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POSITIVELY EXPANSIVE MAPS AND THE LIMIT SHADOWING PROPERTIES

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Abstract. In this paper, the notion of two-sided limit shadowing property is considered for a positively expansive open map. More precisely, let f be a positively expansive open map of a compact metric space X. It is proved that if f is topologically mixing, then it has the two-sided limit shadowing property. As a corollary, we have that if X is connected, then the notions of the two-sided limit shadowing property and the average-shadowing property are equivalent.

1. Introduction

Let X be a compact metrizable space, and let f be a continuous map of X onto itself. We say that f is *positively expansive* if there exist a metric d and a constant $c > 0$ such that $d(f^i(x), f^i(y)) \leq c$ $(x, y \in X)$ for all $i \geq 0$ implies $x = y$. Such a number c is called an *expansive constant*. This property (although not c) is independent of a metric for X. Every one-sided shift map and every expanding differentiable map on a C^∞ closed manifold are positively expansive (see [5], [11] and [14]).

Fix any metric d for X (throughout this paper, this term means that d is a metric compatible with the topology of X). As usual, a sequence $\{x_i\}_{i=0}^\infty$ of points in X is called a *δ-pseudo-orbit* ($\delta > 0$) of f if $d(f(x_i), x_{i+1}) < \delta$ for all $i \geq 0$. We say that f has the (usual) *shadowing property* if for any $\epsilon > 0$, there is $\delta > 0$ such that for any δ-pseudo-orbit $\{x_i\}_{i=0}^\infty$, there exists $y \in X$ ϵ-shadowing the pseudo-orbit, that is, $d(f^i(y), x_i) < \epsilon$ for all $i \geq 0$. This property is also independent of a metric for X. It is known that every positively expansive open map has the shadowing property (see [11], among others).

We say that a sequence of points $\{x_i\}_{i=0}^\infty$ in X is a *limit pseudo-orbit* of f if it satisfies

$$d(f(x_i), x_{i+1}) \to 0 \text{ as } i \to \infty.$$

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A limit pseudo-orbit is *limit-shadowed* if there exists a point $y \in X$ such that

$$d(f^i(y), x_i) \to 0 \quad \text{as} \quad i \to \infty.$$

We say that f has the *limit shadowing property* (see [6], among others) if every limit pseudo-orbit of f is limit-shadowed (this property is also independent of a metric). In general, the value $d(f(x_i), x_{i+1})$ may be large for small i, but tends to 0 as $i \to \infty$. This property is really weaker than the usual shadowing property (see [10, p. 65, Example 1.19]).

We can define the same property for negative limit pseudo-orbits of f if it is injective. A sequence of points $\{x_{-i}\}_{i=0}^{\infty} \subset X$ is a *negative limit pseudo-orbit* if it satisfies

$$d(f(x_{-i}), x_{-i+1}) \to 0 \quad \text{as} \quad i \to \infty.$$

A negative limit pseudo-orbit is *limit-shadowed* if there exists $y \in X$ such that

$$d(f^{-i}(y), x_{-i}) \to 0 \quad \text{as} \quad i \to \infty.$$

We say that a sequence of points $\{x_i\}_{i=-\infty}^{\infty} \subset X$ is a *two-sided limit pseudo-orbit* of f if it satisfies

$$d(f(x_i), x_{i+1}) \to 0 \quad \text{as} \quad |i| \to \infty.$$

A two-sided limit pseudo-orbit is *two-sided limit shadowed* if there is a point $y \in X$ such that

$$d(f^i(y), x_i) \to 0 \quad \text{as} \quad i \to \pm\infty.$$

We say that f has the *two-sided limit shadowing property* (cf. [10, p. 63, Section 1.4]) if every two-sided limit pseudo-orbit is two-sided limit shadowed.

In recent few years, the notion of the two-sided limit shadowing property is intensively studied by [2], [3] and [4], and interesting characterizations are obtained for injective case. For instance, it has shown therein that every Anosov diffeomorphism on a compact and connected manifold is topologically transitive if and only if it has the two-sided limit shadowing property. The C^1-interior in the set of all diffeomorphisms possessing the two-sided limit shadowing property is characterized as the set of topologically transitive Anosov diffeomorphisms and so on.

The author of [9] introduced the notion of two-sided limit shadowing property for a continuous map f, and studied the dynamical properties of it if f possesses the two-sided limit shadowing property. For instance, the existence of a kind of Smale and Bowen spectral decompositions was proved therein for the map f. In this paper, we consider the two-sided limit shadowing property and its relationship to the other topological properties such as the average-shadowing property for positively expansive open maps. More precisely, let f be a positively expansive open map of a compact metric space X. It is proved that if f is topologically mixing, then f has the two-sided limit shadowing property. As a corollary, we have that if X is connected, then the notions of the two-sided limit shadowing property and the average-shadowing property are equivalent.

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2. Definitions and the statement of the results

As before, let X be a compact metrizable space and f be a continuous map of X onto itself. We say that f has the *Lipschitz shadowing property* if there are a metric d for X and positive constants L, ϵ_0 such that for any $0 < \epsilon < \epsilon_0$ and any ϵ-pseudo-orbit $\{x_i\}_{i=0}^{\infty}$ of f, there exists $y \in X$ such that $d(f^i(y), x_i) < L\epsilon$ for all $i \geq 0$ (see [10] and [13]). This property depends on a metric for X.

Fix any metric d for X. The notion of two-sided limit shadowing property for continuous maps is defined as in the following way. Let $\{x_{-i}\}_{i=0}^{\infty} \subset X$ be a *backward orbit* of f; that is, $f(x_{-i}) = x_{-i+1}$ for all $i \geq 0$, and denote by $\boldsymbol{X}_f$ the set of all backward orbits. A negative limit pseudo-orbit is *limit-shadowed* if there exists $\{y_{-i}\}_{i=0}^{\infty} \in \boldsymbol{X}_f$ such that

$$d(y_{-i}, x_{-i}) \to 0 \ \text{ as } \ i \to \infty.$$

We say that a continuous map f has the *two-sided limit shadowing property* (cf. [9, §2.1]) if for any two-sided limit pseudo-orbit $\{x_i\}_{i=-\infty}^{\infty} \subset X$ of f, there exists $\{y_{-i}\}_{i=-\infty}^{\infty} \in \boldsymbol{X}_f$ such that

$$d(f^i(y_0), x_i) \to 0 \ \text{ and } \ d(y_{-i}, x_{-i}) \to 0 \ \text{ as } \ i \to \infty.$$

To state our results, let us recall that f is said to be *topologically transitive* if there is a dense orbit; that is, $X = \overline{\{f^n(x) : n \geq 0\}}$ for some $x \in X$. We say that f is *topologically mixing* if for every pair of non-empty open sets U and V of X, there exists $N \geq 0$ such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq N$. It is known that if f is topologically mixing, then it is topologically transitive (see [15]). The main result of this paper is the following.

Theorem 2.1. *Let $f : X \to X$ be a positively expansive open map on a compact metrizable space. If f is topologically mixing, then f has the two-sided limit shadowing property with respect to some metric.*

Let $f : X \to X$ and d be as before. For $\delta > 0$, a sequence $\{x_i\}_{i=0}^{\infty}$ of points in X is called a *δ-average-pseudo-orbit* of f if there is a number $N = N(\delta) > 0$ such that for all $n \geq N$ and $k \geq 0$,

$$\frac{1}{n}\sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta.$$

The notion of average-pseudo-orbits is a certain generalization of the notion of pseudo-orbits and is arising naturally in the realizations of independent Gaussian random perturbations with zero mean and so on (see [1]).

We say that f has the *average shadowing property* if for every $\epsilon > 0$, there is $\delta > 0$ such that every δ-average-pseudo-orbit $\{x_i\}_{i=0}^{\infty}$ is ϵ-shadowed in average by some point $y \in X$; that is,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(y), x_i) < \epsilon.$$

This property also depends on a metric for X.

It was stated in [13, Theorem 2] that every positively expansive open map f has the average shadowing property under the assumption that f is topologically transitive. Recently, however, Kwietniak and Oprocha [7] have pointed out that there exists an example of a positively expansive open map which is topologically transitive but does not have the average shadowing property. They also have shown that to prove [13, Theorem 2], it is enough to assume that the map f is topologically weakly mixing.

Furthermore, the authors obtained more general result for continuous maps with the shadowing property. To be precise, let $f : X \to X$ be a continuous map of a compact metric space with the shadowing property. Then it is proved in [7, Theorem 1] that the following conditions are mutually equivalent:

(1) f is totally transitive,
(2) f is topologically weakly mixing,
(3) f is topologically mixing,
(4) f has the specification property, and
(5) f has the average shadowing property.

In this paper, we restate [13, Theorem 2] by replacing the topologically transitive condition with topologically mixing (for more description, see the first part of Section 4). The following is proved.

Theorem 2.2. *Let $f : X \to X$ be a positively expansive open map on a compact metrizable space. If f is topologically mixing, then f has the average shadowing property with respect to some metric.*

Remark that the author has also shown in [13, pp. 29–30] that if f has the average shadowing property, then f is topologically transitive. On the other hand, it is stated in [11, pp. 150–151] that in case X is connected, if a positively expansive open map $g : X \to X$ is topologically transitive, then g is topologically mixing. Thus, combining those two facts with Theorems 2.1-2.2, we obtain the following corollary. Observe that every expanding differentiable map on a C^∞ closed manifold ensures the assumption of this corollary.

Corollary 2.3. *Let $f : X \to X$ be a positively expansive open map on a compact metric space with metric d, and suppose that X is connected. Then the following conditions are mutually equivalent*;

(i) *f has the two-sided limit shadowing property,*
(ii) *f has the average shadowing property, and*
(iii) *f is topologically transitive.*

Remark that since every positively expansive open map has the shadowing property, we can add the above properties (1)-(4) of [7, Theorem 1] to the list of Corollary 2.3.

In Section 3, we prove preliminary results. In Section 4, we first prove Proposition 4.1, the main technical result of this paper, and then, by applying this proposition, Theorems 2.1-2.2 and Corollary 2.3 will be proved.

3. Preliminary results

In this section, let f be a positively expansive map on a compact metrizable space X. By [13, Lemma 1], there exist a metric d for X, and constants K, $\delta_0 > 0$ and $\lambda > 1$ such that for any $x, y \in X$,

(3.1) $0 < d(x, y) < \delta_0$ implies $\lambda d(x, y) < d(f(x), f(y))$,

(3.2) $d(f(x), f(y)) \leq K d(x, y)$.

We may suppose that $K \geq \lambda > 1$. Hereafter, we fix both the above metric and the constants, and assume further that f is an open map. Then, by (3.1),

(3.3) there exists $0 < \delta_1 < \delta_0/2$ such that for every $0 < \delta \leq \delta_1$, if $d(f(x), y) < \delta$, then

$$B_{\delta/\lambda}(x) \cap f^{-1}(y) = \{\text{single point}\}$$

(see [5, Lemma 1]).

Let $\Omega(f)$ be the non-wandering set of f. Then it is easy to see that

(3.4) the set of periodic points, $P(f)$, of f is dense in $\Omega(f)$

(cf. [13]), and from this, we have

(3.5) $f(\Omega(f)) = \Omega(f)$.

(3.6) $\Omega(f)$ is decomposed into a finite disjoint union of closed f-invariant sets $\{\Lambda_i\}_{i=1}^{\ell}$; that is, $\Omega(f) = \Lambda_1 \cup \cdots \cup \Lambda_\ell$ such that $f_{|\Lambda_j}$ is topologically transitive for $1 \leq j \leq \ell$.

Such a set Λ_j is called a *basic set* (cf. [11] and [13]).

The first proposition is proved by [13, Proposition 1].

Proposition 3.1. *f has the Lipschitz shadowing property.*

For $\epsilon > 0$, define the *local stable set* of $x \in X$, $W^s_\epsilon(x)$, by

$$W^s_\epsilon(x) = \{y \in X : d(f^n(x), f^n(y)) \leq \epsilon \text{ for all } n \geq 0\}$$

as usual. Remark that if $\epsilon \leq c$, then $W^s_\epsilon(x) = \{x\}$ for all $x \in X$, where c is an expansive constant for f.

Proposition 3.2. *f has the limit shadowing property.*

Proof. The conclusion is obtained by modifying the technique displayed in [10, pp. 66–67] (see also [8]). By Proposition 3.1, f has the Lipschitz shadowing property with two constants L and ε_0. We may suppose $0 < \epsilon_0 < c$ reducing ϵ_0 if necessary.

Put $\epsilon_1 = \varepsilon_0/2L$, and let $\{x_i\}_{i=0}^{\infty}$ be a given limit pseudo-orbit of f; that is, $d(f(x_i), x_{i+1}) \to 0$ as $i \to \infty$. Then, there exists $I > 0$ such that $i \geq I$ implies $d(f(x_i), x_{i+1}) < \epsilon_1$. Let us fix $\{z_{-i}\}_{i=0}^{\infty} \in \boldsymbol{X}_f$ with $z_0 = x_I$. Since

$$\{z_{-I}, z_{-I+1}, \ldots, z_{-1}, x_I, x_{I+1}, x_{I+2}, \ldots\}$$

is an ϵ_1-pseudo-orbit of f, there exists $y \in X$ such that $d(f^i(y), x_i) < L\epsilon_1$ for all $i \geq I$ by the Lipschitz shadowing property.

Now, for any $0 < \delta < \epsilon_1$, there exists $I_\delta \geq I$ such that $d(f(x_i), x_{i+1}) < \delta$ for all $i \geq I_\delta$. Thus $\{x_i\}_{i=I_\delta}^{\infty}$ is a δ-pseudo-orbit of f. Take any $\{z'_{-i}\}_{i=0}^{\infty} \in \boldsymbol{X}_f$ with $z'_0 = x_{I_\delta}$. Then, since

$$\{z'_{-I_\delta}, z'_{-I_\delta+1}, \ldots, z'_{-1}, x_{I_\delta}, x_{I_\delta+1}, x_{I_\delta+2}, \ldots\}$$

is a δ-pseudo-orbit of f, there exists $y_\delta \in X$ such that $d(f^i(y_\delta), x_i) < L\delta$ for all $i \geq I_\delta$ by the Lipschitz shadowing property.

Observe that $d(f^i(y), x_i) < L\epsilon_1$ for all $i \geq I_\delta$, we see

$$\begin{aligned} d(f^i(y), f^i(y_\delta)) &\leq d(f^i(y), x_i) + d(x_i, f^i(y_\delta)) \\ &< L(\epsilon_1 + \delta) < \epsilon_0 < c \end{aligned}$$

for all $i \geq I_\delta$. Since $W_c^s(f^{I_\delta}(y_\delta)) = \{f^{I_\delta}(y_\delta)\}$, we obtain $f^{I_\delta}(y) = f^{I_\delta}(y_\delta)$. Therefore

$$d(f^i(y), x_i) = d(f^i(y_\delta), x_i) < L\delta$$

for all $i \geq I_\delta$. Since δ is arbitrary, $d(f^i(y), x_i) \to 0$ as $i \to \infty$. □

The assertion of the next proposition seems to be true since the dynamics of backward of f is essentially contracting by (3.3). Denote by $\{(w_{-j}, n_j)\}_{j=0}^{\infty}$ the sequence of segments of the orbits of w_{-j} with length n_j, that is,

$$(w_{-j}, n_j) = \{w_{-j}, f(w_{-j}), f^2(w_{-j}), \ldots, f^{n_j-1}(w_{-j})\}$$

for all $j \geq 0$.

Proposition 3.3. *f has the negative limit shadowing property.*

Proof. Let $\lambda > 1$ and $\delta_1 > 0$ be as in (3.1), and recall that f has the Lipschitz shadowing property with constants L, ϵ_0. For a given negative limit pseudo-orbit $\{x_{-i}\}_{i=0}^{\infty}$ of f; that is, $d(f(x_{-i}), x_{-i+1}) \to 0$ as $i \to \infty$, there is $I > 0$ such that $i \geq I$ implies

$$d(f(x_{-i}), x_{-i+1}) < \frac{1}{L} \cdot \frac{\delta_1}{4}.$$

Since f has the Lipschitz shadowing property, we can construct a sequence of segments of orbits $\{(w_{-j}, n_j)\}_{j=0}^{\infty}$ such that $d(f^{n_j-1}(w_{-j}), w_{-j+1}) \to 0$ as $j \to \infty$. In the following construction, we can choose n_j as much as large for each j.

Put $w_0 = x_{-I}$ and $n_0 = 0$, and choose $n_1 > 0$ such that

$$\lambda^{n_1} > 4 \quad \text{and} \quad d(f(x_{-I-i-1}), x_{-I-i}) < \frac{1}{L} \cdot \frac{\delta_1}{4^2}$$

for $i \geq n_1$. Since a segment of the pseudo-orbit

$$\{x_{-I-n_1}, x_{-I-n_1+1}, \dots, x_{-I-1}, x_{-I}\}$$

is a $\frac{\delta_1}{4L}$-pseudo-orbit of f by the choice of I, there is $w_{-1} \in X$ $\frac{\delta_1}{4}$-shadowing the segment by the Lipschitz shadowing property, so that

$$d(f^k(w_{-1}), x_{-I-n_1+k}) < \frac{\delta_1}{4} \text{ for } 0 \leq k \leq n_1.$$

By the same manner, choose $n_2 \geq n_1$ (and thus, $\lambda^{n_2} > 4$) such that

$$d(f(x_{-I-n_1-i-1}), x_{-I-n_1-i}) < \frac{1}{L} \cdot \frac{\delta_1}{4^3} \quad \text{for} \quad i \geq n_2.$$

Since a segment of the pseudo-orbit

$$\{x_{-I-n_2-n_1}, x_{-I-n_2-n_1+1}, \dots, x_{-I-n_1-1}, x_{-I-n_1}\}$$

is a $\frac{\delta_1}{4^2 L}$-pseudo-orbit of f, there is $w_{-2} \in X$ $\frac{\delta_1}{4^2}$-shadowing the segment. It is easy to see that

$$d(f^k(w_{-2}), x_{-I-n_2-n_1+k}) < \frac{\delta_1}{4^2} \quad \text{for} \quad 0 \leq k \leq n_2,$$

and

$$\begin{aligned} d(f^{n_2}(w_{-2}), w_{-1}) &\leq d(f^{n_2}(w_{-2}), x_{-I-n_1}) + d(x_{-I-n_1}, w_{-1}) \\ &\leq \frac{\delta_1}{4^2} + \frac{\delta_1}{4} = \frac{5\delta_1}{4^2}. \end{aligned}$$

Continuing this way, we can obtain $\{(w_{-j}, n_j)\}_{j=0}^{\infty}$ $(n_{j+1} \geq n_j)$ such that $w_{-j} \in X$ $\frac{\delta_1}{4^j}$-shadowing the segment

$$\{x_{-I-n_j-n_{j-1}}, x_{-I-n_j-n_{j-1}+1}, \dots, x_{-I-n_{j-1}-1}, x_{-I-n_{j-1}}\}.$$

Since

$$d(f^k(w_{-j}), x_{-I-n_j-n_{j-1}+k}) < \frac{\delta_1}{4^j} \quad \text{for} \quad 0 \leq k \leq n_j \tag{1}$$

and

$$\begin{aligned} d(f^{n_j}(w_{-j}), w_{-j+1}) &\leq d(f^{n_j}(w_{-j}), x_{-I-n_{j+1}}) + d(x_{-I-n_{j+1}}, w_{-j+1}) \\ &\leq \frac{\delta_1}{4^j} + \frac{\delta_1}{4^{j-1}} = \frac{5\delta_1}{4^j} \end{aligned} \tag{2}$$

for all $j \geq 1$, we have $d(f^{n_j}(w_{-j}), w_{-j+1}) \to 0$ as $j \to \infty$.

Now let us construct the backward orbit $\{z_{-i}\}_{i=0}^{\infty} \in \boldsymbol{X}_f$ with $z_0 = w_0 = x_{-I}$ such that

$$d(z_{-j}, x_{-I-j}) \to 0 \quad \text{as} \quad j \to \infty$$

along the above $\{(w_{-j}, n_j)\}_{j=0}^{\infty}$ by applying (3.3). Note that $\lambda^{n_j} > 4$ for all j.

Put $z_0 = w_0$. Then by (3.3), we can take z_{-1} nearby $f^{n_1-1}(w_{-1})$ such that $f(z_{-1}) = z_0$ and $d(z_{-1}, f^{n_1-1}(w_{-1})) < \frac{1}{\lambda} \cdot \frac{\delta_1}{4}$. By repeating the same manner, we have a segment of a backward orbit

$$\{z_{-n_1}, z_{-n_1+1}, \dots, z_{-1}, z_0 = w_0\} \tag{3}$$

of f such that

$$d(f^k(w_{-1}), z_{-n_1+k}) < \frac{1}{\lambda^{n_1-k}} \cdot \frac{\delta_1}{4} \leq \frac{\delta_1}{4} \quad \text{for} \quad 0 \leq k \leq n_1.$$

By the same fashion, since $d(f^{n_2}(w_{-2}), w_{-1}) \leq \frac{5\delta_1}{4^2}$ by (2), making use of (3.3) with respect to w_{-1}, we can find a segment of a backward orbit

$$\{z'_{-n_2}, z'_{-n_2+1}, \ldots, z'_{-1}, z'_0 = w_{-1}\} \tag{4}$$

of f such that

$$d(f^k(w_{-2}), z'_{-n_2+k}) < \frac{1}{\lambda^{n_1-k}} \cdot \frac{5\delta_1}{4^2} \leq \frac{5\delta_1}{4^2} \quad \text{for} \quad 0 \leq k \leq n_1.$$

Next, we extend (3) along (4) as in the following way. Since z_{-n_1} is near w_{-1}, by (3.3), there is z_{-n_1-1} nearby z'_{-1} such that $f(z_{-n_1-1}) = z_{-n_1}$ and

$$d(z_{-n_1-1}, z'_{-1}) < \frac{1}{\lambda} \cdot \frac{\delta_1}{4}.$$

By the same manner, we can find a segment of a backward orbit

$$\{z_{-n_2}, z_{-n_2+1}, \ldots, z_{-n_1-1}, z_{-n_1}\}$$

of f so that a segment of the extended backward orbit

$$\{z_{-n_2-n_1}, z_{-n_2-n_1+1}, \ldots, z_{-n_1-1}, z_{-n_1}, z_{-n_1+1}, \ldots, z_{-1}, z_0 = w_0\}$$

of f is obtained. Thus

$$d(f^k(w_{-2}), z_{-n_2-n_1+k}) \leq \frac{1}{\lambda^{n_2-k+n_1}} \cdot \frac{\delta_1}{4} + \frac{1}{\lambda^{n_2-k}} \cdot \frac{5\delta_1}{4^2} \leq \frac{6}{4^3}\delta_1 \quad \text{for} \quad 0 \leq k \leq n_2.$$

By the same way, we can see that

$$\begin{aligned} d(f^k(w_{-3}), z_{-n_3-n_2-n_1+k}) &\leq \frac{1}{\lambda^{n_3-k+n_2+n_1}} \cdot \frac{\delta_1}{4} \\ &\quad + \frac{1}{\lambda^{n_3-k+n_2}} \cdot \frac{5\delta_1}{4^2} + \frac{1}{\lambda^{n_3-k}} \cdot \frac{5\delta_1}{4^3} \\ &\leq \frac{11}{4^4}\delta_1 \end{aligned} \tag{5}$$

for $0 \leq k \leq n_3$.

Continuing this fashion, we can construct a backward orbit $\{z_{-i}\}_{i=0}^{\infty} \in \boldsymbol{X}_f$ with $z_0 = w_0 = x_{-I}$ such that

$$d(f^k(w_{-j}), z_{-n_j-n_{j-1}-\cdots-n_1+k}) \leq \frac{1 + 5(j-1)}{4^j}\delta_1 \quad \text{for} \quad 0 \leq k \leq n_j$$

for all j. It is not hard to show that

$$d(z_{-j}, x_{-I-j}) \to 0 \quad \text{as} \quad j \to \infty$$

by (1) and (5). □

4. Main proposition and the proofs of the results

Throughout this section, let $f : (X, d) \to (X, d)$ be a positively expansive open map with constant $c > 0$. We may suppose that f possesses all the properties (3.1)-(3.6) exhibited in the previous section. Our two theorems will be proved by applying the following main proposition.

As explained before, this proposition has already stated in [13, Lemma 3] under the assumption that f is topologically transitive. However, it has been shown by Kwietniak and Oprocha that this assumption is not enough (see [7, Example 3]), and then they repaired it therein (see [7, Theorem 14]). More precisely, they gave a new proof for [13, Lemma 3] under the assumption that f is weakly mixing. We say that a continuous map $g : X \to X$ is *weakly mixing* if the product map $g \times g : X \times X \to X \times X$ is topologically transitive. Note that in our setting, weakly mixing implies topologically mixing (see [7, Lemma 8]).

In what follows, let us give a new proof for [13, Lemma 3] by modifying the original proof for completeness.

Proposition 4.1. *If we assume further that f is topologically mixing, then there exists a constant $B > 1$ such that for each pair $(\{x_{-i}\}_{i=0}^{\infty}, y) \in \boldsymbol{X}_f \times X$, there exists a backward orbit $r(\{x_{-i}\}_{i=0}^{\infty}, y) = \{z_{-i}\}_{i=0}^{\infty} \in \boldsymbol{X}_f$ satisfying*

(i) $z_0 = y$,
(ii) $d(x_{-i}, z_{-i}) \leq B\lambda^{-i} d(x_0, y)$ *for all* $i \geq 0$.

The following lemma, whose assertion is stated in [11, pp. 150–151, Exercises 4(b)], can be proved by modifying the reasoning used in [12].

Lemma 4.2. *f is topologically mixing if and only if for every non-empty open set U of X, there exists $N > 0$ such that $f^N(U) = X$.*

Proof of Proposition 4.1. Let $\delta_1 > 0$ and $\lambda > 1$ be the constants as in (3.3), and let $(\{x_{-i}\}_{i=0}^{\infty}, y) \in \boldsymbol{X}_f \times X$ be given.

Case 1. $d(x_0, y) < \delta_1$.

Since $f(x_{-1}) = x_0$ and $d(f(x_{-1}), y) < \delta_1$, by (3.3), there exists $z_{-1} \in X$ such that

$$d(x_{-1}, z_{-1}) \leq \frac{\delta_1}{\lambda} \quad \text{and} \quad f(z_{-1}) = y.$$

Especially, $\lambda d(x_{-1}, z_{-1}) < d(f(x_{-1}), f(z_{-1})) = d(x_0, y)$. Since $\delta_1/\lambda < \delta_1$ and $f(x_{-2}) = x_{-1}$, by (3.3), there exists $z_{-2} \in X$ such that $d(x_{-2}, z_{-2}) \leq \delta_1/\lambda^2$, $f(z_{-2}) = z_{-1}$ and $\lambda d(x_{-2}, z_{-2}) < d(x_{-1}, z_{-1})$.

Repeating the process, we can find $z_{-i} \in X$ such that $d(x_{-i}, z_{-i}) \leq \delta_1/\lambda^i$, $f(z_{-i}) = z_{-i+1}$ and

$$d(x_{-i}, z_{-i}) < \lambda^{-i} d(x_0, y) \quad \text{for all} \quad i \geq 1.$$

Let $z_0 = y$ and set $r(\{x_{-i}\}_{i=0}^{\infty}, y) = \{z_{-i}\}_{i=0}^{\infty}$.

Case 2. $d(x_0, y) \geq \delta_1$.

Let $K > 0$ be as in (3.2), and let $0 < \epsilon = \epsilon(\delta_1) < \delta_1$ be the number as in the definition of the shadowing property of f. Denote by $\mathcal{U}$ a finite open cover $\{U_1, U_2, \ldots, U_l\}$ of X such that the diameter of each U_i is less than ϵ. For each U_i, by Lemma 4.2, there exists $N_i > 0$ such that $f^{N_i}(U_i) = X$. Put

$$N = \max_{1 \le i \le l} N_i,$$

then it is easy to see that $f^N(U_i) = X$ for all i.

Now for a given pair $(\{x_{-i}\}_{i=0}^{\infty}, y) \in \boldsymbol{X}_f \times X$, fix $U_j \in \mathcal{U}$ such that $x_{-N} \in U_j$. Then, since $f^N(U_j) = X$, there exists $y' \in U_j$ satisfying $f^N(y') = y$. Thus the sequence

$$\{\ldots, x_{-N-2}, x_{-N-1}, y', f(y'), \ldots, f^{N-1}(y'), y, f(y), \ldots\}$$

is an ϵ-pseudo-orbit of f. Using the shadowing property we can find $\{z_{-i}\}_{i=0}^{\infty} \in \boldsymbol{X}_f$ such that $z_0 = y$ and

$$\begin{cases} d(x_{-N-i}, z_{-N-i}) < \delta_1 & \text{for all } \ i \ge 0, \\ d(f^{N-j}(w), z_{-j}) < \delta_1 & \text{for all } \ 0 \le j \le N. \end{cases}$$

Put $r(\{x_{-i}\}_{i=0}^{\infty}, y) = \{z_{-i}\}_{i=0}^{\infty}$. Then, by (3.1)

$$d(x_{-N-1-i}, z_{-N-1-i}) < \lambda^{-i} d(x_{-N-1}, z_{-N-1}) < \lambda^{-i}\delta_1$$

for all $i \ge 0$. Thus

$$\begin{aligned} d(x_{-i}, z_{-i}) &= d(f^{N+1}(x_{-N-1-i}), f^{N+1}(z_{-N-1-i})) \\ &\le K^{N+1} d(x_{-N-1-i}, z_{-N-1-i}) \\ &< K^{N+1}\lambda^{-i}\delta_1 \\ &\le K^{N+1}\lambda^{-i} d(x_0, y) \end{aligned}$$

by (3.2). Finally, we set $B = K^{N+1}$, and the proof of the lemma is complete. □

We are in a position to prove Theorems 2.1 and 2.2.

Proof of Theorem 2.1. Let $\{x_i\}_{i=-\infty}^{\infty}$ be a two-sided limit pseudo-orbit of f, that is, $d(f(x_i), x_{i+1}) \to 0$ as $i \to \pm\infty$. By Proposition 3.2, there exists $y_+ \in X$ such that $d(f^i(y_+), x_i) \to 0$ as $i \to \infty$. By Proposition 3.3, there exists $\{y_{-i}\}_{i=0}^{\infty} \in \boldsymbol{X}_f$ such that

$$d(y_{-i}, x_{-i}) \to 0 \ \text{ as } \ i \to \infty.$$

Let $r(\{y_{-i}\}_{i=0}^{\infty}, y_+) = \{z_{-i}\}_{i=0}^{\infty} \in \boldsymbol{X}_f$ be the backward orbit given by Proposition 4.1 for the above $(\{y_{-i}\}_{i=0}^{\infty}, y_+) \in \boldsymbol{X}_f \times X$. Then it is easy to see that $\{z_{-i}\}_{i=0}^{\infty}$ is two-sided limit shadowing $\{x_i\}_{i=-\infty}^{\infty}$ by items (i) and (ii). □

Proof of Theorem 2.2. As stated before, the assumption of [13, Lemma 3] was not enough. In this paper we give the corrected variant, Proposition 4.1, of [13, Lemma 3] by assuming that f is topologically mixing. Theorem 2.2 follows from [13, pp. 27–29, Proof of Theorem 2] by replacing [13, Lemma 3] with Proposition 4.1. □

We need two lemmas to prove Corollary 2.3. The first lemma, whose assertion is stated in [11, pp. 150–151, Exercises 4(c)], can be also proved by modifying reasoning used in [12].

Lemma 4.3. *Let X be connected. If we assume further that f is topologically transitive, then for every non-empty open set U of X, there exists $N > 0$ such that $f^N(U) = X$, and thus, f is topologically mixing.*

The following lemma is proved in [13, Lemma 4].

Lemma 4.4. *Let $\Omega(f) = \bigcup_{j=1}^{\ell} \Lambda_j$ be as in* (3.6)*. Then $\omega(x) \cap \Lambda_j = \emptyset$ for any $x \in B_{\delta_0}(\Lambda_j) \setminus \Omega(f)$. Here $\omega(x)$ is the ω-limit set of x.*

Proof of Corollary 2.4. Let $f : X \to X$ be as before, and suppose that X is connected. If, in addition, f is topologically transitive, then f is topologically mixing by Lemma 4.3. On the other hand, as explained in Section 1, it has shown in [13, pp. 29–30] that if f has the average shadowing property, then f is topologically transitive. Thus, to get the conclusion, it is enough to show that if f has the two-sided limit shadowing property, then f is topologically transitive.

Let $\Omega(f) = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_\ell$ be as in (3.6).

Claim. Under the above notation, we have $\ell = 1$.

If this claim is established, then $\Omega(f) = \Lambda_1$, so that $X = \Omega(f)$ (for the proof of this equality, see [13, p. 29]). Thus we have $X = \Lambda_1$ so that f is topologically transitive.

To prove the claim, assuming that $\ell \geq 2$, we lead a contradiction. For simplicity, suppose $\ell = 2$ (the other case is treated similarly). Pick $x^j \in \Lambda_j$ for $j = 1, 2$, and choose $\{x_{-i}\}_{i=0}^{\infty} \in \boldsymbol{X}_f$ $(x_0 = x^1)$ in Λ_1. Then

$$\{x_{-i}\}_{i=0}^{\infty} \cup \{f^i(x^2)\}_{i=0}^{\infty}$$

is a two-sided limit pseudo-orbit of f. Since f has the two-sided limit shadowing property, there exists $y \in X$ two-sided limit shadowing the above two-sided limit pseudo-orbit. Observe that $y \notin \Omega(f)$, and thus, $\omega(y) \cap \Lambda_2 = \emptyset$ by Lemma 4.4. On the other hand, since $\{f^i(x^2)\}_{i=0}^{\infty} \subset \Lambda_2$ and $d(f^i(y), f^i(x^2)) \to 0$ as $i \to \infty$, we must have $\omega(y) \cap \Lambda_2 \neq \emptyset$. This is a contradiction. □

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