J. Korean Math. Soc. **58** (2021), No. 1, pp. 133–147
https://doi.org/10.4134/JKMS.j190874
pISSN: 0304-9914 / eISSN: 2234-3008

# MONOTONICITY CRITERION AND FUNCTIONAL INEQUALITIES FOR SOME $q$-SPECIAL FUNCTIONS

Khaled Mehrez

Abstract. Our aim in this paper is to derive several new monotonicity properties and functional inequalities of some functions involving the $q$-gamma, $q$-digamma and $q$-polygamma functions. More precisely, some classes of functions involving the $q$-gamma function are proved to be logarithmically completely monotonic and a class of functions involving the $q$-digamma function is showed to be completely monotonic. As applications of these, we offer upper and lower bounds for this special functions and new sharp upper and lower bounds for the $q$-analogue harmonic number harmonic are derived. Moreover, a number of two-sided exponential bounding inequalities are given for the $q$-digamma function and two-sided exponential bounding inequalities are then obtained for the $q$-tetragamma function.

## 1. Introduction

The Euler's gamma function is defined for positive real numbers $x$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt,$$

and its $q$-analogue, $\Gamma_q(x)$, introduced by Jackson. The $q$-gamma function is defined for $x > 0$ by

$$\Gamma_q(x) = (1-q)^{1-x}\prod_{j=0}^{\infty}\frac{1-q^{j+1}}{1-q^{j+x}},\ 0<q<1, \tag{1}$$

and

$$\Gamma_q(x) = (q-1)^{1-x}q^{\frac{x(x-1)}{2}}\prod_{j=0}^{\infty}\frac{1-q^{-(j+1)}}{1-q^{-(j+x)}},\ q>1. \tag{2}$$

The $q$-gamma function $\Gamma_q(z)$ has the following basic property:

$$\Gamma_q(z) = q^{\frac{(x-1)(x-2)}{2}}\Gamma_{\frac{1}{q}}(z). \tag{3}$$

Received December 30, 2019; Revised April 27, 2020; Accepted May 14, 2020.
2010 *Mathematics Subject Classification.* 33D05, 33B15, 39B72.
*Key words and phrases.* Logarithmically completely monotonic function, completely monotonic function, $q$-gamma function, $q$-digamma function, $q$-trigamma function.

The close connection between gamma and $q$-gamma functions is given by the limit relations

$$\lim_{q\to 1^-}\Gamma_q(z)=\lim_{q\to 1^+}\Gamma_q(z)=\Gamma(z). \tag{4}$$

The most important function related to the gamma function is the digamma (or psi) function which is defined as the logarithmic derivative of gamma function, namely, $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$. The derivatives $\psi^{(n)}, n=1,2,3,\ldots$ are known to be the polygamma functions in the literature. Particularly, $\psi'$ and $\psi''$ are called the trigamma and tetragamma functions, respectively. The $q$-digamma function $\psi_q$, the $q$-analogue of the psi or digamma function $\psi$ is defined for $0<q<1$ by

$$\begin{aligned}\psi_q(x)&=-\log(1-q)+\log q\sum_{k=0}^{\infty}\frac{q^{k+x}}{1-q^{k+x}}\\&=-\log(1-q)+\log q\sum_{k=1}^{\infty}\frac{q^{kx}}{1-q^{k}}.\end{aligned} \tag{5}$$

For $q>1$ and $x>0$, the $q$-digamma function $\psi_q$ is defined by

$$\begin{aligned}\psi_q(x)&=-\log(q-1)+\log q\left[x-\frac{1}{2}-\sum_{k=0}^{\infty}\frac{q^{-(k+x)}}{1-q^{-(k+x)}}\right]\\&=-\log(q-1)+\log q\left[x-\frac{1}{2}-\sum_{k=1}^{\infty}\frac{q^{-kx}}{1-q^{-kx}}\right].\end{aligned} \tag{6}$$

From the previous definitions, for a positive $x$ and $q>1$, we get

$$\psi_q(x)=\frac{2x-3}{2}\log(q)+\psi_{1/q}(x). \tag{7}$$

Similarly, the derivative $\psi_q',\psi_q'',\ldots$ are called the $q$-polygamma functions. In particular the functions $\psi_q'$ and $\psi_q''$ are called $q$-trigamma and $q$-tetragamma functions, respectively. For some basic properties for the $q$-gamma, $q$-digamma and $q$-polygamma functions, we refer [1–5, 8, 12, 13, 15] and references therein. The dilogarithm function $\operatorname{Li}_2(z)$ defined for complex argument $z$ by

$$\operatorname{Li}_2(z)=-\int_0^z\frac{\log(1-t)}{t}dt,\ z\notin(0,\infty). \tag{8}$$

Simple computation shows that

$$\left(\frac{\operatorname{Li}_2(1-q^x)}{\log(q)}\right)'=\frac{xq^x\log(q)}{1-q^x}. \tag{9}$$

Moreover, we have [8]

$$\lim_{q\to 1}\frac{\operatorname{Li}_2(1-q^x)}{\log(q)}=-x. \tag{10}$$

Each of the following definitions will be used in the remainder of our investigation.

**Definition.** A real valued function $f$, defined on an interval $I$, is called completely monotonic, if $f$ has derivatives of all orders and satisfies

$$(-1)^n f^{(n)}(x) \geq 0, \; n \in \mathbb{N}_0, x \in I, \tag{11}$$

where $\mathbb{N}$ the set of all positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The celebrated Bernstein Characterization Theorem gives a necessary and sufficient condition that the function $f$ should be completely monotonic for $0 < x < \infty$ is that

$$f(x) = \int_0^\infty e^{-xt} d\mu(t), \tag{12}$$

where $\mu(t)$ is non-decreasing and the integral converges for $0 < x < \infty$.

**Definition.** A non-negative function $f : (0, \infty) \to [0, \infty)$ is a Bernstein function, if it is infinitely differentiable and satisfies

$$(-1)^{n-1} f^{(n)}(x) \geq 0, \; n \in \mathbb{N}, x > 0.$$

**Definition.** A positive function $f$ is said to be logarithmically completely monotonic on an interval $I$ if its logarithm $\log f$ satisfies

$$(-1)^n \left(\log f(x)\right)^{(n)} \geq 0, \; x \in I, n \in \mathbb{N}.$$

In [6, Theorem 4], it was found and verified once again that a logarithmically completely monotonic function must be completely monotonic, but not conversely.

The paper is organized as follows: In Section 2, we prove new monotonicity properties and functional inequalities of some functions involving the $q$-gamma function. In particular, two classes of functions associated the $q$-gamma function are proved to be logarithmically completely monotonic and the monotonicity of ratios for a class of functions related to the $q$-gamma function is showed. As consequences of them we establish various new sharp upper and lower bounds for the $q$-gamma function. In Section 3, new complete monotonicity properties are obtained for a class of function related to the $q$-digamma functions. As applications, a functions class of Bernstein and logarithmically completely monotonic functions related to the $q$-gamma function are investigated. In addition, sharp bounds for the $q$-digamma and $q$-trigamma functions are derived. Moreover, new sharp upper and lower bounds for the Harmonic number and $q$-analogue of Harmonic number. In Section 4, two sets of two-sided bounding inequalities are given for the $q$-digamma and $q$-tetragamma functions.

## 2. Monotonicity properties and functional inequalities for the $q$-gamma function

Motivated by the definition of the $q$-gamma function (1) we have

$$\Gamma_q(x) = \frac{(1-q)^{2-x}}{1-q^x} \prod_{j=1}^{\infty} \frac{1-q^{j+1}}{1-q^{j+x}},\ 0 < q < 1, \tag{13}$$

combining with the recurrence formula

$$\Gamma_q(x+1) = \frac{1-q^x}{1-q} \Gamma_q(x), \tag{14}$$

we get the following inequality

$$(1-q)^{1-x} \leq \Gamma_q(x+1) \tag{15}$$

for $0 < x \leq 1$ and $0 < q < 1$. Motivated by the above inequality we determine the real numbers $\alpha$ and $\beta$ which are independent of $x$ such that we have the following inequalities

$$\alpha(1-q)^{1-x} \leq \Gamma_q(x+1) \leq \beta(1-q)^{1-x}, \tag{16}$$

holds true for all $0 < x \leq 1$ and $0 < q < 1$, where $\alpha$ and $\beta$ are the best possible constants.

**Theorem 2.1.** *Let $0 < q < 1$, and $0 < x \leq 1$. Then the inequalities* (16) *hold true, with the best possible constants $\alpha = 1$ and $\beta = \frac{1}{1-q}$. Furthermore, the left hand side of inequalities* (16) *is reversed when $x \geq 1$.*

*Proof.* We define the function $S_q(x)$ by

$$S_q(x) = (1-q)^{x-1}\Gamma_q(x+1),\ x > 0,\ 0 < q < 1.$$

Differentiating $S_q(x)$ gives

$$\begin{aligned} S_q'(x) &= (1-q)^{x-1}[\Gamma_q'(x+1) + \log(1-q)\Gamma_q(x+1)] \\ &= (1-q)^{x-1}\Gamma_q(x+1)[\psi_q(x+1) + \log(1-q)] \\ &= (1-q)^{x-1}\Gamma_q(x+1)W_q(x) \text{ (say)}. \end{aligned} \tag{17}$$

By using the definition of $q$-digamma function (5), we have

$$W_q(x) = \log(q) \sum_{k=0}^{\infty} \frac{q^{k+x+1}}{1-q^{k+x+1}} < 0 \tag{18}$$

for all $0 < q < 1$. This implies that the function $S_q(x)$ is decreasing on $(0, \infty)$ and consequently the following inequalities

$$S_q(1) \leq S_q(x) \leq S_q(0),$$

holds true for all $0 < x \leq 1$ and $0 < q < 1$. So (16) holds true. Moreover, if $x \geq 1$, then $S_q(x) \leq S_q(1)$ and consequently the left hand side of inequalities (16) is reversed when $x \geq 1$. The proof of Theorem 2.1 is completes. $\square$

Moreover, by using the definition of the $q$-gamma function (2) when $q > 1$, we easily get the following inequality

$$\Gamma_q(x+1) \geq (q-1)^{1-x} q^{\frac{(x-1)(x+2)}{2}}, \ q > 1. \tag{19}$$

In addition, the above inequality is reversed when $x \geq 1$. The question that arises: prove the best constants $\mu$ and $\nu$ such that the following inequalities

$$\mu(q-1)^{1-x} q^{\frac{(x-1)(x+2)}{2}} \leq \Gamma_q(x+1) \leq \nu(q-1)^{1-x} q^{\frac{(x-1)(x+2)}{2}}, \tag{20}$$

are true for all $q > 1$ and $0 < x \leq 1$. An affirmative answer to this question is proved in the following Theorem.

**Theorem 2.2.** *Let $q > 1$. The inequalities* (20) *hold true for all $0 < x \leq 1$, with the best possible constants $\mu = 1$ and $\nu = \frac{q}{q-1}$. In addition, the left hand side of inequalities* (20) *is reversed when $x \geq 1$.*

*Proof.* Let $q > 1$. We consider the function $T_q(x)$ defined by

$$T_q(x) = (q-1)^{x-1} q^{\frac{(1-x)(x+2)}{2}} \Gamma_q(x+1), x > 0.$$

Hence

$$\begin{aligned} T'_q(x) &= T_q(x)\left[\psi_q(x+1) + \log(q-1) - (x+1/2)\log(q)\right] \\ &= T_q(x) W_q^1(x) \text{ (say)}. \end{aligned} \tag{21}$$

By using the definition (6), we get

$$W_q^1(x) = -\log(q) \sum_{k=0}^{\infty} \frac{q^{-k(x+1)}}{1-q^{-k}} > 0 \tag{22}$$

for $q > 1$. This implies that the function $T_q(x)$ is decreasing on $(0, \infty)$. Hence, $T_q(1) \leq Tq(x) \leq T_q(0)$, and thus the proof of the inequalities (20) is done, such that the constants $\mu = 1$ and $\nu = \frac{q}{q-1}$ are the best possible. □

**Theorem 2.3.** *Let $0 < q < 1$. Then the function $\chi_q(x)$ defined by*

$$\chi_q(x) = (1-q)^x \Gamma_q(x+1), \tag{23}$$

*logarithmically completely monotonic on $(0, \infty)$. Furthermore, the following inequalities*

$$\Gamma_q(x+1)\Gamma_q(y+1) \leq \Gamma_q(x+y+1), \ x, y > 0, \tag{24}$$

$$\left[\frac{\Gamma_q(x+1)}{\Gamma_q(y+1)}\right]^{\frac{1}{x-y}} \leq \frac{1}{1-q}, \ 0 < y < x, \tag{25}$$

*hold true for all $0 < q < 1$. In addition, the inequality* (25) *is sharp.*

*Proof.* We set

$$\phi_q(x) = \log \chi_q(x) = x \log(1-q) + \log \Gamma_q(x+1).$$

Therefore,

$$\phi_q'(x) = W_q(x), \text{ and } \phi_q''(x) = \psi_q'(x+1).$$

Since the $q$-trigamma function $\psi_q'(x)$ is completely monotonic on $(0,\infty)$ for $0 < q < 1$, we deduce that the function $\phi_q''(x)$ is completely monotonic on $(0,\infty)$. In addition, the function $\phi_q'(x) < 0$ by (18), and consequently the result is obvious. Now, we proved the inequality (24). It is clear that the function $\chi_q(x)$ maps to $(0,\infty)$ to $(0,1)$ and it is completely monotonic on $(0,\infty)$. On the other hand, according to Kimberling [8], if a function $f$, defined on $(0,\infty)$, is continuous and completely monotonic and maps $(0,\infty)$ into $(0,1)$, then $\log f$ is super-additive, that is for all $x, y > 0$ we have

$$f(x+y) \geq f(x)f(y).$$

Therefore we conclude the asserted inequality (24). As the function $\chi_q(x)$ is logarithmically completely monotonic, $\chi_q(x)$ is decreasing on $(0,\infty)$, then for $0 < y < x$ we get

$$\log \Gamma_q(x+1) - \log \Gamma_q(y+1) \leq (x-y)\log(1/(1-q)),$$

which is equivalent to (25). Now, we define the function

$$K_q(x,y) = \begin{cases} \left[\frac{\Gamma_q(x+1)}{\Gamma_q(y+1)}\right]^{\frac{1}{x-y}} & \text{if } x \neq y, \\ \exp(\psi_q(x+1)) & \text{if } x = y. \end{cases} \tag{26}$$

From (5), we find that

$$\lim_{x\to\infty} \psi_q(x) = -\log(1-q), \ 0 < q < 1, \tag{27}$$

and consequently

$$\lim_{x\to\infty} K_q(x,x) = \frac{1}{1-q}.$$

This evidently completes the proof of Theorem 2.3. □

*Remark* 2.4. Ismail et al. [7, Theorem 2.2] show that the function $f(q;x)$ defined by

$$f(q;x) = (1-q)^x \Gamma_q(x),$$

is completely monotonic on $(0,\infty)$ for all $0 < q < 1$. In our main result in Theorem 2.3, we obtain that the function

$$x \mapsto \left(\frac{1-q^x}{1-q}\right) f(q;x) := \chi_q(x), \tag{28}$$

is logarithmically completely monotonic on $(0,\infty)$ for each $0 < q < 1$. Moreover, by Theorem 2.3, we deduce that the function $f(q;x)$ is completely monotonic on $(0,\infty)$. Indeed, we see that the function $x \mapsto \frac{1}{1-q^x}$ is completely monotonic on $(0,\infty)$ for all $0 < q < 1$. So, by (28), we deduce that the function $f(q;x)$ is completely monotonic for all $0 < q < 1$, as product of two completely monotonic functions.

**Theorem 2.5.** *Let $q > 1$. Then the function $\chi_q^1(x)$ defined by*

$$\chi_q^1(x) = (q-1)^x q^{\frac{(1-x)(x+2)}{2}} \Gamma_q(x+1),$$

*is logarithmically completely monotonic on $(0,\infty)$. Moreover, the following inequalities*

$$q^{2+2xy}\Gamma_q(x+1)\Gamma_q(y+1) \le \Gamma_q(x+y+1), x, y > 0, \tag{29}$$

$$\left[\frac{q^{\frac{(1-x)(x+2)}{2}}\Gamma_q(x+1)}{q^{\frac{(1-y)(y+2)}{2}}\Gamma_q(y+1)}\right]^{\frac{1}{x-y}} \le \frac{1}{q-1},\ 0 < y < x, \tag{30}$$

*hold true for all $q > 1$. The inequality (30) is sharp.*

*Proof.* Upon putting

$$\phi_q^1(x) = \log\chi_q^1(x) = \log\Gamma_q(x+1) + x\log(q-1) + \frac{(1-x)(x+2)}{2}\log(q).$$

Differentiating $\phi_q^1(x)$ yields

$$[\phi_q^1(x)]' = W_q^1(x) \text{ and } [\phi_q^1(x)]'' = \psi_q'(x+1) - \log(q) = \psi_{\frac{1}{q}}'(x+1).$$

Using the fact that the function $[\phi_q^1(x)]''$ is completely monotonic and the function $[\phi_q^1(x)]'$ is negative for all $q > 1$ we deduce that the function $\chi_q^1(x)$ is logarithmically completely monotonic on $(0,\infty)$. Again, by applying the Kimberling results, we deduce that the inequality (29) is valid for all $x, y > 0$ and each $q > 1$. Now let us focus on inequality (30). Since the function $\chi_q^1(x)$ is decreasing on $(0,\infty)$ for each $q > 1$, we have

$$\frac{1}{x-y}\log\left[\frac{\Gamma_q(x+1)}{\Gamma_q(y+1)}\right] + \frac{(1-x)(x+2)-(1-y)(y+2)}{2(x-y)}\log(q) \le \log(\frac{1}{q-1}) \tag{31}$$

and thus the inequality (30) is obvious. We consider the function

$$K_q^1(x,y) = \begin{cases} \left[\frac{q^{\frac{(1-x)(x+2)}{2}}\Gamma_q(x+1)}{q^{\frac{(1-y)(y+2)}{2}}\Gamma_q(y+1)}\right]^{\frac{1}{x-y}} & \text{if } x \neq y, \\ \exp(\psi_q(x+1) - (x+1/2)\log(q)) & \text{if } x = y. \end{cases} \tag{32}$$

By using (6), we get

$$\lim_{x\to\infty}(\psi_q(x+1) - (x+1/2)\log(q)) = -\log(q-1). \tag{33}$$

Then

$$\lim_{x\to\infty} K_q^1(x,x) = \frac{1}{q-1}.$$

This completes the proof of Theorem 2.5. $\square$

*Remark* 2.6. In [7, Theorem 2.3], the authors was proved that the function $g(q;x)$ defined by

$$g(q;x) = q^{(x-x^2)/2}(q-1)^x\Gamma_q(x),$$

is completely monotonic on $(0,\infty)$ for all $q>1$. We note that in Theorem 2.5, we obtain that the function

$$x \mapsto \left(\frac{q^{1-x}(1-q^x)}{1-q}\right) g(q;x) := \chi_q^1(x)$$

is logarithmically completely monotonic on $(0,\infty)$ for all $q>1$, and consequently is completely monotonic on $(0,\infty)$.

**Theorem 2.7.** *Let $a, b, A$ and $q$ be positive real numbers such that $0<q<1$ and $b \geq a$. Then the function $G(x)$ defined by*

$$G(x) = \frac{\Gamma_q(Ax+a)}{\Gamma_q(Ax+b)} - \frac{\Gamma_q(Ax+A+a)}{\Gamma_q(Ax+A+b)}, \tag{34}$$

*is non-negative and decreasing on $(0,\infty)$. Moreover, the following inequalities fold true*

$$\frac{\Gamma_q(Ax+A+a)}{\Gamma_q(Ax+A+b)} \leq \frac{\Gamma_q(Ax+a)}{\Gamma_q(Ax+b)} \leq \frac{\Gamma_q(Ax+A+a)}{\Gamma_q(Ax+A+b)} + \epsilon(a,b,A), \tag{35}$$

*where*

$$\epsilon(a,b,A) = \frac{\Gamma_q(a)}{\Gamma_q(b)} - \frac{\Gamma_q(A+a)}{\Gamma_q(A+b)},$$

*is the best possible constant.*

*Proof.* Differentiation yields

$$\begin{aligned} G'(z) = & \frac{A\Gamma_q(a+Az)}{\Gamma_q(b+Az)}\left[\psi_q(a+Az)-\psi_q(b+Az)\right] \\ & - \frac{A\Gamma_q(a+A+Az)}{\Gamma_q(b+A+Az)}\left[\psi_q(a+A+Az)-\psi_q(b+A+Az)\right]. \end{aligned} \tag{36}$$

On the other hand, due to log-convexity property of the Gamma function $\Gamma_q(z)$, the ratio $x \mapsto \frac{\Gamma_q(x+\alpha)}{\Gamma_q(x)}$ is increasing on $(0,\infty)$, when $\alpha>0$. This implies that the following inequality

$$\frac{\Gamma_q(x+\alpha)}{\Gamma_q(x+\alpha+\beta)} \leq \frac{\Gamma_q(x)}{\Gamma_q(x+\beta)}, \tag{37}$$

hold true for all $\alpha, \beta, z > 0$. Setting $x = a+Az,\ \alpha = A$, and $\beta = b-a$ in (37), we get

$$\frac{\Gamma_q(a+A+Az)}{\Gamma_q(b+A+Az)} \leq \frac{\Gamma_q(a+Az)}{\Gamma_q(b+Az)}. \tag{38}$$

Hence, by using the fact that the $q$-digamma function $\psi_q(z)$ is increasing on $(0,\infty)$ and in view of inequalities (36) and (38), we obtain

$$
\begin{aligned}
G'(z) &\leq \frac{A\Gamma_q(a+A+Az)}{\Gamma_q(b+A+Az)}\big[\psi_q(b+A+Az)-\psi_q(a+A+Az) \\
&\qquad + \psi_q(a+Az)-\psi_q(b+Az)\big] \\
&= \frac{\Gamma_q(a+A+Az)}{\Gamma_q(b+A+Az)}\Phi^A_{a,b}(q;z) \text{ (say).}
\end{aligned} \tag{39}
$$

By using the $q$-integral representation of $q$-digamma function [14, Formula (2.4)]

$$\psi_q(x)=\psi_q(1)-\frac{\log(q)}{1-q}\int_0^q \frac{1-t^{x-1}}{1-t}d_qt,\ x>0, \tag{40}$$

we obtain

$$\Phi^A_{a,b}(q;z)=-\frac{\log(q)}{1-q}\int_0^q \frac{t^{Ax-1}(1-t^A)(t^b-t^a)}{1-t}d_qt \leq 0 \tag{41}$$

for all $z>0$ and $b\geq a\geq 0$. Keeping (39) and (41) in mind we deduce that the function $G(x)$ is decreasing on $(0,\infty)$. □

*Remark* 2.8. We note that Theorem 2.7 is the $q$-version of Lemma 5 in [11].

## 3. A class of completely monotonic functions related to the $q$-digamma functions and its applications

The first main aim of this section is to investigate the complete monotonicity property of the function $F_a(q;x)$ defined by

$$F_a(q;x)=\psi_q(x+a)-\log\left(\frac{1-q^{x+a-\frac{1}{2}}}{1-q}\right), \tag{42}$$

where $q,x>0$ and $a\geq\frac{1}{2}$.

**Theorem 3.1.** *Let $q$ and $a$ be real numbers such that $q>0$ and $a\geq\frac{1}{2}$. Then the function $F_a(q;x)$ is completely monotonic on $(0,\infty)$.*

*Proof.* Let $0<q<1$. Differentiating (42) with respect to $x$, using (5), yields

$$
\begin{aligned}
F'_a(q;x) &= \psi'_q(x+a)+\frac{q^{x+a-\frac{1}{2}}\log(q)}{1-q^{x+a-\frac{1}{2}}} \\
&= \sum_{k=1}^{\infty}\frac{kq^{k(x+a)}\log^2(q)}{1-q^k}+\log(q)\sum_{k=1}^{\infty}q^{k\left(x+a-\frac{1}{2}\right)} \\
&= \sum_{k=1}^{\infty}\frac{q^{k(x+a)}f(q^k)\log(q)}{1-q^k},
\end{aligned} \tag{43}
$$

where

$$f(y) = \frac{1 - y + \sqrt{y}\log(y)}{\sqrt{y}},\ 0 < y < 1.$$

It is obvious to proved that the function $f(y)$ is non-negative on $(0,1)$. This yields that

$$(-1)^n F_a^{(n)}(q;x) = \sum_{k=1}^{\infty} \frac{q^{k(x+a)} f(q^k) \log^n(1/q)}{1-q^k} \geq 0 \tag{44}$$

for all $0 < q < 1$ and $n \in \mathbb{N}$. This implies that the function $F_a(q;x)$ is decreasing on $(0,\infty)$, from (27) we find

$$\lim_{x\to\infty} F_a(q;x) = 0,\ 0 < q < 1, \tag{45}$$

and consequently

$$F_a(q;x) \geq \lim_{x\to\infty} F_a(q;x) = 0,\ 0 < q < 1. \tag{46}$$

This implies that the function $F_a(q;x)$ is completely monotonic on $(0,\infty)$ for $0 < q < 1$. Now, assume that $q > 1$, from (10), we have

$$\begin{aligned} F_a(1/q;x) &= \psi_{1/q}(x+a) - \log\left(\frac{1 - q^{x+a-\frac{1}{2}}}{1-q}\right) + (x+a-3/2)\log(q) \\ &= \psi_q(x+a) - \log\left(\frac{1 - q^{x+a-\frac{1}{2}}}{1-q}\right) = F_a(q;x). \end{aligned} \tag{47}$$

Therefore, the function $F_a(q;x)$ is completely monotonic on $(0,\infty)$ for all $q > 0$ and $a \geq \frac{1}{2}$. This completes the proof. $\square$

*Remark* 3.2. If we let $q \to 1$ in the above Theorem, we deduce that the function $F_a(x)$ defined by

$$F_a(x) = \psi(x+a) - \log(x+a-1/2), \tag{48}$$

is completely monotonic on $(0,\infty)$ for each $a \geq \frac{1}{2}$.

*Remark* 3.3. The above result is shown to be a generalization of result which were obtained by Batir in [4, Theorem 2.2], that is, the function

$$x \mapsto \psi_q(x+1) - \log\left(\frac{1 - q^{x+\frac{1}{2}}}{1-q}\right),$$

is completely monotonic on $(0,\infty)$ for $q > 0$.

**Corollary 3.4.** *Let $q > 0$. Then the function $H(x;q)$ defined by*

$$H(x;q) = \log\Gamma_q(x+1/2) - x\log\left(\frac{1-q^x}{1-q}\right) - \frac{Li_2(1-q^x)}{\log(q)}, \tag{49}$$

*is a Bernstein function on $(0,\infty)$.*

*Proof.* Differentiating $H(x;q)$ yields that

$$H'(x;q) = F_{1/2}(q;x).$$

By using Theorem 3.1, we deduce that the function $H'(x;q)$ is completely monotonic on $(0,\infty)$ and consequently the function $H(x;q)$ is a Bernstein function on $(0,\infty)$. □

**Corollary 3.5.** *Let $q > 0$. Then the function $x \mapsto \frac{1}{H(x;q)}$ is logarithmically completely monotonic on* $(0,\infty)$.

*Proof.* By means of Corollary3.4, the function $H(x;q)$ is increasing on $(0,\infty)$. Moreover, by using the fact that the $q$-gamma function is strictly decreasing on $(0,x^*)$ where $x^*$ is the abscissa of the minimum of the $q$-gamma function $\Gamma_q(x)$, such that $x^* \in (1,2)$, (see [13, Lemma 2.2, p. 1668]) we deduce that $\Gamma_q(1/2) > \Gamma_q(1) = 1$. This implies that the function $H(x;q)$ is positive on $(0,\infty)$. Then the hypotheses of Theorem 6 in [10] are fulfilled. □

*Remark* 3.6. If we set $q \to 1$ in Corollary 3.4 and using (10) we deduce that the function $H(x)$ defined by

$$H(x) = \log\Gamma(x+1/2) - x\log(x) + x,\ x > 0,$$

is a Bernstein function. Moreover, by using Corollary 3.5, we obtain that the function $1/H(x)$ is logarithmically completely monotonic on $(0,\infty)$.

In the next results, we present new upper and lower bounds for the $q$-digamma and $q$-trigamma functions.

**Corollary 3.7.** *Let $q, x$ and $a$ be real numbers such that $q, x > 0$ and $a \geq \frac{1}{2}$. Then the following inequalities hold true*:

$$\alpha + \log\left(\frac{1-q^{x+a-\frac{1}{2}}}{1-q}\right) \leq \psi_q(x+a) \leq \log\left(\frac{1-q^{x+a-\frac{1}{2}}}{1-q}\right) + \beta, \tag{50}$$

$$\mu - \frac{q^{x+a-\frac{1}{2}}\log(q)}{1-q^{x+a-\frac{1}{2}}} \leq \psi_q'(x+a) \leq \nu - \frac{q^{x+a-\frac{1}{2}}\log(q)}{1-q^{x+a-\frac{1}{2}}}, \tag{51}$$

*where $\alpha = \nu = 0, \beta = \psi_q(a) - \log\left(\frac{1-q^{x+a-\frac{1}{2}}}{1-q}\right)$, and $\mu = \psi_q'(a) + \frac{q^{a-\frac{1}{2}}\log(q)}{1-q^{a-\frac{1}{2}}}$ are the best possible constants.*

*Proof.* In virtue of Theorem 3.1, the function $F_a(q;x)$ is decreasing on $(0,\infty)$ for all $q > 0$ and $a \geq 1/2$ and consequently we have

$$\alpha = 0 = \lim_{x\to\infty} F_a(q;x) \leq F_a(q;x) \leq F_a(q;0) = \beta.$$

As the function $F_a(q;x)$ is completely monotonic on $(0,\infty)$, the function $F_a'(q;x)$ is increasing on $(0,\infty)$, this implies that

$$\mu = F_a'(q;0) \leq F_a'(q;x) \leq \lim_{x\to\infty} F_a'(q;x) = \nu,$$

which is equivalent to (51). □

*Remark* 3.8. Letting $q \to 1$ in Corollary 3.7, we get the following inequalities

$$\log(x+a-1/2) \le \psi(x+a) \le \log(x+a-1/2) + \psi(a) - \log(a-1/2), \tag{52}$$

and

$$\psi'(a) + \frac{2}{2x+2a-1} - \frac{2}{2a-1} \le \psi'(x+a) \le \frac{2}{2x+2a-1}, \tag{53}$$

hold true for all $x > 0$ and $a \ge 1/2$. Both bounds are sharp.

A $q$-analogue of Harmonic number defined by [15]

$$H_{n,q} = \sum_{k=1}^{n} \frac{q^k}{1-q^k}, \tag{54}$$

which can be related to $\psi_q(n+1)$ for a positive integer $n$ by

$$\psi_q(n+1) = \frac{\log(q)}{1-q}\gamma_q - \log(q) H_{n,q}, \tag{55}$$

where $\gamma_q = \frac{1-q}{\log(q)}\psi_q(1)$ is the $q$-analogue of the Euler-Mascheroni constant [12].

Upon setting $x = n \in \mathbb{N}$ and $a = 1$ in the inequalities (50), we obtain the following new upper and lower bounds for the $q$-analogue of Harmonic numbers.

**Corollary 3.9.** *Let $n$ be a positive integer and $0 < q < 1$. Then, the following inequalities hold true*:

$$\gamma_q + \frac{q-1}{\log(q)} \log\left(\frac{1-q^{n+1/2}}{1-q}\right) \le (1-q)H_{n,q} \le \frac{q-1}{\log(q)} \log\left(\frac{1-q^{n+1/2}}{1-q^{1/2}}\right). \tag{56}$$

*Both bounds are sharp.*

*Remark* 3.10. Letting $x = n \in \mathbb{N}$ and $a = 1$ in (52), we find the new sharp upper and lower bound for the harmonic number

$$\gamma + \log(n+1/2) \le H_n \le \log(2) + \log(n+1/2). \tag{57}$$

## 4. Some two-sided bounding inequalities for the $q$-digamma and q-tetragamma functions

In this section, we present two sets of two-sided bounding inequalities for the $q$-digamma and $q$-tetragamma functions in terms of the $q$-gamma function.

**Theorem 4.1.** *Let $0 < q < 1$. Then the following inequalities*

$$\frac{q^x \log(q)}{1-q^x} + \frac{1 - \exp\left(\frac{(q^x-q)\Gamma_q(x)}{1-q}\right)}{\Gamma_q(x+1)} \le \psi_q(x) \le \frac{\exp\left(\frac{(q-q^x)\Gamma_q(x)}{1-q}\right) - 1}{\Gamma_q(x)} \tag{58}$$

*hold true for all $x > 0$.*

*Proof.* Letting $x > 0$ and $0 < q < 1$. Applying the mean value Theorem to the function $e^{\Gamma_q(t)}$ on the interval $[x, x+1]$, we find that

$$e^{\Gamma_q(x+1)} - e^{\Gamma_q(x)} = \Gamma_q'(x+\epsilon)e^{\Gamma_q(x+\epsilon)},\ 0 < \epsilon < 1. \tag{59}$$

We consider the following auxiliary function

$$U(x) = \Gamma_q'(x+\epsilon)e^{\Gamma_q(x+\epsilon)},\ x > 0,\ 0 < \epsilon, q < 1.$$

Hence

$$U'(x) = [\Gamma_q''(x+\epsilon) + (\Gamma_q'(x+\epsilon))^2]e^{\Gamma_q(x+\epsilon)}.$$

By using the integral representation of the $q$-gamma function [9]

$$\Gamma_q(x) = \int_0^{\frac{1}{1-q}} t^{x-1}E_q(q(q-1)t)d_qt,\ x > 0, \tag{60}$$

where $E_q(t)$ is the $q$-exponential function defined by

$$E_q(t) = \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}}t^k}{(q;q)_k},\ t \in \mathbb{C}.$$

This implies that the function $\Gamma_q''(x)$ is non-negative, and consequently the function $U(x)$ is increasing on $(0, \infty)$. This yields that

$$\Gamma_q'(x)e^{\Gamma_q(x)} \le e^{\Gamma_q(x+1)} - e^{\Gamma_q(x)} \le \Gamma_q'(x+1)e^{\Gamma_q(x+1)},\ x > 0. \tag{61}$$

Combining the left hand side of the above inequalities with the recurrence formula (14), we get

$$\Gamma_q'(x) \le \exp\left(\frac{(q-q^x)\Gamma_q(x)}{1-q}\right) - 1, \tag{62}$$

which is equivalent to the right hand side of inequalities (58). Now, focus on the left hand side of inequalities (58). By (14), we find

$$\Gamma_q'(x+1) = \frac{1-q^x}{1-q}\Gamma_q'(x) - \frac{q^x\log(q)}{1-q}\Gamma_q(x).$$

Keeping in mind the above formula with the right hand side of inequalities (61), we arrive at

$$\frac{1-q}{1-q^x}\left(1 - \exp\left(\frac{(q^x-q)\Gamma_q(x)}{1-q}\right) + \frac{q^x\log(q)}{1-q}\Gamma_q(x)\right) \le \Gamma_q'(x) \tag{63}$$

which proves to the left hand side of inequalities (58). The proof is now completed. □

Letting $q$ tends to 1 in the inequalities (58), we obtain the new two-sided bounding inequalities for the digamma function $\psi(x)$.

**Corollary 4.2.** *For every $x > 0$, it holds*

$$\frac{1 - e^{(1-x)\Gamma(x)}}{\Gamma(x+1)} - \frac{1}{x} \le \psi(x) \le \frac{e^{(x-1)\Gamma(x)} - 1}{\Gamma(x)}. \tag{64}$$

In the next theorem, we present two-sided bounding inequalities of the $q$-tetragamma function $\psi_q''(x)$.

**Theorem 4.3.** *Let $q, x > 0$. The following two-sided bounding inequalities hold true*:
(65)
$$1-\exp\left(\frac{q^x\cdot\log^2(q)}{(q^x-1)^2}\right)-\frac{q^{3x}(q^x+1)\log^3(q)}{(q^x-1)^3}\leq\psi_q''(x)\leq\exp\left(-\frac{q^x\cdot\log^2(q)}{(q^x-1)^2}\right)-1.$$

*Proof.* Let $0 < q < 1$. By again the mean value theorem applying to the function $e^{\psi'_q(t)}$ in the interval $[x, x+1]$, therefore we have

$$e^{\psi'_q(x+1)}-e^{\psi'_q(x)}=\psi_q''(x+\epsilon_2)e^{\psi'_q(x+\epsilon_2)},\ 0<\epsilon_2<1. \tag{66}$$

We set
$$U_3(x)=\psi_q''(x)e^{\psi'_q(x)}.$$
Differentiation gives
$$U_3'(x)=(\psi_q^{(3)}(x)+[\psi_q^{(2)}(x)]^2)e^{\psi'_q(x)}.$$
Since $\psi_q^{(3)}(x)+[\psi_q^{(2)}(x)]^2\geq 0$, we deduce that the function $U_3(x)$ is increasing on $(0,\infty)$, which readily implies that the following inequalities hold true:

$$\psi_q''(x)e^{\psi'_q(x)}\leq e^{\psi'_q(x+1)}-e^{\psi'_q(x)}\leq\psi_q''(x+1)e^{\psi'_q(x+1)}. \tag{67}$$

We take logarithm of both sides of (14) and then differentiate, we find

$$\psi_q(x+1)-\psi_q(x)=\frac{q^x\log(q)}{q^x-1},\ \ \psi'_q(x+1)-\psi'_q(x)=-\frac{q^x\log^2(q)}{(q^x-1)^2} \tag{68}$$

$$\psi_q''(x+1)-\psi_q''(x)=\frac{q^{3x}\log^3(q)}{(q^x-1)^3}. \tag{69}$$

By applying the above equations with (67) we easily get the two–sided inequalities (65) asserted by Theorem 4.3. □

Letting $q \to 1$ in Theorem 4.3 leads to the new two–sided inequalities for the trigamma function.

**Corollary 4.4.** *Let $x > 0$. It holds*

$$1-e^{\frac{1}{x}}-\frac{2}{x^3}\leq\psi''(x)\leq e^{-\frac{1}{x}}-1. \tag{70}$$

## References

[1] H. Alzer, *Sharp bounds for the ratio of q-gamma functions*, Math. Nachr. **222** (2001), 5–14.
[2] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, *Inequalities for quasiconformal mappings in space*, Pacific J. Math. **160** (1993), no. 1, 1–18. http://projecteuclid.org/euclid.pjm/1102624560
[3] N. Batir, *q-extensions of some estimates associated with the digamma function*, J. Approx. Theory **174** (2013), 54–64. https://doi.org/10.1016/j.jat.2013.06.002

[4] ———, *Monotonicity properties of q-digamma and q-trigamma functions*, J. Approx. Theory **192** (2015), 336–346. https://doi.org/10.1016/j.jat.2014.12.013

[5] J. El Kamel and K. Mehrez, *A function class of strictly positive definite and logarithmically completely monotonic functions related to the modified Bessel functions*, Positivity **22** (2018), no. 5, 1403–1417. https://doi.org/10.1007/s11117-018-0584-3

[6] B.-N. Guo and F. Qi, *A property of logarithmically absolutely monotonic functions and the logarithmically complete monotonicity of a power-exponential function*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **72** (2010), no. 2, 21–30.

[7] M. E. H. Ismail, L. Lorch, and M. E. Muldoon, *Completely monotonic functions associated with the gamma function and its q-analogues*, J. Math. Anal. Appl. **116** (1986), no. 1, 1–9. https://doi.org/10.1016/0022-247X(86)90042-9

[8] C. H. Kimberling, *A probabilistic interpretation of complete monotonicity*, Aequationes Math. **10** (1974), 152–164. https://doi.org/10.1007/BF01832852

[9] T. H. Koornwinder, *q-Special functions*, a tutorial, arXiv:math/9403216v2

[10] K. Mehrez, *A class of logarithmically completely monotonic functions related to the q-gamma function and applications*, Positivity **21** (2017), no. 1, 495–507. https://doi.org/10.1007/s11117-016-0431-3

[11] ———, *Some geometric properties of a class of functions related to the Fox-Wright functions*, Banach J. Math. Anal. (2020), https://doi.org/10.1007/s43037-020-00059-w

[12] A. Salem, *A q-analogue of the exponential integral*, Afr. Mat. **24** (2013), no. 2, 117–125. https://doi.org/10.1007/s13370-011-0046-6

[13] ———, *A certain class of approximations for the q-digamma function*, Rocky Mountain J. Math. **46** (2016), no. 5, 1665–1677. https://doi.org/10.1216/RMJ-2016-46-5-1665

[14] ———, *Generalized the q-digamma and the q-polygamma functions via neutrices*, Filomat **31** (2017), no. 5, 1475–1481. https://doi.org/10.2298/FIL1705475S

[15] C. Wei and Q. Gu, *q-generalizations of a family of harmonic number identities*, Adv. in Appl. Math. **45** (2010), no. 1, 24–27. https://doi.org/10.1016/j.aam.2009.11.007

Khaled Mehrez
Department of Mathematics
Faculty of Sciences of Tunis
University of Tunis El Manar
1060 Tunis, Tunisia
*Email address*: k.mehrez@yahoo.fr