F-fixed point algebra

$$\Lambda^F = \{a \in \Lambda \mid F(a) = a\}$$

is an $\mathbb{F}_q$-algebra, called the valued canonical algebra associated with weight permutation σ.

Since Λ is a finite dimensional k-algebra, we have by Section 2 the Frobenius (twist) functor

$$(\ )^{[1]} = (\ )^{[1]}_{\text{mod-}\Lambda} : \text{mod-}\Lambda \longrightarrow \text{mod-}\Lambda.$$

Furthermore, the Frobenius twist functor can be lifted step by step to the chain complex level, homotopy level and finally to the derived category level:

$$(\ )^{[1]} = (\ )^{[1]}_{D^b(\text{mod-}\Lambda)} : D^b(\text{mod-}\Lambda) \longrightarrow D^b(\text{mod-}\Lambda).$$

Moreover, $(\ )^{[1]}_{D^b(\text{mod-}\Lambda)}$ is an equivalence of triangulated categories.

In view of the lemmas above, we identify $(\mathrm{End}_{\text{coh-}\mathbb{X}}(T))^{\mathrm{op}}$ with $\Lambda = \Lambda(\boldsymbol{p}, \boldsymbol{\lambda})$. Then the functor

$$\mathrm{Hom}_{\text{coh-}\mathbb{X}}(T, -) : \text{coh-}\mathbb{X} \longrightarrow \text{mod-}\Lambda$$

induces an equivalence $\operatorname{add} T \xrightarrow{\sim} \operatorname{proj} \Lambda$ and admits a left adjoint

$$- \otimes_\Lambda T : \text{mod-}\Lambda \longrightarrow \text{coh-}\mathbb{X}.$$

Given a Λ-module M with a projective resolution

$$P_1 = \mathrm{Hom}_\Lambda(T, T_0) \longrightarrow P_0 = \mathrm{Hom}_\Lambda(T, T_1) \longrightarrow M \longrightarrow 0,$$

$M \otimes_\Lambda T$ is by definition the cokernel of the corresponding morphism $T_1 \to T_0$ in $\operatorname{add} T$.

Lemma 4.3 ([10])**.** *Let $\mathbb{X} = (\mathbb{P}^1_k, \boldsymbol{p}, \boldsymbol{\lambda})$ be a weighted projective line, $T = \bigoplus_{0 \le \vec{x} \le \vec{c}} \mathcal{O}(\vec{x})$, and $\Lambda = \Lambda(\boldsymbol{p}, \boldsymbol{\lambda}) = (\mathrm{End}_{\text{coh-}\mathbb{X}}(T))^{\mathrm{op}}$. Then the derived functor*

$$- \otimes^{\boldsymbol{L}}_\Lambda T : D^b(\text{mod-}\Lambda) \xrightarrow{\sim} K^b(\text{proj}\Lambda) \longrightarrow D^b(\text{coh-}\mathbb{X})$$

is an equivalence of triangulated categories which takes a complex $\mathcal{P}$ of projective Λ-modules to $\mathcal{P} \otimes_\Lambda T$, and its right adjoint $\boldsymbol{R}\mathrm{Hom}_{\text{coh-}\mathbb{X}}(T, -)$ is a quasi-inverse.

Theorem 4.4. *There are canonical natural isomorphisms*

$$(\ )^{[1]}_{D^b(\text{mod-}\Lambda)} \circ \boldsymbol{R}\mathrm{Hom}_{\text{coh-}\mathbb{X}}(T, -) \cong \boldsymbol{R}\mathrm{Hom}_{\text{coh-}\mathbb{X}}(T, -) \circ (\ )^{[1]}_{D^b(\text{coh-}\mathbb{X})},$$

$$(\ )^{[1]}_{D^b(\text{coh-}\mathbb{X})} \circ (- \otimes^{\boldsymbol{L}}_\Lambda T) \cong (- \otimes^{\boldsymbol{L}}_\Lambda T) \circ (\ )^{[1]}_{D^b(\text{mod-}\Lambda)}.$$

Proof. By Proposition 3.4, $T^{[1]} \cong T$ with isomorphism $\phi = \phi_T$. Thus, for each $X \in$ coh-$\mathbb{X}$, there is a canonical k-vector space isomorphism

$$\operatorname{Hom}_{\text{coh-}\mathbb{X}}(T, X)^{[1]} \cong \operatorname{Hom}_{\text{coh-}\mathbb{X}}(T^{[1]}, X^{[1]}) \cong \operatorname{Hom}_{\text{coh-}\mathbb{X}}(T, X^{[1]}),$$
$$f \otimes_{\mathfrak{f}} 1 \longmapsto f^{[1]} \longmapsto f^{[1]}\phi^{-1}.$$

This is also a Λ-module isomorphism. Indeed, the action of Λ is explicitly given by

$$\theta \cdot (f \otimes_{\mathfrak{f}} 1) = (fF^{-1}(\theta)) \otimes_{\mathfrak{f}} 1,$$
$$\theta \cdot f^{[1]} = f^{[1]}\phi^{-1}\theta\phi,$$
$$\theta \cdot (f^{[1]}\phi^{-1}) = f^{[1]}\phi^{-1}\theta,$$

where $\theta \in \Lambda$, and F is the Frobenius morphism on Λ given by $F(\theta) = \phi\theta^{[1]}\phi^{-1}$. Moreover, there are natural isomorphisms

$$\operatorname{Hom}_{\text{coh-}\mathbb{X}}(T, -)^{[1]} \cong \operatorname{Hom}_{\text{coh-}\mathbb{X}}(T^{[1]}, -^{[1]}) \cong \operatorname{Hom}_{\text{coh-}\mathbb{X}}(T, -^{[1]}),$$

since, for any $X, Y \in$ coh-$\mathbb{X}$ and every homomorphism $\xi : X \to Y$, there is a commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\text{coh-}\mathbb{X}}(T, X)^{[1]} \arrow[r, "\sim"] \arrow[d, "\operatorname{Hom}_{\text{coh-}\mathbb{X}}(T,\xi)^{[1]}"] & \operatorname{Hom}_{\text{coh-}\mathbb{X}}(T^{[1]}, X^{[1]}) \arrow[r, "\sim"] \arrow[d, "\operatorname{Hom}_{\text{coh-}\mathbb{X}}(T^{[1]},\xi^{[1]})"] & \operatorname{Hom}_{\text{coh-}\mathbb{X}}(T, X^{[1]}) \arrow[d, "\operatorname{Hom}_{\text{coh-}\mathbb{X}}(T,\xi^{[1]})"] \\
\operatorname{Hom}_{\text{coh-}\mathbb{X}}(T, Y)^{[1]} \arrow[r, "\sim"] & \operatorname{Hom}_{\text{coh-}\mathbb{X}}(T^{[1]}, Y^{[1]}) \arrow[r, "\sim"] & \operatorname{Hom}_{\text{coh-}\mathbb{X}}(T, Y^{[1]})
\end{tikzcd}$$

where

$$\operatorname{Hom}_{\text{coh-}\mathbb{X}}(T, \xi)^{[1]}(f \otimes_{\mathfrak{f}} 1) = \xi f \otimes_{\mathfrak{f}} 1,$$
$$\operatorname{Hom}_{\text{coh-}\mathbb{X}}(T^{[1]}, \xi^{[1]})(f^{[1]}) = \xi^{[1]} f^{[1]},$$
$$\operatorname{Hom}_{\text{coh-}\mathbb{X}}(T, \xi^{[1]})(f^{[1]}\phi^{-1}) = \xi^{[1]} f^{[1]}\phi^{-1}.$$

In other words, we have the commutative diagram (up to a natural isomorphism):

$$\begin{tikzcd}
\text{coh-}\mathbb{X} \arrow[r, "\operatorname{Hom}_{\text{coh-}\mathbb{X}}(T,-)"] \arrow[d, "(\ )^{[1]}"] & \text{mod-}\Lambda \arrow[d, "(\ )^{[1]}"] \\
\text{coh-}\mathbb{X} \arrow[r, "\operatorname{Hom}_{\text{coh-}\mathbb{X}}(T,-)"] & \text{mod-}\Lambda
\end{tikzcd}$$

Taking their derived functors give the first required natural isomorphism.

By the uniqueness of left adjoint functor, we obtain a natural isomorphism

$$(- \otimes_\Lambda T)^{[1]} \cong -^{[1]} \otimes_\Lambda T.$$

That is, we have the commutative diagram (up to a natural isomorphism):

$$\begin{tikzcd}
\text{mod-}\Lambda \arrow[r, "-\otimes_\Lambda T"] \arrow[d, "(\ )^{[1]}"] & \text{coh-}\mathbb{X} \arrow[d, "(\ )^{[1]}"] \\
\text{mod-}\Lambda \arrow[r, "-\otimes_\Lambda T"] & \text{coh-}\mathbb{X}
\end{tikzcd}$$

This gives the second required natural isomorphism. $\square$

This theorem together with Theorem 2.4 implies that

$$(D^b(\text{coh-}\mathbb{X}))^F \cong (D^b(\text{mod-}\Lambda))^F \cong D^b(\text{mod-}\Lambda^F). \tag{5}$$

Recall from the previous section the $\mathbb{F}_q$-category coh-$(\mathbb{X}, \sigma; q)$ associated with $(\mathbb{X}, \sigma)$. Then the embedding coh-$\mathbb{X} \hookrightarrow D^b$(coh-$\mathbb{X}$) induces an embedding

$$\text{coh-}(\mathbb{X}, \sigma; q) \hookrightarrow (D^b(\text{coh-}\mathbb{X}))^F \cong (D^b(\text{mod-}\Lambda))^F.$$

Hence, we can apply the method in [6, Thm. 5.6] to construct indecomposable objects in $(D^b(\text{coh-}\mathbb{X}))^F$.

Proposition 4.5. *There exists a triangle equivalence*

$$G : D^b(\text{coh-}(\mathbb{X}, \sigma; q)) \longrightarrow (D^b(\text{coh-}\mathbb{X}))^F.$$

Proof. Since coh-$\mathbb{X}$ is a hereditary abelian category, it follows that for each $\mathcal{M} \in D^b$(coh-$\mathbb{X}$), there is an isomorphism

$$\mathcal{M} \cong \coprod_{n\in\mathbb{Z}} \Sigma^{-n} H^n(\mathcal{M}),$$

where Σ denotes the shift functor, see, e.g., [2, Lem. 2.2.1]. In particular, each indecomposable object in D^b(coh-$\mathbb{X}$) has the form $M[n]$ for some indecomposable object $M \in$ coh-$\mathbb{X}$ and $n \in \mathbb{Z}$. Therefore, each indecomposable object in $(D^b(\text{coh-}\mathbb{X}))^F$ has the form $Y[n]$, where Y is an indecomposable object in coh-$(\mathbb{X}, \sigma; q)$.

Consider the standard bounded t-structure $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ on D^b(coh-$\mathbb{X}$), where $\mathcal{D}^{\leqslant 0}$ (resp., $\mathcal{D}^{\geqslant 0}$) is the full subcategory of D^b(coh-$\mathbb{X}$) consisting of $\mathcal{M}$ such that $H^i(\mathcal{M}) = 0$ for all $i > 0$ (resp., $i < 0$). Then the corresponding heart is $\mathcal{D}^{\leqslant 0} \cap \mathcal{D}^{\geqslant 0} =$ coh-$\mathbb{X}$. Clearly, both $\mathcal{D}^{\leqslant 0}$ and $\mathcal{D}^{\geqslant 0}$ are closed under taking the Frobenius twist. This gives two subcategories $(\mathcal{D}^{\leqslant 0})^F$ and $(\mathcal{D}^{\geqslant 0})^F$ of $(D^b(\text{coh-}\mathbb{X}))^F$. It is routine to check that $((\mathcal{D}^{\leqslant 0})^F, (\mathcal{D}^{\geqslant 0})^F)$ defines a bounded t-structure of $(D^b(\text{coh-}\mathbb{X}))^F$. Moreover, its corresponding heart is

$$\text{coh-}(\mathbb{X}, \sigma; q) = (\mathcal{D}^{\leqslant 0})^F \cap (\mathcal{D}^{\geqslant 0})^F.$$

By [12, 3.2] and [4, Sect. 3], there exists a triangle functor

$$G : D^b(\text{coh-}(\mathbb{X}, \sigma; q)) \longrightarrow (D^b(\text{coh-}\mathbb{X}))^F$$

satisfying $G|_{\text{coh-}(\mathbb{X},\sigma;q)} = \text{id}_{\text{coh-}(\mathbb{X},\sigma;q)}$, called the realization functor. Since each indecomposable object in $(D^b(\text{coh-}\mathbb{X}))^F$ has the form $Y[n]$ with Y an indecomposable object in coh-$(\mathbb{X}, \sigma; q)$ and $n \in \mathbb{Z}$, G is full. Applying [1, Thm. B] gives that G is an equivalence. $\square$

By the above proposition and (5), we obtain our main theorem.

Theorem 4.6. *There is a derived equivalence*

$$D^b(\text{coh-}(\mathbb{X}, \sigma; q)) \cong D^b(\text{mod-}\Lambda^F).$$

Corollary 4.7. *The category* coh-$(\mathbb{X},\sigma;q)$ *is a hereditary and Ext-finite* $\mathbb{F}_q$*-category. Moreover, it has Serre duality.*

Proof. By Proposition 4.5, for any two objects $M,N\in$ coh-$(\mathbb{X},\sigma;q)$,

$$\begin{aligned}\mathrm{Ext}^{\,i}_{\text{coh-}(\mathbb{X},\sigma;q)}(M,N) &= \mathrm{Hom}_{\,D^b(\text{coh-}(\mathbb{X},\sigma;q))}(M,\Sigma^i N)\\ &= \mathrm{Hom}_{\,(D^b(\text{coh-}\mathbb{X}))^F}(M,\Sigma^i N)\\ &\subseteq \mathrm{Hom}_{\,D^b(\text{coh-}\mathbb{X})}(M,\Sigma^i N)\\ &= \mathrm{Ext}^{\,i}_{\text{coh-}\mathbb{X}}(M,N).\end{aligned}$$

Therefore, the heredity and Ext-finiteness of coh-$(\mathbb{X},\sigma;q)$ follow from those of coh-$\mathbb{X}$.

In view of Remark 3.5, coh-$(\mathbb{X},\sigma;q)$ has Auslander–Reiten sequences. Since coh-$(\mathbb{X},\sigma;q)$ has neither non-zero projective nor non-zero injective objects, we have by [13, Thm. I.3.3] that coh-$(\mathbb{X},\sigma;q)$ has Serre duality. □

Remark 4.8. Indeed, the derived equivalence in the main theorem is induced by a tilting object in coh-$(\mathbb{X},\sigma;q)$. More precisely, let T be the tilting object in coh-$\mathbb{X}$ given in Lemma 4.1. Then $\phi: T^{[1]}\cong T$ and thus, $T=(T,\phi)\in$ coh-$(\mathbb{X},\sigma;q)$. Applying Proposition 4.5 shows that T is a tilting object in coh-$(\mathbb{X},\sigma;q)$ and $(\mathrm{End}_{\,\text{coh-}(\mathbb{X},\sigma;q)}(T))^{\mathrm{op}}\cong\Lambda^F$. By [10, Thm. 4.6], T gives rise to a derived equivalence

$$D^b(\text{coh-}(\mathbb{X},\sigma;q))\cong D^b(\text{mod-}\Lambda^F).$$

Example 4.9. Let $\mathbb{X}$ be the weighted projective line of type $(2,2,3)$, and $\boldsymbol{\lambda}=(\infty,0,-1)$, with weight permutation $\sigma=(12)$. Then $S=k[X_1,X_2,X_3]/(X_3^3-X_2^2-X_1^2)$, the Frobenius morphism on S is defined by

$$F:S\longrightarrow S,\quad \sum c_{n_1,n_2,n_3}x_1^{n_1}x_2^{n_2}x_3^{n_3}\longmapsto \sum c^q_{n_1,n_2,n_3}x_2^{n_1}x_1^{n_2}x_3^{n_3},$$

and the group homomorphism $\sigma_{\mathbb{L}}$ on $\mathbb{L}$ is given by

$$\sigma_{\mathbb{L}}:\mathbb{L}\longrightarrow\mathbb{L},\quad l_1\vec{x}_1+l_2\vec{x}_2+l_3\vec{x}_3+l\vec{c}\longmapsto l_1\vec{x}_2+l_2\vec{x}_1+l_3\vec{x}_3+l\vec{c}.$$

This induces the Frobenius functor $(\ )^{[1]}$: coh-$\mathbb{X}\to$ coh-$\mathbb{X}$. By Proposition 3.4, we have

$$\mathcal{O}(l_1\vec{x}_1+l_2\vec{x}_2+l_3\vec{x}_3+l\vec{c})^{[1]}\cong\mathcal{O}(l_1\vec{x}_2+l_2\vec{x}_1+l_3\vec{x}_3+l\vec{c}),$$

$$(S_{1,0})^{[1]}\cong S_{2,0},\ (S_{1,1})^{[1]}\cong S_{2,1},\ (S_{2,0})^{[1]}\cong S_{1,0},\ (S_{2,1})^{[1]}\cong S_{1,1},$$

$$(S_{3,0})^{[1]}\cong S_{3,0},\ (S_{3,1})^{[1]}\cong S_{3,1},\ (S_{3,2})^{[1]}\cong S_{3,2},$$

$$(S_\lambda)^{[1]}=S_{1/\lambda^q},\ \forall\,\lambda\in\mathbb{H}_k.$$

Moreover, the associated canonical algebra Λ is given by the path algebra of the quiver Q

$$\begin{tikzcd}
 & & \vec{x}_1 \arrow[dll, "X_{11}"'] & & \\
\vec{0} & & \vec{x}_2 \arrow[ll, "X_{21}"'] & & \vec{c} \arrow[ull, "X_{12}"'] \arrow[ll, "X_{22}"'] \arrow[dl, "X_{33}"] \\
 & \vec{x}_3 \arrow[ul, "X_{31}"] & & 2\vec{x}_3 \arrow[ll, "X_{32}"] &
\end{tikzcd}$$

modulo the ideal $I = \langle X_3^3 - X_2^2 - X_1^2 \rangle$. The weight permutation $\sigma = (12)$ induces an automorphism σ of Q given by

$$\sigma(\vec{0}) = \vec{0},\ \sigma(\vec{c}) = \vec{c},\ \sigma(\vec{x}_1) = \vec{x}_2,\ \sigma(\vec{x}_2) = \vec{x}_1,$$

$$\sigma(\vec{x}_3) = \vec{x}_3,\ \sigma(2\vec{x}_3) = 2\vec{x}_3,$$

$$\sigma(X_{11}) = X_{21},\ \sigma(X_{12}) = X_{22},\ \sigma(X_{21}) = X_{11},\ \sigma(X_{22}) = X_{12},$$

$$\sigma(X_{31}) = X_{31},\ \sigma(X_{32}) = X_{32},\ \sigma(X_{33}) = X_{33}.$$

Then the valued quiver $\Gamma = \Gamma(Q, \sigma)$ associated with (Q, σ) has the form

$$\begin{tikzcd}
 & & a \arrow[dll, "\alpha"'] & & \\
b & & & & c \arrow[ull, "\beta"'] \arrow[dl, "\eta"] \\
 & d \arrow[ul, "\gamma"] & & e \arrow[ll, "\rho"] &
\end{tikzcd}$$

The valuation is given by $(d_a, d_b, d_c, d_d, d_e) = (2, 1, 1, 1, 1)$, and $(m_\alpha, m_\beta, m_\gamma, m_\rho, m_\eta) = (2, 2, 1, 1, 1)$. Moreover, $\Lambda^F = T(\mathscr{Q})/I^F$, where $T(\mathscr{Q})$ is the tensor algebra of the associated modulated quiver $\mathscr{Q} = \mathscr{Q}_{Q,\sigma;q}$ (see [7, Sect. 3.5] for notations).

Finally, applying the theorem above gives the derived equivalence

$$D^b(\text{coh-}(\mathbb{X}, \sigma; q)) \cong D^b(\text{mod-}\Lambda^F).$$

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References

[1] X.-W. Chen, Z. Han, and Y. Zhou, *Derived equivalences via HRS-tilting*, Adv. Math. **354** (2019), 106749. 26 pp. `https://doi.org/10.1016/j.aim.2019.106749`

[2] X.-W. Chen and H. Krause, *Introduction to coherent sheaves over weighted projective lines*, arXiv: 0911.4473.

[3] Z. Chen and Y. Lin, *A realization of elliptic Lie algebras of type $F_4^{(2,2)}$ by the Ringel-Hall approach*, J. Algebra **350** (2012), 108–131. `https://doi.org/10.1016/j.jalgebra.2011.10.008`

[4] X.-W. Chen and C. M. Ringel, *Hereditary triangulated categories*, J. Noncommut. Geom. **12** (2018), no. 4, 1425–1444. `https://doi.org/10.4171/JNCG/311`

[5] B. Deng and J. Du, *Frobenius morphisms and representations of algebras*, Trans. Amer. Math. Soc. **358** (2006), no. 8, 3591–3622. `https://doi.org/10.1090/S0002-9947-06-03812-8`

[6] ______, *Folding derived categories with Frobenius functors*, J. Pure Appl. Algebra **208** (2007), no. 3, 1023–1050. https://doi.org/10.1016/j.jpaa.2006.05.001

[7] B. Deng, J. Du, B. Parshall, and J. Wang, *Finite dimensional algebras and quantum groups*, Mathematical Surveys and Monographs, 150, American Mathematical Society, Providence, RI, 2008. https://doi.org/10.1090/surv/150

[8] A. D. Elagin, *On equivariant triangulated categories*, arXiv:1403.7027v2, 2015.

[9] W. Geigle and H. Lenzing, *A class of weighted projective curves arising in representation theory of finite-dimensional algebras*, in Singularities, representation of algebras, and vector bundles (Lambrecht, 1985), 265–297, Lecture Notes in Math., 1273, Springer, Berlin, 2006. https://doi.org/10.1007/BFb0078849

[10] D. Happel, I. Reiten, and S. O. Smalø, *Tilting in abelian categories and quasitilted algebras*, Mem. Amer. Math. Soc. **120** (1996), no. 575, viii+ 88 pp. https://doi.org/10.1090/memo/0575

[11] D. Happel and C. M. Ringel, *Tilted algebras*, Trans. Amer. Math. Soc. **274** (1982), no. 2, 399–443. https://doi.org/10.2307/1999116

[12] B. Keller and D. Vossieck, *Sous les catégories dérivées*, C. R. Acad. Sci. Paris Sér. I Math. **305** (1987), no. 6, 225–228.

[13] I. Reiten and M. Van den Bergh, *Noetherian hereditary abelian categories satisfying Serre duality*, J. Amer. Math. Soc. **15** (2002), no. 2, 295–366. https://doi.org/10.1090/S0894-0347-02-00387-9

[14] C. M. Ringel, *The canonical algebras*, in Topics in algebra, Part 1 (Warsaw, 1988), 407–432, Banach Center Publ., 26, Part 1, PWN, Warsaw, 1990.

[15] J.-P. Serre, *Faisceaux algébriques cohérents*, Ann. of Math. (2) **61** (1955), 197–278. https://doi.org/10.2307/1969915

Lina Han
Yau Mathematical Sciences Center
Tsinghua University
Beijing 100084, P. R. China
Email address: hln15@mails.tsinghua.edu.cn

Xintian Wang
College of Science
China University of Mining and Technology (Beijing)
Beijing 100083, P. R. China
Email address: wangxintian0916@126.com