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STRONG HYPERCYCLICITY OF BANACH SPACE OPERATORS

Mohammad Ansari, Karim Hedayatian, and Bahram Khani-Robati

Dedicated to Kit Chan and Vladimir Troitsky

Abstract. A bounded linear operator T on a separable infinite dimensional Banach space X is called strongly hypercyclic if

$$X\backslash\{0\} \subseteq \bigcup_{n=0}^{\infty} T^n(U)$$

for all nonempty open sets $U \subseteq X$. We show that if T is strongly hypercyclic, then so are T^n and cT for every $n \geq 2$ and each unimodular complex number c. These results are similar to the well known Ansari and León-Müller theorems for hypercyclic operators. We give some results concerning multiplication operators and weighted composition operators. We also present a result about the invariant subset problem.

1. Introduction

Let X be a separable infinite dimensional Banach space and $B(X)$ be the space of all bounded linear operators on X. An operator $T \in B(X)$ is said to be *hypercyclic* if there is some $x \in X$ for which

$$\text{orb}(T, x) = \{T^n x : n \in \mathbb{N}_0\}$$

is dense in X. In that case, x is called a *hypercyclic vector* for T. Here $\mathbb{N}_0$ is the set of all nonnegative integers and $T^0 = I$, the identity operator on X.

The set of all hypercyclic vectors for T is denoted by $HC(T)$ and it is known that if T is hypercyclic, then $HC(T)$ is dense in X. An operator $T \in B(X)$ is called *hypertransitive* if $HC(T) = X\backslash\{0\}$. A set $M \subseteq X$ is called an invariant subset for T if $T(M) \subseteq M$. It is clear that T is hypertransitive if and only if T lacks nontrivial closed invariant subsets. The Read operator on ℓ^1 is an example of such operators [14].

An operator T is called *topologically transitive* if, for any nonempty open sets $U, V \subseteq X$, there is some $n \in \mathbb{N}_0$ such that $T^n(U) \cap V \neq \emptyset$. It is well known

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that an operator T is topologically transitive if and only if it is hypercyclic [6]. Two excellent sources for studying linear dynamics are [4] and [9].

Recently, the authors in [3] have introduced the notion of strong topological transitivity for continuous linear operators. The similar notion of strong transitivity in (nonlinear) dynamical systems was presented in [10]. Here (since topological transitivity and hypercyclicity are equivalent) we use the title *strong hypercyclicity* for strong topological transitivity. Before going to the definition, note that an operator $T \in B(X)$ is topologically transitive if and only if $\overline{\bigcup_{n=0}^{\infty} T^n(U)} = X$ for any nonempty open set $U \subseteq X$.

Definition 1.1. An operator $T \in B(X)$ is called strongly hypercyclic if $X\backslash\{0\} \subseteq \bigcup_{n=0}^{\infty} T^n(U)$ for any nonempty open set $U \subseteq X$.

In [1], strong hypercyclicity of some well known classes of operators has been investigated. For example, it has been shown that the operator λB, the scalar multiple of the backward shift B on ℓ^p $(1 \leq p < \infty)$ and c_0, is strongly hypercyclic if and only if it is hypercyclic (i.e., if and only if $|\lambda| > 1$), some hypercyclic weighted backward shifts on ℓ^p $(1 \leq p < \infty)$ and c_0 are strongly hypercyclic and some fail to be, and no composition operator on a Banach space of analytic functions on (the open unit disk) $\mathbb{D}$ can be strongly hypercyclic.

For the sake of the reader's convenience, we bring the statements of some basic results in strong hypercyclicity.

Proposition 1.2 (Proposition 4 of [3]). *Every strongly hypercyclic operator is surjective.*

Proposition 1.3 (Proposition 6 of [3]). *An invertible operator T is strongly hypercyclic if and only if T^{-1} is hypertransitive.*

Theorem 1.4 (Theorem 2.1 of [1]). *Let $T \in B(X)$ be surjective with a right inverse map S. Then T is strongly hypercyclic if and only if, for every nonzero vector $x \in X$ and any $y \in X$, there exist sequences $(n_k)_k$ in $\mathbb{N}_0$ and $(w_k)_k$ in X such that $w_k \in KerT^{n_k}$ and $S^{n_k}x + w_k \to y$ as $k \to \infty$.*

The *generalized kernel* of an operator $T \in B(X)$ is defined by $\bigcup_{n=1}^{\infty} \mathrm{Ker}T^n$. It is clear that the generalized kernel is a subspace of X. The following result shows that the generalized kernel of a strongly hypercyclic operator is either (0) or a dense subspace.

Proposition 1.5. *Assume that $T \in B(X)$ is strongly hypercyclic. Then the generalized kernel of T is either (0) or a dense subspace of X.*

Proof. If T is injective, then the generalized kernel of T is (0). To see the other possibility, suppose T is not injective. Let U be a nonempty open subset of X and choose a nonzero vector $x \in \bigcup_{n=1}^{\infty} \mathrm{Ker}T^n$. Then $T^k x = 0$ for some $k \geq 1$. But, by the definition of strong hypercyclicity, there is some $m \geq 0$ such that $x \in T^m(U)$. Then $0 = T^k x \in T^{k+m}(U)$ and so, there is some $y \in U$ such that $T^{k+m}y = 0$. Hence, $y \in U \cap (\bigcup_{n=1}^{\infty} \mathrm{Ker}T^n)$ and we are done. $\square$

Note that since we do not know whether there exist invertible hypertransitive operators, by Proposition 1.3 the existence of invertible strongly hypercyclic operators too remains as an open question.

The following result shows that strong hypercyclicity (at least for non-invertible operators) is strictly stronger than satisfying *the hypercyclicity criterion.* Recall that the hypercyclicity criterion which was first introduced in [11], is a good tool to prove the hypercyclicity of many classes of operators. Here, we use that version which has been presented in [5]. It says that, if for an operator T acting on a separable $\mathcal{F}$-space (i.e., a complete metrizable topological vector space) X, there are two dense sets D_1, $D_2 \subseteq X$, a sequence of nonnegative integers $(n_k)_k$ and a sequence of maps $(S_{n_k})_k$ defined on D_2 such that

$$\begin{array}{ll} T^{n_k}x \to 0 & \forall x \in D_1, \\ S^{n_k}(y) \to 0 & \forall y \in D_2, \text{ and} \\ T^{n_k}S^{n_k}(y) \to y & \forall y \in D_2, \end{array}$$

then T is hypercyclic.

Proposition 1.6. *Suppose that $T \in B(X)$ is not invertible. If T is strongly hypercyclic, then T satisfies the hypercyclicity criterion but the converse is not true.*

Proof. If T is strongly hypercyclic, then it is hypercyclic and, by Proposition 1.5, it has dense generalized kernel. Then by Proposition 2.11 of [5], T satisfies the hypercyclicity criterion. To show that the converse is not true, we refer to the weighted backward shift operator presented in Example 2.11 of [1]. Note that the hypercyclicity of weighted backward shifts is proved by the hypercyclicity criterion. □

In Section 2, we present two theorems which are analogues of the well known Ansari and León-Müller theorems for hypercyclic operators. In Section 3, we generalize the notion of strong hypercyclicity to semigroups of operators. Then we define strong supercyclicity and give some results on it. In Section 4, we see that, unlike supercyclic and hypercyclic operators, strongly supercyclic and strongly hypercyclic operators are scalar multiples of one another. Then we present some results about the adjoints of multiplication and weighted composition operators. Finally, in Section 5, a result is given concerning the invariant subset problem.

2. Analogues of Ansari and León-Müller theorems

We use the following remark in the proofs of Theorems 2.2 and 2.3.

Remark 2.1. In Theorem 1.4, if T is strongly hypercyclic, then we can claim that there is a *strictly increasing* sequence $(n_k)_k$ such that $S^{n_k}x + w_k \to y$. In fact, by Proposition 1.2 and the open mapping theorem, $T^k(U)$ is open for every $k \geq 1$ and each open set U in X. Thus, by choosing $T^k(U)$ instead of U in the definition of strong hypercyclicity, we have $X\backslash\{0\} \subseteq \bigcup_{n=k}^{\infty} T^n(U)$ for every $k \geq 1$. We use this fact in the proof of Theorem 2.1 of [1] to obtain a strictly increasing sequence $(n_k)_k$. Indeed, for the neighborhood basis $\{U_k\}_{k=1}^{\infty}$ at y satisfying $U_1 \supset U_2 \supset U_3 \supset \cdots$, we write $X\backslash\{0\} \subseteq \bigcup_{n=0}^{\infty} T^n(U_1)$ to find some $y_1 \in U_1$ and $n_1 \geq 0$ such that $x = T^{n_1}y_1$. Then, assuming that n_k has been found such that $x = T^{n_k}y_k$ for some $y_k \in U_k$, we write $X\backslash\{0\} \subseteq \bigcup_{n>n_k}^{\infty} T^n(U_{k+1})$ to find $n_{k+1} > n_k$ and $y_{k+1} \in U_{k+1}$ such that $x = T^{n_{k+1}}y_{k+1}$.

Recall that the famous Ansari theorem [2] says that if T is a hypercyclic operator, then, for any integer $n \geq 2$, T^n is also hypercyclic with the same hypercyclic vectors. We present the following similar result.

Theorem 2.2. *If $T \in B(X)$ is a strongly hypercyclic operator, then so is T^n for every $n \geq 2$.*

Proof. Fix an integer $n \geq 2$ and let S be a right inverse map of T. If T is invertible, then $S = T^{-1}$ is hypertransitive by Proposition 1.3 and hence so is S^n by Ansari theorem. Then $T^n = (S^n)^{-1}$ is strongly hypercyclic by Proposition 1.3. Now, suppose T is not invertible. If $x \neq 0$ and y are arbitrary vectors in X, then by Theorem 1.4 and Remark 2.1 there is a strictly increasing sequence $(n_k)_k$ in $\mathbb{N}_0$ and a sequence $(w_k)_k$ in X with $w_k \in \mathrm{Ker}T^{n_k}$ such that $S^{n_k}x+w_k \to y$. Suppose $n_k = nq_k+r_k$ $(k \geq 1, 0 \leq r_k \leq n-1)$. Then there is an integer $R \in [0, n-1]$ for which there are subsequences of $(n_k)_k$ and $(q_k)_k$, again denoted by $(n_k)_k$ and $(q_k)_k$, such that $n_k = nq_k+R$. Then $S^R(S^n)^{q_k}(x)+w_k \to y$ and so $(S^n)^{q_k}(x) + u_k \to T^R y$ where $u_k = T^R(w_k) \in \mathrm{Ker}(T^n)^{q_k}$. On the other hand, by Proposition 1.5, there is a sequence $(v_k)_k$ in $\bigcup_{m=1}^{\infty} \mathrm{Ker}T^m$ such that $v_k \to y - T^R y$. Passing through a subsequence of $(q_k)_k$, we can assume that $v_k \in \mathrm{Ker}(T^n)^{q_k}$ (since $n_k \to \infty$ and $n_k = nq_k + R$ we see that $q_k \to \infty$). Then we have $(S^n)^{q_k}(x) + t_k \to y$ where $t_k = u_k + v_k \in \mathrm{Ker}(T^n)^{q_k}$. Thus, T^n is strongly hypercyclic by Theorem 1.4. □

By the well known León-Müller theorem [12], if T is a hypercyclic operator on a complex topological vector space, then, for any unimodular number $c \in \mathbb{C}$, the operator cT is also hypercyclic with the same hypercyclic vectors. We give the following similar result. Note that here X is assumed to be a complex Banach space.

Theorem 2.3. *If $T \in B(X)$ is strongly hypercyclic, then so is cT for every unimodular $c \in \mathbb{C}$.*

Proof. Assume that $|c| = 1$ and $TS = I$. If T is invertible, then $S = T^{-1}$ is hypertransitive by Proposition 1.3 and so, by the León-Müller theorem, $\frac{1}{c}S$ is also hypertransitive. Hence, $cT = (\frac{1}{c}S)^{-1}$ is strongly hypercyclic by Proposition 1.3. Now, suppose T is not invertible. Let $x \neq 0$ and y be arbitrary vectors in X. By Theorem 1.4 and Remark 2.1 there are a strictly increasing sequence $(n_k)_k$ in $\mathbb{N}_0$ and a sequence $(w_k)_k$ in X with $w_k \in \mathrm{Ker}T^{n_k}$ such that $S^{n_k}x + w_k \to y$. There is a subsequence of $(n_k)_k$, which we denote it again by $(n_k)_k$, such that the sequence $(c^{n_k})_k$ is convergent; say $c^{n_k} \to c$. Then

$\frac{1}{c^{n_k}}S^{n_k}x + \frac{1}{c^{n_k}}w_k \to \frac{1}{c}y$. On the other hand, by Proposition 1.5, there is a sequence $(u_k)_k$ in $\bigcup_{n=1}^{\infty} \mathrm{Ker}T^n$ such that $u_k \to y - \frac{1}{c}y$. Passing through a subsequence of $(n_k)_k$ (again denoted by $(n_k)_k$), we can assume that $u_k \in \mathrm{Ker}T^{n_k}$. Now, putting $v_k = \frac{1}{c^{n_k}}w_k + u_k$, we have $v_k \in \mathrm{Ker}(cT)^{n_k}$ and $(\frac{1}{c}S)^{n_k}x + v_k \to y$. Thus, cT is strongly hypercyclic by Theorem 1.4. □

3. Semigroups of operators and strong supercyclicity

We generalize the notion of strong hypercyclicity to arbitrary (multiplicative) semigroups of operators.

Definition 3.1. A semigroup $\mathcal{T} \subset B(X)$ is called strongly hypercyclic if $X\backslash\{0\} \subseteq \bigcup_{T\in\mathcal{T}} T(U)$ for any nonempty open set $U \subseteq X$.

Definition 1.1 shows that strong hypercyclicity of an operator T is in fact strong hypercyclicity of the semigroup $\mathcal{T} = \{T^n : n \in \mathbb{N}_0\}$.

The next result is a generalization of Proposition 1.5. The proof is similar and so we omit it.

Proposition 3.2. *Let $\mathcal{T} \subseteq B(X)$ be a strongly hypercyclic semigroup. Then $\bigcup_{T\in\mathcal{T}} KerT$ is either* (0) *or a dense subset* (*subspace if $\mathcal{T}$ is commutative*) *of X.*

Let us call $\bigcup_{T\in\mathcal{T}} \mathrm{Ker}T$ the generalized kernel of $\mathcal{T}$. We show that both cases may happen in Proposition 3.2. Recall that $\mathcal{T}$ is called *strictly transitive* if to each pair x, y of nonzero vectors in X, there corresponds an operator $T \in \mathcal{T}$ such that $Tx = y$. It is clear that if $\mathcal{T}$ is strictly transitive, then it is strongly hypercyclic. Now, it is easy to see that the semigroup of invertible operators is strictly transitive. Meanwhile, the generalized kernel of this semigroup is (0). For the other possibility, consider the semigroup $\{T^n : n \in \mathbb{N}_0\}$ where T is a non-invertible strongly hypercyclic operator. Then the generalized kernel is not (0) and hence it is dense by Proposition 1.5.

A semigroup $\mathcal{S} \subseteq B(X)$ is called hypercyclic if $\mathrm{orb}(\mathcal{S}, x) = \{Tx : T \in \mathcal{S}\}$ is dense in X for some $x \in X$. Then x is called a hypercyclic vector for $\mathcal{S}$ and the set of all hypercyclic vectors for $\mathcal{S}$ is denoted by $HC(\mathcal{S})$. If $HC(\mathcal{S}) = X\backslash\{0\}$, then $\mathcal{S}$ is called hypertransitive and, it is easy to show that $\mathcal{S}$ is hypertransitive if and only if $X\backslash\{0\} \subseteq \bigcup_{T\in\mathcal{S}} T^{-1}(U)$ for any nonempty open set $U \subseteq X$. If the operators in $\mathcal{S}$ are all invertible and we define

$$\mathcal{S}^{-1} = \{T^{-1} : T \in \mathcal{S}\},$$

then we will use the following fact many times in the rest of this paper.

Fact 3.3. *Let $\mathcal{S}$ be a semigroup of operators whose members are all invertible. Then $\mathcal{S}$ is strongly hypercyclic if and only if $\mathcal{S}^{-1}$ is hypertransitive.*

Let us put $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Recall that an operator $T \in B(X)$ is said to be *supercyclic* if there is some nonzero $x \in X$ such that the set

$$\{cT^n x : c \in \mathbb{C}^*, n \in \mathbb{N}_0\}$$

is dense in X. In other words, T is called supercyclic if the semigroup

$$\mathcal{S} = \{cT^n : c \in \mathbb{C}^*, n \in \mathbb{N}_0\}$$

is hypercyclic. Therefore, we can define strong supercyclicity in a similar way based on the notion of strong hypercyclicity.

Definition 3.4. An operator $T \in B(X)$ is said to be strongly supercyclic if the semigroup $\{cT^n : c \in \mathbb{C}^*, n \in \mathbb{N}_0\}$ is strongly hypercyclic.

It is clear that strong hypercyclicity implies strong supercyclicity but not vice-versa. Indeed, if T is a strongly hypercyclic operator, then $T/\|T\|$ is strongly supercyclic but not strongly hypercyclic (in fact it is not even hypercyclic).

Note that we can define strong cyclicity as well. An operator T is called *strongly cyclic* if the semigroup $\{p(T) : p \text{ a polynomial}\}$ is strongly hypercyclic. In this paper, we only deal with strong hypercyclicity and strong supercyclicity.

Proposition 3.5. *If $T \in B(X)$ is strongly supercyclic, then T is surjective.*

Proof. Given a nonzero $x \in X$ and a nonempty open subset U of X, the set $V = U \backslash \mathbb{C}x$ is another nonempty open subset of X. Thus, by the definition of strong supercyclicity, we have $x \in cT^k(V)$ for some $c \in \mathbb{C}^*$ and $k \geq 0$. But, k cannot be zero by the construction of V. So, $k \geq 1$ and this shows that $x \in \text{Ran}T$. □

As applications of Fact 3.3 and Proposition 3.5, we give Proposition 3.6 and Theorem 3.7.

Recall that for an operator $T \in B(X)$, a point $x \in X$ is called *periodic* if $T^k x = x$ for some positive integer k. We say that T is *chaotic* if it is hypercyclic and has a dense set of periodic points. The following proposition shows that injective chaotic operators are not strongly supercyclic (this shows that hypercyclicity does not imply strong supercyclicity).

For a set $Y \subseteq X$, by $\mathbb{C}^*Y$ we mean the set $\{cy : c \in \mathbb{C}^*, y \in Y\}$.

Proposition 3.6. *Suppose $T \in B(X)$ is injective. If T is strongly supercyclic, then $\sigma_p(T^k) = \emptyset$ for all $k \geq 1$. In particular, injective chaotic operators are not strongly supercyclic.*

Proof. To get a contradiction, assume that there is some $k \in \mathbb{N}$ such that $T^k x = ax$ for some nonzero $x \in X$ and some $a \in \mathbb{C}^*$. If T is strongly supercyclic, then it is surjective by Proposition 3.5 and so T is invertible. If $\mathcal{S} = \{cT^n : c \in \mathbb{C}^*, n \in \mathbb{N}_0\}$, then $\mathcal{S}$ is strongly hypercyclic and meanwhile, each operator in $\mathcal{S}$ is invertible. Then by Fact 3.3, $\mathcal{S}^{-1}$ is hypertransitive. But, since $(T^{-1})^k(x) = (1/a)x$ we have

$$\text{orb}(\mathcal{S}^{-1}, x) = \mathbb{C}^*\{x, T^{-1}x, T^{-2}x, \ldots, T^{-k+1}x\}$$

which cannot be dense in X. This shows that $x \notin HC(\mathcal{S}^{-1})$, a contradiction. □

A Hilbert space of analytic functions on $\mathbb{D}$ in which the polynomials are dense and the monomials $1, z, z^2, \ldots$ constitute an orthogonal set of nonzero vectors is called a *weighted Hardy space*. Each weighted Hardy space is characterized by its weight sequence β defined by $\beta(j) = \|z^j\|$ for $j \geq 0$. The weighted Hardy space $H^2(\beta)$ consists of those functions f analytic on $\mathbb{D}$ whose Maclaurin coefficients $(\hat{f}(j))$ satisfy

$$\sum_{j=0}^{\infty} |\hat{f}(j)|^2 \beta(j)^2 < \infty.$$

The inner product of $H^2(\beta)$ is defined by

$$\langle f, g \rangle = \sum_{j=0}^{\infty} \hat{f}(j)\overline{\hat{g}(j)}\beta(j)^2.$$

If $\beta(j) = 1$ for all j, then $H^2(\beta)$ is the classical Hardy space $H^2(\mathbb{D})$; the cases $\beta(j) = (j+1)^{-1/2}$ and $\beta(j) = (j+1)^{1/2}$ give the Bergman and Dirichlet spaces respectively. Recall that for the reproducing kernels K_z ($z \in \mathbb{D}$) in $H^2(\beta)$ we have $\langle f, K_z \rangle = f(z)$ for all $f \in H^2(\beta)$.

We say that the $H^2(\beta)$ is *automorphism invariant* provided that $f \circ \phi \in H^2(\beta)$ whenever $f \in H^2(\beta)$ and ϕ is an automorphism of $\mathbb{D}$. All widely used classical weighted Hardy spaces are automorphism invariant.

The weighted composition operator $C_{w,\phi}$ on $H^2(\beta)$ is defined by $C_{w,\phi}(f) = w(f \circ \phi)$, i.e., $C_{w,\phi}(f)(z) = w(z)f(\phi(z))$. It is easy to see that the nth iterate of $C_{w,\phi}$ is

$$C_{w,\phi}^n(f) = w(w \circ \phi)(w \circ \phi_2) \cdots (w \circ \phi_{n-1})(f \circ \phi_n),$$

where ϕ_k is the kth iterate of ϕ.

Theorem 3.7. *Let $H^2(\beta)$ be an automorphism invariant weighted Hardy space. Then no weighted composition operator on $H^2(\beta)$ is strongly supercyclic.*

Proof. To get a contradiction, assume that $C_{w,\phi}$ is strongly supercyclic on $H^2(\beta)$. Then $w(z) \neq 0$ for all $z \in \mathbb{D}$. To prove this claim, assume that f is a supercyclic vector for $C_{w,\phi}$. Then there are sequences $(b_k)_k$ in $\mathbb{C}^*$ and $(n_k)_k$ in $\mathbb{N}_0$ such that $b_k C_{w,\phi}^{n_k}(f) \to g$ where $g(z) = 1$ ($z \in \mathbb{D}$). Then the pointwise convergence at any $z \in \mathbb{D}$ gives

$$b_k w(z) w(\phi(z)) w(\phi_2(z)) \cdots w(\phi_{n_k - 1}(z)) f(\phi_{n_k}(z)) \to 1$$

which shows that $w(z) \neq 0$ for all $z \in \mathbb{D}$. Meanwhile, it is easy to see that ϕ cannot be a constant map. Indeed, if ϕ is constant, then for any $f \in H^2(\beta)$ we have

$$\{bC_{w,\phi}^n f : b \in \mathbb{C}^*, n \in \mathbb{N}_0\} \subseteq \mathbb{C}^*\{w\}$$

which shows that f cannot be a supercyclic vector. This contradicts our assumption and so, ϕ must be nonconstant. Then $C_{w,\phi}$ is injective and so it is invertible by Proposition 3.5. Hence, by Theorem 3.5 of [7], ϕ is an automorphism of the disk. Then C_ϕ is invertible and $(C_\phi)^{-1} = C_{\phi^{-1}}$. Now, the composition $C_{w,\phi}C_{\phi^{-1}}$ of two bounded invertible operators on $H^2(\beta)$ gives the multiplication operator M_w $(f \mapsto wf)$. Then it is easy to see that w is a bounded function. Indeed, for all $z \in \mathbb{D}$, by using Cauchy-Schwarz inequality we have

$$|w(z)| = |\langle w^n, K_z\rangle|^{1/n} = |\langle M_w^n(1), K_z\rangle|^{1/n} \leq \|M_w\|\|K_z\|^{1/n}$$

for all $n \in \mathbb{N}$. Then letting n tend to infinity, we obtain $|w(z)| \leq \|M_w\|$. On the other hand, it is easy to verify that $C_{w,\phi}C_{1/w\circ\phi^{-1},\phi^{-1}} = I$ which says that $C_{w,\phi}^{-1} = C_{1/w\circ\phi^{-1},\phi^{-1}}$. Let us put $\psi = 1/w \circ \phi^{-1}$ and $\rho = \phi^{-1}$. Then ψ is bounded away from zero. Now, if we put $\mathcal{S} = \{bC_{w,\phi}^n : b \in \mathbb{C}^*, n \in \mathbb{N}_0\}$, by Fact 3.3 the semigroup

$$\mathcal{S}^{-1} = \{bC_{\psi,\rho}^n : b \in \mathbb{C}^*, n \in \mathbb{N}_0\}$$

must be hypertransitive, but we prove that this is not true. First assume that ρ has a fixed point $a \in \mathbb{D}$. If $\mathcal{S}^{-1}$ is hypertransitive, the function $f(z) = z - a$ $(z \in \mathbb{D})$ must be a hypercyclic vector for it and so there are sequences $(b_k)_k$ in $\mathbb{C}^*$ and $(n_k)_k$ in $\mathbb{N}_0$ such that $b_kC_{\psi,\rho}^{n_k}f \to g$ where $g(z) = 1$ $(z \in \mathbb{D})$. Then the pointwise convergence at $z = a$ gives

$$b_k(\psi(a))^{n_k}f(a) \to 1$$

which cannot hold. Thus, ρ does not have any fixed point in $\mathbb{D}$. Now, we show that the constant function $f(z) = 1$ $(z \in \mathbb{D})$ cannot be a hypercyclic vector for $\mathcal{S}^{-1}$. In fact, if $f \in HC(\mathcal{S}^{-1})$, there are sequences $(b_k)_k$ in $\mathbb{C}^*$ and $(n_k)_k$ in $\mathbb{N}_0$ such that $b_kC_{\psi,\rho}^{n_k}f \to g$ where $g(z) = \rho(0) - z$ $(z \in \mathbb{D})$. Then the pointwise convergence at $z = 0$ and $z = \rho(0)$ give

$$b_k\psi(0)\psi(\rho(0))\cdots\psi(\rho_{n_k-1}(0)) \to \rho(0),$$

$$b_k\psi(\rho(0))\psi(\rho_2(0))\cdots\psi(\rho_{n_k}(0)) \to 0.$$

Now, by dividing the two sequences we have $\psi(\rho_{n_k}(0)) \to 0$ which contradicts that ψ is bounded away from zero. Hence, $\mathcal{S}^{-1}$ cannot be hypertransitive and we are done. □

4. Strong hypercyclicity and strong supercyclicity

We know that there are supercyclic operators whose scalar multiples are not hypercyclic [4, Example 1.15]. Surprisingly, we will see in Corollary 4.4 that this never happens for non-invertible strongly supercyclic operators.

Proposition 4.1 (Corollary 2.3 of [1]). *Suppose $T \in B(X)$ has dense generalized kernel and S is a right inverse map for T. If for all nonzero vectors $x \in X$ the sequence $(S^m x)_m$ is convergent, then T is strongly hypercyclic.*

For any positive number r, let us put $B(r) = \{x \in X : \|x\| < r\}$ and $B = B(1)$.

Theorem 4.2. *Assume that $T \in B(X)$ is not invertible. Then T is surjective with dense generalized kernel if and only if cT is strongly hypercyclic for all $c \in \mathbb{C}^*$ with $|c| > 1/\epsilon$ where $\epsilon = \inf\{\|y\| : y \in X \backslash T(B)\}$.*

Proof. If cT is strongly hypercyclic for some $c \in \mathbb{C}^*$, then it is surjective by Proposition 1.2 and has dense generalized kernel by Proposition 1.5. Thus, T itself is surjective with dense generalized kernel. To prove the necessity, suppose T is surjective and has dense generalized kernel. Then by the open mapping theorem, $T(B)$ is open and hence it contains the open ball $B(\eta)$ for some $\eta > 0$. In fact, it is easy to see that if $\eta < \epsilon$, then $\overline{B(\eta)} \subset T(B)$. Thus, if we put $\epsilon_k = \frac{k}{k+1}\epsilon$ ($k \in \mathbb{N}$), then $\overline{B(\epsilon_k)} \subset T(B)$. Now, let k be a fixed positive integer. Then for any $x \in X$ with $\|x\| = \epsilon_k$ there is some y (we choose one of them) with $\|y\| < 1$, which we call $S_k x$, such that $Ty = TS_k x = x$. Now, for all nonzero $x \in X$ we define

$$S_k x = \frac{\|x\|}{\epsilon_k} S_k(\frac{\epsilon_k}{\|x\|} x),$$

with $S_k 0 = 0$. Then S_k is well defined, and for any $x \in X$ we have $TS_k x = x$ and $\|S_k x\| \le \|x\|/\epsilon_k$. Hence

$$\|(\frac{1}{c} S_k)^n x\| \le \frac{1}{|c|^n {\epsilon_k}^n} \|x\| \to 0$$

for any $x \in X$ and all complex numbers c with $|c| > 1/\epsilon_k$. On the other hand, the operator cT has dense generalized kernel since T has by our assumption. Therefore, by Proposition 4.1, cT is strongly hypercyclic for all $c \in \mathbb{C}$ with $|c| > 1/\epsilon_k$. Now, suppose that $|c| > 1/\epsilon$. Then there is some $k \in \mathbb{N}$ such that $|c| > 1/\epsilon_k$ and so cT is strongly hypercyclic. □

Since every nonzero scalar multiple of a strongly hypercyclic operator is strongly supercyclic, we can obtain the following corollaries from Theorem 4.2, Proposition 3.2 (together with Definition 3.4) and Proposition 3.5.

Corollary 4.3. *Suppose T is not invertible. Then T is strongly supercyclic if and only if T is surjective and has dense generalized kernel.*

Corollary 4.4. *Suppose T is not invertible. Then T is strongly supercyclic if and only if aT is strongly hypercyclic for some $a > 1$.*

Remark 4.5. By Corollary 4.4, if we want to show that a non-invertible operator T is not strongly supercyclic we can prove that aT is not strongly hypercyclic for any $a > 1$. The usefulness of this fact is that instead of working with sequences $(c_k)_k$ in $\mathbb{C}^*$ while dealing with convergence affairs, we can consider the very much easier case of dealing with the sequence $(a^{n_k})_k$. For example, if we have $\|c_k T^{n_k} x\| \to 0$ we cannot generally say anything about the behavior of the sequence $(\|T^{n_k} x\|)_k$, but if $a^{n_k}\|T^{n_k} x\| \to 0$ for some $a > 1$, then we can claim that $\|T^{n_k} x\| \to 0$.

Regarding Theorem 4.2 and Corollary 4.3, whenever we know that a non-invertible operator T is strongly supercyclic, we can try to compute the value of ϵ or at least a lower estimate $r \leq \epsilon$. Then it is clear that for any $c \in \mathbb{C}^*$ with $|c| > 1/r$ the operator cT is strongly hypercyclic.

The following example includes two parts. In the first part, we find the value of ϵ, but in the second one, we find a lower estimate for ϵ.

Example 4.6. (i) Let T be the backward shift operator on ℓ^p $(1 \leq p < \infty)$ or c_0. Then T is strongly supercyclic by Corollary 4.3. On the other hand, since $\|T\| = 1$, it is easy to see that $T(B) = B$. Indeed, $T(B) \subseteq B$ is clear and to show that $B \subseteq T(B)$, suppose $y = (a_1, a_2, \ldots) \in B$. Then if we choose $x = (0, a_1, a_2, \ldots)$, we see that $x \in B$ since $\|x\| = \|y\|$ and $Tx = y$. Thus, $y \in T(B)$. Now, $T(B) = B$ shows that $\epsilon = 1$ and hence cT is strongly hypercyclic for all $c \in \mathbb{C}^*$ such that $|c| > 1 = 1/\epsilon$.

(ii) Suppose T is a weighted backward shift on ℓ^p $(1 \leq p < \infty)$ or c_0 whose weight sequence $(w_n)_n$ satisfies $w_n \geq r$ for some $r > 0$. Pick a number $a > 1$ such that $ar > 1$. Then for the weighted backward shift operator $aT = B_{W'}$, where $W' = (w'_n)_n = (aw_n)_n$, we have $\limsup_{k\to\infty}(w'_1 w'_2 \cdots w'_k) = \infty$. Thus, aT is hypercyclic [16]. On the other hand, since $w'_n w'_{n+1} \cdots w'_{n+k-1} > 1$ for all positive integers n, k, Proposition 2.8 of [1] shows that aT is strongly hypercyclic. Hence, T is strongly supercyclic by Corollary 4.4. Moreover, it is easy to see that $\epsilon \geq r$. So, we can claim that cT is strongly hypercyclic for all $c \in \mathbb{C}^*$ satisfying $|c| > 1/r$.

We use a symbolic language to show, at a glance, the relationships between the sets of all supercyclic and all hypercyclic operators on one hand, and the sets of all non-invertible strongly supercyclic and all non-invertible strongly hypercyclic operators on the other hand. Let $L_s(X)$, $L_h(X)$, $L_{ss}(X)$ and $L_{sh}(X)$ be the above-mentioned sets respectively. Moreover, let $L_{sd}(X)$ be the set of all surjective operators with dense generalized kernel. Then we have

$$L_h(X) \subsetneqq L_s(X) \nsubseteq \mathbb{C}^* L_h(X),$$
$$L_{sh}(X) \subsetneqq L_{ss}(X) = L_{sd}(X) = \mathbb{C}^* L_{sh}(X).$$

4.1. Adjoints of multiplication and weighted composition operators

In this subsection, we give some results which enable us to find strongly hypercyclic and strongly supercyclic adjoints of multiplication and weighted composition operators on $H^2(\mathbb{D})$. Before giving Proposition 4.9, we need the following results.

Proposition 4.7. *Suppose $T \in B(X)$ has dense generalized kernel and S is a right inverse map for T. If there are positive numbers M, ρ such that $\|S^n(x)\| \leq M\rho^n\|x\|$ for each $n \in \mathbb{N}$ and all nonzero $x \in X$, then cT is strongly hypercyclic for all complex numbers c with $|c| > \rho$.*

Proof. Let c be a fixed complex number with $|c| > \rho$. Then $(1/c)S$ is a right inverse map for cT and meanwhile, $\|((1/c)S)^n(x)\| \to 0$ for all $x \in X$. Then cT is strongly hypercyclic by Proposition 4.1. □

In what follows, all multiplication operators are assumed to act on $H^2(\mathbb{D})$. Note that when we say M_w is a multiplication operator on $H^2(\mathbb{D})$, we are implicitly assuming that w is a bounded analytic function on $\mathbb{D}$.

Lemma 4.8. *If w has a zero in $\mathbb{D}$, then the operator M_w^* has dense generalized kernel.*

Proof. Put $T = M_w^*$ and $E_n = \mathrm{Ker} T^n$ $(n \geq 1)$. To prove that $\overline{\bigcup_{n\geq 1} E_n} = H^2(\mathbb{D})$ we show that $(\bigcup_{n\geq 1} E_n)^\perp = (0)$. If $f \in (\bigcup_{n\geq 1} E_n)^\perp$, then for all $n \geq 1$ we have

$$f \in (E_n)^\perp = \overline{\mathrm{Ran} M_w^n} = \overline{\mathrm{Ran} M_{w^n}}.$$

Then for any $n \geq 1$ there is a sequence $(f_{n,j})_j$ in $\mathrm{Ran} M_{w^n}$ such that $f_{n,j} \to f$ in $H^2(\mathbb{D})$ as $j \to \infty$. If $w(a) = 0$ for some $a \in \mathbb{D}$, then for each $n \geq 1$ we have $f_{n,j}^{(n-1)}(a) = 0$ for all $j \in \mathbb{N}$. Here $f^{(0)} = f$ and for any $k \geq 1$, $f^{(k)}$ is the kth derivative of f. Hence, we must have $f^{(n-1)}(a) = 0$ for all $n \geq 1$. Thus, f should be the zero function and we are done. □

We know that no multiplication operator M_w on $H^2(\mathbb{D})$ is hypercyclic and the adjoint M_w^* is hypercyclic if and only if w is nonconstant and $w(\mathbb{D}) \cap \partial\mathbb{D} \neq \emptyset$ [8]. Here, $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$.

Sometimes we had better use another definition of the Hardy space $H^2(\mathbb{D})$ (we have already considered it as the weighted Hardy space $H^2(\beta)$ for which $\beta(j) = 1$ for all $j \geq 0$). The other form of introducing this space is to say that $H^2(\mathbb{D})$ is the set of all $f \in H(\mathbb{D})$ (the space of all analytic functions on $\mathbb{D}$) such that

$$(\|f\|_2)^2 = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty.$$

The inner product is given by

$$\langle f, g \rangle = \sup_{0<r<1} \int_0^{2\pi} f(re^{i\theta}) \overline{g(re^{i\theta})} \frac{d\theta}{2\pi}.$$

Recall that the radial limit of a function $f \in H(\mathbb{D})$ at a point $e^{i\theta} \in \partial\mathbb{D}$ is defined by $f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ if the limit exists. A function $w \in H^\infty(\mathbb{D})$ (the space of all bounded analytic functions on $\mathbb{D}$) is called an *inner function* if $|w^*(e^{i\theta})| = 1$ for almost every $\theta \in [0, 2\pi]$.

It is easy to verify that if w is an inner function, then the multiplication operator M_w is an isometry on $H^2(\mathbb{D})$. Now, inner functions having a zero in $\mathbb{D}$ give strongly hypercyclic adjoints of multiplication operators.

Proposition 4.9. *If w is an inner function with a zero in $\mathbb{D}$, then for any complex number c with $|c| > 1$ the operator cM_w^* is strongly hypercyclic.*

Proof. By Lemma 4.8, the operator M_w^* has dense generalized kernel. On the other hand, M_w is an isometry and so $S = M_w$ is a right inverse for M_w^* and $\|S^n(f)\| = \|f\|$. Thus, by Proposition 4.7, for any $c \in \mathbb{C}$ with $|c| > 1$ the operator cM_w^* is strongly hypercyclic. □

The above proposition together with the following results will enable us to present a wide class of strongly supercyclic (and hence a wide class of strongly hypercyclic) adjoints of multiplication operators on $H^2(\mathbb{D})$.

Lemma 4.10. *If $A, B \in B(X)$, A is non-invertible strongly supercyclic and $AB = BA$, then AB is strongly supercyclic if and only if B is surjective.*

Proof. Suppose B is surjective. Since A is surjective by Proposition 3.5, then so is AB. On the other hand, A has dense generalized kernel by Proposition 3.2 and Definition 3.4. But $\mathrm{Ker}A^n \subseteq \mathrm{Ker}B^nA^n = \mathrm{Ker}(AB)^n$ for all $n \in \mathbb{N}$ and so AB has dense generalized kernel. Thus, AB is strongly supercyclic by Corollary 4.3. Conversely, if AB is strongly supercyclic, then it is surjective and hence by using the assumption $AB = BA$, it is easy to see that B is surjective. □

Proposition 4.11. *Suppose k is an inner function with a zero in $\mathbb{D}$ and $h \in H^\infty(\mathbb{D})$. Then M_{kh}^* is strongly supercyclic if and only if M_h^* is surjective.*

Proof. Put $A = M_k^*$ and $B = M_h^*$. Then A is not invertible since $0 \in k(\mathbb{D})$ and A is strongly supercyclic by Proposition 4.9. Thus, the result follows by Lemma 4.10 □

Corollary 4.12. *Suppose $w \in H^\infty(\mathbb{D})$ has a zero in $\mathbb{D}$. If $w = pq$ is the inner-outer factorization of w, then M_w^* is strongly supercyclic if and only if M_q^* is surjective.*

Proof. By Theorem 17.17 of [15], $q \in H^\infty(\mathbb{D})$. On the other hand, p has a zero in $\mathbb{D}$ (in fact, $Z(w) = Z(p)$ since the outer functions are zero-free) and hence the proof is complete by Proposition 4.11. □

The following corollary is another consequence of Lemma 4.10.

Corollary 4.13. *If A is non-invertible strongly supercyclic, then so is A^n for all $n \geq 2$.*

Lemma 4.14. *If A is non-invertible, B is invertible and $AB = BA$, then AB is strongly supercyclic if and only if A is strongly supercyclic.*

Proof. If A is strongly supercyclic, then so is AB by Lemma 4.10. To prove the converse, assume that AB is strongly supercyclic. Since AB is not invertible and B^{-1} is surjective, then $A = (AB)B^{-1}$ is strongly supercyclic by lemma 4.10 □

As a direct consequence of Lemma 4.14, we present the following result. Recall that the adjoint M_ϕ^* of a multiplication operator M_ϕ is invertible if and only if ϕ is bounded away from zero, i.e., $0 \notin \overline{\phi(\mathbb{D})}$. In that case $(M_\phi^*)^{-1} = M_{1/\phi}^*$.

Proposition 4.15. *Suppose $k, h \in H^\infty(\mathbb{D})$, $0 \in \overline{k(\mathbb{D})}$ and $0 \notin \overline{h(\mathbb{D})}$. Then M_{kh}^* is strongly supercyclic if and only if M_k^* is strongly supercyclic.*

Example 4.16. Let ϕ be any polynomial on $\mathbb{D}$ of the form

$$\phi(z) = z^m \prod_{i=1}^{n} (z - a_i)$$

where $m, n \geq 1$ and $|a_i| > 1$ $(i = 1, \ldots, n)$. Then M_ϕ^* is strongly supercyclic. In fact, if we put $k(z) = z^m$, $h(z) = \prod_{i=1}^n (z - a_i)$ $(z \in \mathbb{D})$, then $k(0) = 0$ but $0 \notin \overline{h(\mathbb{D})}$ because if $r = \min |a_i|$ $(1 \leq i \leq n)$, then $|h(z)| > (r-1)^n$ for all $z \in \mathbb{D}$. On the other hand, M_k^* is strongly supercyclic by Proposition 4.9 and Corollary 4.4. Thus, $M_\phi^* = M_{kh}^*$ is strongly supercyclic by Proposition 4.15.

Corollary 4.17. *Suppose $w \in H^\infty(\mathbb{D})$. If $w = pq$ is the inner-outer factorization of w, $0 \in \overline{p(\mathbb{D})}$ and $0 \notin \overline{q(\mathbb{D})}$, then M_w^* is strongly supercyclic if and only if M_p^* is strongly supercyclic.*

The following result is a consequence of Proposition 4.9 and Corollary 4.17.

Corollary 4.18. *Suppose $w \in H^\infty(\mathbb{D})$ has a zero in $\mathbb{D}$. If $w = pq$ is the inner-outer factorization of w and $0 \notin \overline{q(\mathbb{D})}$, then M_w^* is strongly supercyclic.*

The reader may have noticed that we cannot use Lemmas 4.10 and 4.14 to investigate the strong supercyclicity of the adjoint $C_{w,\phi}^*$ of weighted composition operator $C_{w,\phi}$ on $H^2(\mathbb{D})$ where w is a bounded function. In fact, if we put $A = M_w^*$, $B = C_\phi^*$, then we may not necessarily have $AB = BA$. In the following lemma, by replacing the condition of commutativity with a weaker condition, we will find strongly supercyclic adjoints.

Lemma 4.19. *Suppose $A, B \in B(X)$, A is non-invertible strongly supercyclic and $BA = SAB$ for some injective operator S which commutes with both A, B. If B is surjective, then AB is strongly supercyclic. Furthermore, if S is invertible, then AB is strongly supercyclic if and only if B is surjective. In this case, BA is also strongly supercyclic.*

Proof. Suppose B is surjective. Since A is surjective by Proposition 3.5, then so is AB. On the other hand, A has dense generalized kernel by Proposition 3.2 and Definition 3.4. Meanwhile, an easy use of the Mathematical induction shows that $B^n A^n = S^{n(n+1)/2}(AB)^n$ for all $n \in \mathbb{N}$. Thus,

$$\text{Ker}A^n \subseteq \text{Ker}B^n A^n = \text{Ker}S^{n(n+1)/2}(AB)^n = \text{Ker}(AB)^n$$

for all $n \in \mathbb{N}$ and so AB has dense generalized kernel. Hence, AB is strongly supercyclic by Corollary 4.3. Now, assume that S is invertible. If AB is strongly supercyclic, it is surjective and hence by using the assumption $BA = SAB$, we conclude that B is surjective. Finally, we prove the last assertion of the lemma. We know that BA is surjective. On the other hand, since $(BA)^n = S^n(AB)^n$ for all positive integers n, we have $\overline{\bigcup_{n=1}^{\infty} \operatorname{Ker}(BA)^n} = X$. Thus, BA is strongly supercyclic by Corollary 4.3. □

Proposition 4.20. *Suppose $w : \mathbb{D} \to \mathbb{C}$ and $\phi : \mathbb{D} \to \mathbb{D}$ are analytic functions such that $\lambda w \circ \phi = w$ for some $\lambda \in \mathbb{C}^*$. If M_w^* is non-invertible strongly supercyclic, then $C_{w,\phi}^*$ is strongly supercyclic if and only if C_ϕ^* is surjective.*

Proof. Since $C_{w,\phi} = M_w C_\phi$ we have $C_{w,\phi}^* = C_\phi^* M_w^*$. On the other hand, the condition $\lambda w \circ \phi = w$ shows that $\lambda C_\phi M_w = M_w C_\phi$. So, putting $S = \overline{\lambda} I$, we have $C_\phi^* M_w^* = S M_w^* C_\phi^*$ which gives our desired result by the above lemma. □

Example 4.21. Let $w(z) = z$, $\phi(z) = bz$ $(z \in \mathbb{D})$ for some $b \in \mathbb{C}^*$ with $|b| = 1$. Then M_w^* is strongly supercyclic by Proposition 4.9 and Corollary 4.4. On the other hand, C_ϕ^* is invertible and meanwhile, for $\lambda = \overline{b}$ we have $\lambda w \circ \phi = w$. Thus, $C_{w,\phi}^*$ is strongly supercyclic by the above proposition.

4.2. Bilateral weighted shifts

In [1] a sufficient condition and a necessary condition have been given for strong hypercyclicity of weighted backward shifts on ℓ^p $(1 \leq p < \infty)$ and c_0. In this subsection, we show that bilateral weighted shifts on Banach spaces $\ell^p(\mathbb{Z})$ $(1 \leq p < \infty)$ and $c_0(\mathbb{Z})$ are not strongly supercyclic. Let $(e_n)_{n \in \mathbb{Z}}$ be the canonical basis of $\ell^p(\mathbb{Z})$ $(1 \leq p < \infty)$ or $c_0(\mathbb{Z})$ and $W = (w_n)_{n \in \mathbb{Z}}$ be a bounded sequence of nonnegative numbers. The bilateral weighted shift operator B_W on these spaces is defined by $B_W e_n = w_n e_{n+1}$ $(n \in \mathbb{Z})$. The sequence W is called the weight sequence of B_W.

Proposition 4.22. *There is no strongly supercyclic bilateral weighted shift operator on $\ell^p(\mathbb{Z})$ $(1 \leq p < \infty)$ and $c_0(\mathbb{Z})$.*

Proof. To get a contradiction, suppose B_W is a strongly supercyclic operator on $\ell^p(\mathbb{Z})$ $(1 \leq p < \infty)$ or $c_0(\mathbb{Z})$ with the weight sequence $W = (w_n)_{n \in \mathbb{Z}}$. Then, it is surjective by Proposition 3.5. If B_W is invertible, then, by Fact 3.3, the semigroup

$$\mathcal{S}^{-1} = \{cB_W^{-n} : c \in \mathbb{C}^*, n \in \mathbb{N}_0\}$$

is hypertransitive. But it is easy to see that no vector e_n is a hypercyclic vector for $\mathcal{S}^{-1}$. Thus, B_W cannot be invertible and so it is not injective. Hence, there is some $k \in \mathbb{Z}$ such that $w_k = 0$. Now, it is easy to see that $e_{k+1} \notin \operatorname{Ran} B_W$. □

5. Hypertransitivity and the invariant subsets

In this section, we show that if T is a hypertransitive operator, then cT is also hypertransitive if and only if $|c| = 1$. In other words, if T does not have any nontrivial closed invariant subset, then cT admits such invariant subsets if and only if $|c| \neq 1$. We use the following theorem to prove our result.

Theorem 5.1 (Theorem 3 of [13])**.** *If $T \in B(X)$ satisfies $\sum_{n=1}^{\infty} 1/\|T^n\| < \infty$, then there is some $x \in X$ such that $\|T^n x\| \to \infty$ as $n \to \infty$.*

Proposition 5.2. *If $T \in B(X)$ is hypertransitive, then cT is hypertransitive if and only if $|c| = 1$.*

Proof. By the León-Müller theorem, cT is hypertransitive for all $c \in \mathbb{C}$ such that $|c| = 1$. To prove the converse, suppose $|c| > 1$. Since $\sup_{n \in \mathbb{N}} \|T^n\| = \infty$ (power bounded operators are not hypercyclic) it is easy to show that $\|T^n\| > 1$ for all $n \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} 1/\|(cT)^n\| < \infty$ and so by Theorem 5.1, there is some $x \in X$ such that $\|(cT)^n x\| \to \infty$. Then x is not a hypercyclic vector for cT and so cT is not hypertransitive. Now, we consider the case $0 \neq |c| < 1$ (if $c = 0$ it is clear that cT is not hypertransitive). If cT is hypertransitive, then $(1/c)(cT) = T$ is not hypertransitive by the proof of previous case, a contradiction. Thus, if $|c| \neq 1$, then cT is not hypertransitive and we are done. □

Since an operator is hypertransitive if and only if it lacks nontrivial closed invariant subsets, Proposition 5.2 can be represented in the following form.

Corollary 5.3. *If $T \in B(X)$ lacks nontrivial closed invariant subsets, then cT admits nontrivial closed invariant subsets if and only if $|c| \neq 1$.*

It is interesting to notice that if $T \in B(X)$ is hypertransitive, $|c| \neq 1$ and M is a nontrivial closed invariant subset for cT, then $M \nsubseteq cM$ because otherwise, $T(M) \subseteq T(cM) = (cT)(M) \subseteq M$ which is not true. Therefore, for example if $|c| > 1$ then for any $r > 0$, no closed ball

$$M = \{x \in X : \|x\| \leq r\}$$

can be cT-invariant and for the case $|c| < 1$ the set

$$M' = \{x \in X : \|x\| \geq r\}$$

cannot be cT-invariant.

Let us write $L_{ht}(X)$ to denote the set of all hypertransitive operators on X. If $\mathcal{A} = B(X) \backslash L_{ht}(X)$ and $\mathcal{A}'$ is the set of limit points of $\mathcal{A}$ in $B(X)$, then we can use Proposition 5.2 to give the following result.

Corollary 5.4. *With the norm topology on $B(X)$, we have $\mathcal{A}' = B(X)$.*

Proof. Let $T \in B(X)$ and $T_n = \frac{n}{n+1} T$ $(n \geq 1)$. Then $T_n \to T$ in the norm topology. But, by Proposition 5.2, at most one of the operators T_n belongs to $L_{ht}(X)$. □

Since by Proposition 1.3, an invertible operator is strongly hypercyclic if and only if its inverse is hypertransitive, we can present the following result as another consequence of Proposition 5.2.

Corollary 5.5. *Suppose $T \in B(X)$ is invertible and strongly hypercyclic. Then cT is strongly hypercyclic if and only if $|c| = 1$.*

We finish this paper by inviting the interested reader to find a sufficient condition for strong supercyclicity of the adjoints of composition operators. Meanwhile, strong cyclicity of operators could also be another research path in this area.

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Mohammad Ansari
Department of Mathematics
Azad University of Gachsaran
Gachsaran, Iran
Email address: m_ansari@shirazu.ac.ir

Karim Hedayatian
Department of Mathematics
Shiraz University
Shiraz, Iran
Email address: hedayati@shirazu.ac.ir

Bahram Khani-Robati
Department of Mathematics
Shiraz University
Shiraz, Iran
Email address: bkhani@shirazu.ac.ir