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WEIGHTED L^p-BOUNDEDNESS OF SINGULAR INTEGRALS WITH ROUGH KERNEL ASSOCIATED TO SURFACES

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Abstract. In this paper, we prove weighted norm inequalities for rough singular integrals along surfaces with radial kernels h and sphere kernels Ω by assuming $h \in \triangle_\gamma(\mathbb{R}_+)$ and $\Omega \in \mathcal{WG}_\beta(\mathrm{S}^{n-1})$ for some $\gamma > 1$ and $\beta > 1$. Here $\Omega \in \mathcal{WG}_\beta(\mathrm{S}^{n-1})$ denotes the variant of Grafakos-Stefanov type size conditions on the unit sphere. Our results essentially improve and extend the previous weighted results for the rough singular integrals and the corresponding maximal truncated operators.

1. Introduction

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$, $n \geq 2$, with normalized Lebesgue measure $d\sigma = d\sigma(\cdot)$. Let Ω be a homogeneous function of degree 0, which is integrable over S^{n-1}, and enjoys the property that

$$\int_{\mathrm{S}^{n-1}} \Omega(u) d\sigma(u) = 0. \tag{1.1}$$

Let $\Gamma(t)$ be a suitable function on the interval $\mathbb{R}^+ := (0, \infty)$. The singular integral operator $T_{\Omega,\Gamma,h}$ associated with Γ is defined by

$$T_{\Omega,\Gamma,h} f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y')h(|y|)}{|y|^n} f(x - \Gamma(|y|)y') dy, \tag{1.2}$$

and the corresponding maximal truncated operator $T^*_{\Omega,\Gamma,h}$ is defined by

$$T^*_{\Omega,\Gamma,h} f(x) = \sup_{\epsilon>0} \Big| \int_{|y|>\epsilon} \frac{\Omega(y')h(|y|)}{|y|^n} f(x - \Gamma(|y|)y') dy \Big|, \tag{1.3}$$

where $y' = y/|y|$ for any $y \neq 0$, $f \in \mathcal{S}(\mathbb{R}^n)$, the space of Schwartz functions, and $h \in \Delta_\gamma(\mathbb{R}^+)$. Here $\Delta_\gamma(\mathbb{R}^+)$, $\gamma \geq 1$, is the set of all measurable functions h

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defined on $\mathbb{R}^+$ satisfying

$$\|h\|_{\Delta_\gamma(\mathbb{R}^+)} := \sup_{R>0} \Big(\frac{1}{R}\int_0^R |h(t)|^\gamma dt\Big)^{1/\gamma} < \infty.$$

It is clear that

$$L^\infty(\mathbb{R}^+) = \Delta_\infty(\mathbb{R}^+) \subsetneq \Delta_{\gamma_2}(\mathbb{R}^+) \subsetneq \Delta_{\gamma_1}(\mathbb{R}^+) \quad \text{for} \quad 1 \le \gamma_1 < \gamma_2 < \infty. \tag{1.4}$$

For the sake of simplicity, we denote $T_{\Omega,\Gamma,h} = T_{\Omega,h}$, $T^*_{\Omega,\Gamma,h} = T^*_{\Omega,h}$ if $\Gamma(t) = t$, $T_{\Omega,\Gamma,h} = T_{\Omega,\Gamma}$, $T^*_{\Omega,\Gamma,h} = T^*_{\Omega,\Gamma}$ if $h(t) = 1$ and $T_{\Omega,\Gamma,h} = T_\Omega$, $T^*_{\Omega,\Gamma,h} = T^*_\Omega$ if $\Gamma(t) = t$ and $h(t) = 1$.

The investigation of the operators T_Ω and T^*_Ω began with Calderón-Zygmund's groundbreaking study in [4, 5] and then continued by many authors (see [6, 13, 15, 20, 22] etc.). In their fundamental work, Calderón and Zygmund established the L^p boundedness of the operators T_Ω and T^*_Ω under the condition that $\Omega \in L\log^+ L(\mathrm{S}^{n-1})$. Later on, Coifman-Wesis [7] and Connett [8] independently extended the result of T_Ω to the case: $\Omega \in H^1(\mathrm{S}^{n-1})$, where $H^1(\mathrm{S}^{n-1})$ denotes the Hardy space on the unit sphere in the sense of Coifman and Weiss. It should be noted that the following relations:

$$L^q(\mathrm{S}^{n-1}) \subsetneq L\log^+ L(\mathrm{S}^{n-1}) \subsetneq H^1(\mathrm{S}^{n-1}) \subsetneq L^1(\mathrm{S}^{n-1}), \quad \forall\, 1 < q \le \infty.$$

Also, in studying the L^p-boundedness of singular integrals and maximal singular integrals with rough kernels, Grafakos and Stefanov [20] introduced the following function class:

$$\mathcal{G}_\beta(S^{n-1}) = \Big\{\Omega \in L^1(S^{n-1}) : \sup_{\xi \in S^{n-1}} \int_{S^{n-1}} |\Omega(y')| \Big(\log \frac{1}{|\langle \xi, y'\rangle|}\Big)^\beta d\sigma(y') < \infty\Big\}$$

for $\beta > 0$, and showed that

$$\mathcal{G}_{\beta_1}(S^{n-1}) \subsetneq \mathcal{G}_{\beta_2}(S^{n-1}), \quad 0 < \beta_2 < \beta_1,$$
$$\bigcup_{q>1} L^q(S^{n-1}) \subsetneq \bigcap_{\beta>0} \mathcal{G}_\beta(S^{n-1}),$$

and

$$\bigcap_{\beta>1} \mathcal{G}_\beta(S^{n-1}) \nsubseteq H^1(S^{n-1}) \nsubseteq \bigcup_{\beta>1} \mathcal{G}_\beta(S^{n-1}). \tag{1.5}$$

Moreover, they proved that T_Ω (resp., T^*_Ω) is bounded on $L^p(\mathbb{R}^n)$, provided that $\Omega \in \mathcal{G}_\beta(S^{n-1})$ for some $\beta > 2$ (resp., $\beta > 3$) and $p \in (1 + 2/\beta, 1 + \beta/2)$ (resp., $2\beta/(2\beta - 3) < p < 2\beta/3$). Soon later, Fan, Guo and Pan [14] improved and extended the above results as follows.

Theorem A (cf. [14]). *Let $\Omega \in \mathcal{G}_\beta(S^{n-1})$ for some $\beta > 1$ and satisfy* (1.1). *Then*

(i) *If $\beta > 2$, then $T_{\Omega,\Gamma}$ is bounded on $L^p(\mathbb{R}^n)$ for $p \in (\beta', \beta)$.*

(ii) *If $\beta > 3$, then $T^*_{\Omega,\Gamma}$ is bounded on $L^p(\mathbb{R}^n)$ for $p \in (\frac{\beta-1}{\beta-2}, \beta - 1)$.*

Here $\Gamma(t) = P_N(t)$, $P_N(t)$ is a real polynomial on $\mathbb{R}$ of degree N satisfying

$P(0) = 0$, and the bounds of the above operators are independent of the coefficients of P_N.

The operator $T_{\Omega,h}$, whose kernel has the additional roughness due to the presence of h, was first studied by Fefferman [18] and subsequently by many other authors (see [3,13,15,17,24] etc.). In particular, we can find the following results.

Theorem B. *Let $h \in \Delta_\gamma(\mathbb{R}^+)$ for $\gamma > 1$, $\Omega \in L^1(S^{n-1})$ and satisfy* (1.1).

(i) (cf. [15]) *If $\Omega \in H^1(S^{n-1})$, then $T_{\Omega,h}$ is bounded on $L^p(\mathbb{R}^n)$ for $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$.*

(ii) (cf. [17]) *If $\Omega \in \mathcal{WG}_\beta(S^{n-1})$ for some $\beta > \max\{2, \gamma'\}$, then $T_{\Omega,h}$ is bounded on $L^p(\mathbb{R}^n)$ for $|1/p - 1/2| < \min\{1/2, 1/\gamma'\} - 1/\beta$, where $\mathcal{WG}_\beta(\mathrm{S}^{n-1})$ for $\beta > 0$ denotes the set of all functions $\Omega : \mathrm{S}^{n-1} \to \mathbb{R}$ satisfying*

$$(1.6)\quad \sup_{\xi' \in \mathrm{S}^{n-1}} \iint_{\mathrm{S}^{n-1}\times\mathrm{S}^{n-1}} |\Omega(x')\Omega(y')| \left(\log \frac{1}{|\langle x'-y', \xi'\rangle|}\right)^\beta d\sigma(x')d\sigma(y') < \infty.$$

It follows from [17, Lemma 1] and [23, Lemma A.2] that

$$\mathcal{G}_\beta(\mathrm{S}^1) \subset \mathcal{WG}_\beta(\mathrm{S}^1),$$

and for $n \geq 2$,

$$(1.7)\qquad \mathcal{WG}_{2\beta}(\mathrm{S}^{n-1}) \setminus \mathcal{G}_\beta(\mathrm{S}^{n-1}) \neq \emptyset \quad \text{for } \beta > 0,$$

$$\bigcup_{r>1} L^r(\mathrm{S}^{n-1}) \subset \mathcal{WG}_{\beta_2}(\mathrm{S}^{n-1}) \subset \mathcal{WG}_{\beta_1}(\mathrm{S}^{n-1}) \quad \text{for } 0 < \beta_1 < \beta_2 < \infty.$$

In this paper, we will focus on the weighted L^p estimates of $T_{\Omega,\Gamma,h}$ and $T^*_{\Omega,\Gamma,h}$. Duoandikoetxea and Rubio de Francia [13] first showed the weighted L^p-bounds of T_Ω with $\Omega \in L^\infty(S^{n-1})$. Subsequently, the weighted L^p bounds of T_Ω and T^*_Ω with $\Omega \in L^q(S^{n-1})$ were given by Watson [27] and Duoandikoetxea [12] successively. Moreover, Duoandikoetxea [12] proved that T_Ω and T^*_Ω are bounded on $L^p(\omega)$ for $1 < p < \infty$, provided that $\Omega \in L\log^+L(\mathrm{S}^{n-1})$ and $\omega \in \tilde{A}_p(\mathbb{R}^+)$, a special class of radial Muckenhoupt weights. Precisely, for $1 < p < \infty$, we say $\omega \in \tilde{A}(\mathbb{R}^+)$, if $\omega(x) = \nu_1(|x|)\nu_2(|x|)^{1-p}$, where either $\nu_i \in A_1(\mathbb{R}^+)$ is decreasing, or $\nu_i^2 \in A_1(\mathbb{R}^+)$, $i = 1, 2$.

In 1999, Fan, Pan and Yang [16] improved and extended the result of [12] as follows.

Theorem C (cf. [16])**.** *Let $h \in \Delta_\gamma(\mathbb{R}^+)$ for $\gamma \geq 2$, $1 < p < \infty$. Suppose that $\Gamma \in \mathfrak{F}_1$ or $\mathfrak{F}_2$, $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$ with $p \geq \gamma'$. If $\Omega \in H^1(S^{n-1})$, then $T_{\Omega,\Gamma,h}$ and $T^*_{\Omega,\Gamma,h}$ are bounded on $L^p(\omega)$.*

Remark D. (i) In [16], the authors proved Theorem C, provided $\omega \in \tilde{A}^I_{p/\gamma'}(\mathbb{R}^+) := \tilde{A}_{p/\gamma'}(\mathbb{R}^+) \cap A^I_{p/\gamma'}$ (see [16] for the definition). However, we know from [28, Theorem 4] that $\tilde{A}_p(\mathbb{R}^+) \subset A^I_p(\mathbb{R}^n)$ for any $1 < p < \infty$, which indicates that $\tilde{A}^I_p(\mathbb{R}^+) = \tilde{A}_p(\mathbb{R}^+)$. Meanwhile, if $\omega(t) \in \tilde{A}_p(\mathbb{R}^+)$, we know from [12] that the Hardy-Littlewood function M is bounded on $L^p(\mathbb{R}^n, \omega(|x|)dx)$. Therefore, if $\omega(t) \in \tilde{A}_p(\mathbb{R}^+)$, then $\omega(|x|) \in A_p(\mathbb{R}^n)$.

(ii) (cf. [16]) The class of functions $\mathfrak{F}_1$ and $\mathfrak{F}_2$ in Theorem C are given as follows.

A nonnegative $C^1(\mathbb{R}^+)$ function $\Gamma \in \mathfrak{F}_1$ if it satisfies the following conditions:

(a) Γ is strictly increasing on $\mathbb{R}^+$, $\Gamma(2t) \geq \lambda\Gamma(t)$ for all $t > 0$ and some fixed $\lambda > 1$.
(b) Γ satisfies a doubling condition, $\Gamma(2t) \leq c\Gamma(t)$ for all $t > 0$ and some constant $c \geq \lambda > 1$,
(c) $\Gamma'(t) \geq C_1\Gamma(t)/t$ on $\mathbb{R}^+$ for some fixed $C_1 \in (0, \log_2 c])$ and $\Gamma'(t)$ is monotone on $(0, \infty)$.

A nonnegative $C^1(\mathbb{R}^+)$ function $\Gamma \in \mathfrak{F}_2$ if satisfies the following conditions:

(a') Γ is strictly decreasing on $\mathbb{R}^+$, $\Gamma(t) \geq \lambda\Gamma(2t)$ for all $t > 0$ and some fixed $\lambda > 1$.
(b') $\Gamma(t) \leq c\Gamma(2t)$ for all $t > 0$ and some constant $c \geq \lambda > 1$,
(c') $|\Gamma'(t)| \geq C_1\Gamma(t)/t$ on $\mathbb{R}^+$ for all $t > 0$ and some fixed $C_1 \in (0, \log_2 c])$ and $\Gamma'(t)$ is monotone on $(0, \infty)$.

We remark that model examples for the functions $\mathfrak{F}_1$ are $\Gamma(t) = t^\delta$ with $\delta > 0$, and their linear combinations with positive coefficients. Model examples for the functions $\mathfrak{F}_2$ are t^δ with $\delta < 0$, and their linear combinations with positive coefficients.

On the other hand, Al-Qassem [1] obtained the following weighted result.

Theorem E (cf. [1]). *Let $\Omega \in \mathcal{G}_\beta(S^{n-1})$ for some $\beta > 1$ and satisfy* (1.1). *Suppose that $\Gamma \in \mathfrak{F}_1$ or $\mathfrak{F}_2$, and $\omega \in \tilde{A}_p(\mathbb{R}^+)$ for $1 < p < \infty$.*

(i) *If $\beta > 2$, then $T_{\Omega,\Gamma}$ is bounded on $L^p(\omega)$ for $p \in (\beta', \beta)$.*

(ii) *If $\beta > 3$, then $T^*_{\Omega,\Gamma}$ is bounded on $L^p(\omega)$ for $p \in (\frac{\beta-1}{\beta-2}, \beta - 1)$.*

In addition, Ding, Xue and Yabuta [10] introduced the following conditions on Γ:

Suppose Γ is a nonnegative monotone $C^1(\mathbb{R}^+)$ function such that $\phi(t) := \frac{\Gamma(t)}{t\Gamma'(t)}$ is bounded. We say $\Gamma \in \mathfrak{F}$ if Γ satisfies one of the following conditions:

(i) Γ is increasing on $\mathbb{R}^+$, and $\Gamma(2t) \leq C_1\Gamma(t)$.
(ii) Γ is increasing on $\mathbb{R}^+$, and $t\Gamma'(t)$ is increasing on $\mathbb{R}^+$.
(iii) Γ is decreasing on $\mathbb{R}^+$, and $\Gamma(t) \leq C_2\Gamma(2t)$.
(iv) Γ is decreasing on $\mathbb{R}^+$ and convex.

Remark F. It is worthwhile to note that $\mathfrak{F}_1$, $\mathfrak{F}_2 \subset \mathfrak{F}$, and there is a function Γ which satisfies $\mathfrak{F}$ but does not satisfy $\mathfrak{F}_1$ or $\mathfrak{F}_2$. For example, $\Gamma(t) = \sqrt{t}\log(1 + t) \in \mathfrak{F}$, but is not in $\mathfrak{F}_1$.

In light of the aforementioned facts concerning the singular integrals and the assumptions on Γ, a question that arise naturally is the following.

Question. *Are the operators $T_{\Omega,\Gamma,h}$ and $T^*_{\Omega,\Gamma,h}$ bounded on $L^p(\omega)$, provided that $\Gamma \in \mathfrak{F}$, $h \in \triangle_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$, $\Omega \in \mathcal{WG}_\beta(\mathrm{S}^{n-1})$ for some $\beta > 1$, and $\omega \in \tilde{A}_p(\mathbb{R}^+)$ for some $1 < p < \infty$?*

The main purpose of this paper is to address the above question. Our results can be formulated as follows.

Theorem 1.1. *Let $\gamma \in (2,\infty]$, $\beta \in (2,\infty)$ and $\Gamma \in \mathfrak{F}$. If $h \in \triangle_\gamma(\mathbb{R}^+)$, $\Omega \in \mathcal{WG}_\beta(S^{n-1})$ and satisfies (1.1), then for $p \in (\frac{\gamma'\beta}{\gamma'+\beta-2}, \frac{\gamma'\beta}{\gamma'+(\beta-2)(\gamma'-1)})$, and $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$ or $\tilde{A}_{p'/\gamma'}(\mathbb{R}^+)$, $T_{\Omega,\Gamma,h}$ is bounded on $L^p(\omega)$.*

Theorem 1.2. *Let $\gamma \in (2,\infty]$, $\beta \in (3,\infty)$ and $\Gamma \in \mathfrak{F}$. If $h \in \triangle_\gamma(\mathbb{R}^+)$, $\Omega \in \mathcal{WG}_\beta(S^{n-1})$ and satisfies (1.1), then for $p \in (\frac{\gamma'(\beta-1)}{\gamma'+\beta-3}, \frac{\gamma'(\beta-1)}{\gamma'+(\gamma'-1)(\beta-3)})$ and $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$, $T^*_{\Omega,\Gamma,h}$ is bounded on $L^p(\omega)$.*

Remark 1.3. (1) If $\Gamma(t) = t$, $\omega \equiv 1$ and $\Omega \in \mathcal{WG}_\beta(S^{n-1})$ for some $\beta > 2$, then Theorem 1.1 reduces to (ii) of Theorem B in the case $\gamma \in (2,\infty]$.

(2) Comparing with Theorem C, Theorems 1.1 and 1.2 relax the conditions imposed on Γ and have themselves interesting because of (1.5) and (1.7).

(3) It should be pointed out that the inclusion relation between $\mathcal{WG}_\beta(\mathrm{S}^{n-1})$ and $\mathcal{G}_\beta(\mathrm{S}^{n-1})$ is not clear for $n \geq 2$ at present. But even so, our results also present a meaningful extension to Theorem E because of (1.7), even for $h \equiv 1$.

Moreover, we remark that for $h(t) \equiv 1$, Theorem 1.1 or 1.2 is true, provided that $\gamma' = 1$ and $\Omega \in \mathcal{G}_\beta(S^{n-1})$ for some $\beta > 2$, or $\beta > 3$. This represents an improvement and extension to Theorem E since $\mathfrak{F}_1$, $\mathfrak{F}_2 \subsetneq \mathfrak{F}$. We leave the details to the interested readers.

The paper is organized as follows. In Section 2, we will give several auxiliary lemmas. The proofs of Theorems 1.1 and 1.2 will be based on two general criterions on the weighted inequality of the convolution operators, which will be given in Section 3. In Section 4, we will extend Theorems 1.1-1.2 to the more general cases. Finally, we will present two results related to Marcinkiewicz integrals in Section 5. We would like to remark that the main ideas of our proofs are taken from [1, 12, 14, 16].

Throughout the paper, the letter C, sometimes with certain parameters, will stand for positive constants not necessarily the same one at each occurrence, but are independent of the essential variables. In what follows, for $p \in (1,\infty)$, we denote p' by the conjugate index of p, that is, $1/p + 1/p' = 1$. For a measure σ, we denote by $|\sigma|$ the total variation of σ.

2. Preliminaries

In this section, we will establish some auxiliary lemmas. First, we introduce some relevant notations and definitions. Let b be a positive constant, Γ be a nonnegative monotone C^1 function on $\mathbb{R}^+$ such that

$$\phi(t) := \frac{\Gamma(t)}{t\Gamma'(t)}, \quad \text{and} \quad |\phi(t)| \le b.$$

Then for any $k \in \mathbb{Z}$, we know from [10, Lemma 2.8, Lemma 2.9] that if Γ is positive, increasing, $\Gamma(t)/(t\Gamma'(t)) \le b$, then $\Gamma(2^{k+1})/\Gamma(2^k) \ge 2^{1/b}$. If Γ is positive, decreasing, $-\Gamma(t)/(t\Gamma'(t)) \le b$, then $\Gamma(2^{-(k+1)})/\Gamma(2^{-k}) \ge 2^{1/b}$, which indicates both $\{\Gamma(2^k)\}_{k\in\mathbb{Z}}$ and $\{\Gamma(2^{-k})\}_{k\in\mathbb{Z}}$ are lacunary sequences. In the sequence, we denote $\Gamma(2^k)$ by a_k if Γ is decreasing and denote $\Gamma(2^k)$ by a_k^{-1} if Γ is decreasing.

We respectively define two sequences of measures $\{\sigma_{k,\Omega,\Gamma,h} : k \in \mathbb{Z}\}$ and $\{|\sigma_{k,\Omega,\Gamma,h}| : k \in \mathbb{Z}\}$ related to Γ by

$$\int_{\mathbb{R}^n} f d\sigma_{k,\Omega,\Gamma,h} = \int_{2^k \le |x| < 2^{k+1}} f(\Gamma(|x|)x')\frac{\Omega(x)h(|x|)}{|x|^n}dx,$$

and

$$\int_{\mathbb{R}^n} f d|\sigma_{k,\Omega,\Gamma,h}| = \int_{2^k \le |x| < 2^{k+1}} f(\Gamma(|x|)x')\frac{|\Omega(x)h(|x|)|}{|x|^n}dx.$$

Thus, by dyadic decomposition, we can write $T_{\Omega,\Gamma,h}f$ as follows:

$$T_{\Omega,\Gamma,h}f(x) = \sum_{k\in\mathbb{Z}} \int_{2^k \le |y| < 2^{k+1}} f(x-\Gamma(|y|)y')\frac{\Omega(y)h(|y|)}{|y|^n}dy = \sum_{k\in\mathbb{Z}} \sigma_{k,\Omega,\Gamma,h} * f(x).$$

Also, we define the maximal operators $M_{\sigma,\Omega,\Gamma,h}$ on $\mathbb{R}^n$ by

$$M_{\sigma,\Omega,\Gamma,h}f(x) = \sup_{k\in\mathbb{Z}} \big||\sigma_{k,\Omega,\Gamma,h}| * f(x)\big|.$$

Now we give the following several lemmas, which will play key roles in our proofs.

Lemma 2.1. *Let Γ be a positive function $\mathbb{R}^+$ with $|\phi(t)| \le b$ and $\Gamma \in \mathfrak{F}$. Suppose $h \in \triangle_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$, $\Omega \in L^1(\mathrm{S}^{n-1})$. Then*

$$\|M_{\sigma,\Omega,\Gamma,h}f\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}$$

for all $p \in (\gamma', \infty)$ and $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$.

Proof. The proof is only a simple imitation of [16, Lemma 2.4] and [10, Lemma 3.2] and we omit the details here. □

Lemma 2.2. *Let Γ be a nonnegative monotone C^1 function on $\mathbb{R}^+$. Suppose $h \in \triangle_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$, $\Omega \in L^1(\mathrm{S}^{n-1})$ and satisfies* (1.1). *Then*

(i) $\|\sigma_{k,\Omega,\Gamma,h}\| \le C$.

(ii) *If Γ is increasing,* $|\widehat{\sigma_{k,\Omega,\Gamma,h}}(\xi)| \le C|a_{k+1}\xi|$.

(iii) *If Γ is decreasing,* $|\widehat{\sigma_{k,\Omega,\Gamma,h}}(\xi)| \le C|a_k^{-1}\xi|$.

Proof. (i) is obvious. For (ii), by using the definition of the measure $\{\sigma_{k,\Omega,\Gamma,h} : k \in \mathbb{Z}\}$ and the vanishing condition of Ω, we have

$$\begin{aligned}
|\widehat{\sigma_{k,\Omega,\Gamma,h}}(\xi)| &= \Big| \int_{S^{n-1}} \int_{2^k}^{2^{k+1}} \Omega(\theta) h(r) e^{-2\pi i \Gamma(r)\langle \xi,\theta\rangle} \frac{dr}{r} d\sigma(\theta) \Big| \\
&= \Big| \int_{2^k}^{2^{k+1}} h(r) \int_{S^{n-1}} \Omega(\theta) e^{-2\pi i \Gamma(r)\langle \xi,\theta\rangle} d\sigma(\theta) \frac{dr}{r} \Big| \\
&\le \int_{2^k}^{2^{k+1}} |h(r)| \int_{S^{n-1}} |\Omega(\theta)| \big| e^{-2\pi i \Gamma(r)\langle \xi,\theta\rangle} - 1 \big| d\sigma(\theta) \frac{dr}{r} \\
&\le C_\Omega |a_{k+1}\xi| \int_{2^k}^{2^{k+1}} \frac{|h(r)|}{r} dr \\
&\le C_\Omega |a_{k+1}\xi| \Big(\int_{2^k}^{2^{k+1}} |h(r)|^\gamma \frac{dr}{r} \Big)^{1/\gamma} \Big(\int_{2^k}^{2^{k+1}} \frac{dr}{r} \Big)^{1/\gamma'} \\
&\le C_{\Omega,\gamma,h} |a_{k+1}\xi|.
\end{aligned}$$

Similarly, for (iii), if Γ is decreasing, we also have

$$|\widehat{\sigma_{k,\Omega,\Gamma,h}}(\xi)| \le C_{\Omega,\gamma,h} |a_k^{-1}\xi|.$$

□

Lemma 2.3. *Let Γ be a positive function $\mathbb{R}^+$ with $|\phi(t)| \le b$ and $\Gamma \in \mathfrak{F}$. Suppose $h \in \triangle_\gamma(\mathbb{R}^+)$ for some $\gamma > 2$, $\Omega \in \mathcal{WG}_\beta(\mathrm{S}^{n-1})$ for some $\beta > 2$. Then*

(i) *If Γ is increasing on $\mathbb{R}^+$, and $\Gamma(2t) \le C_1\Gamma(t)$, then*

$$|\widehat{\sigma_{k,\Omega,\Gamma,h}}(\xi)| \le C_{\Omega,\gamma,h} \big(\log |a_k\xi| \big)^{-\beta/2}, \quad \text{if } |a_k\xi| > 1.$$

(ii) *If Γ is increasing on $\mathbb{R}^+$, and $t\Gamma'(t)$ is increasing on $\mathbb{R}^+$, then*

$$|\widehat{\sigma_{k,\Omega,\Gamma,h}}(\xi)| \le C_{\Omega,\gamma,h} \big(\log |a_k\xi| \big)^{-\beta/2}, \quad \text{if } |a_k\xi| > 1.$$

(iii) *If Γ is decreasing on $\mathbb{R}^+$, and $\Gamma(t) \le C_2\Gamma(2t)$, then*

$$|\widehat{\sigma_{k,\Omega,\Gamma,h}}(\xi)| \le C_{\Omega,\gamma,h} \big(\log |a_{k+1}^{-1}\xi| \big)^{-\beta/2}, \quad \text{if } |a_{k+1}^{-1}\xi| > 1.$$

(iv) *If Γ is decreasing on $\mathbb{R}^+$ and convex, then*

$$|\widehat{\sigma_{k,\Omega,\Gamma,h}}(\xi)| \le C_{\Omega,\gamma,h} \big(\log |a_{k+1}^{-1}\xi| \big)^{-\beta/2}, \quad \text{if } |a_{k+1}^{-1}\xi| > 1.$$

Proof. The main ideas of the proof are taken from [10]. For (i), using the definition of the measure $\{\sigma_{k,\Omega,\Gamma,h}\}_{k\in\mathbb{Z}}$ and Hölder's inequality leads to

$$\begin{aligned}
&|\widehat{\sigma_{k,\Omega,\Gamma,h}}(\xi)| \\
&\le \Big(\int_{2^k}^{2^{k+1}} |h(r)|^\gamma \frac{dr}{r} \Big)^{1/\gamma} \Big(\int_{2^k}^{2^{k+1}} \Big| \int_{S^{n-1}} |\Omega(\theta)| e^{-2\pi i \Gamma(r)\langle \xi,\theta\rangle} d\sigma(\theta) \Big|^{\gamma'} \frac{dr}{r} \Big)^{1/\gamma'} \\
&\le C_{h,\gamma} \Big(\int_{2^k}^{2^{k+1}} \Big| \int_{S^{n-1}} \Omega(\theta) e^{-2\pi i \Gamma(r)\langle \xi,\theta\rangle} d\sigma(\theta) \Big|^{\gamma'} \frac{dr}{r} \Big)^{1/\gamma'}
\end{aligned}$$

$$\le C_{h,\gamma}\Big(\int_{2^k}^{2^{k+1}}\Big|\int_{S^{n-1}}\Omega(\theta)e^{-2\pi i\Gamma(r)\langle\xi,\theta\rangle}d\sigma(\theta)\Big|^2\frac{dr}{r}\Big)^{1/2}$$
$$= C_{h,\gamma}\Big|\int\int_{S^{n-1}\times S^{n-1}}\Omega(\theta)\overline{\Omega}(\nu)\int_{2^k}^{2^{k+1}}e^{-2\pi i\Gamma(r)\langle\xi,\theta-\nu\rangle}\frac{dr}{r}d\sigma(\theta)d\sigma(\nu)\Big|^{1/2}$$
$$= C_{h,\gamma}\Big|\int\int_{S^{n-1}\times S^{n-1}}\Omega(\theta)\overline{\Omega}(\nu)$$
$$\times\int_{\Gamma(2^k)}^{\Gamma(2^{k+1})}e^{-2\pi is\langle\xi,\theta-\nu\rangle}\frac{s}{\Gamma^{-1}(s)\Gamma'(\Gamma^{-1}(s))}\frac{ds}{s}d\sigma(\theta)d\sigma(\nu)\Big|^{1/2}$$
$$\le C_{h,\gamma}b\Big|\int\int_{S^{n-1}\times S^{n-1}}\Omega(\theta)\overline{\Omega}(\nu)\int_{\Gamma(2^k)}^{\Gamma(2^{k+1})}e^{-2\pi is\langle\xi,\theta-\nu\rangle}\frac{ds}{s}d\sigma(\theta)d\sigma(\nu)\Big|^{1/2}.$$

Here we used the change of variable $r=\Gamma(s)$ in the penultimate equation.

Obviously,

$$\Big|\int_{\Gamma(2^k)}^{\Gamma(2^{k+1})}e^{-2\pi is\langle\xi,\theta-\nu\rangle}\frac{ds}{s}\Big|\le\log\frac{\Gamma(2^{k+1})}{\Gamma(2^k)}=\log C_1.$$

On the other hand, by the Van der Corput lemma, we have

$$\Big|\int_{\Gamma(2^k)}^{\Gamma(2^{k+1})}e^{-2\pi is\langle\xi,\theta-\nu\rangle}\frac{ds}{s}\Big|\le C\frac{1}{|a_k\xi||\langle\xi',\theta-\nu\rangle|}.$$

For $|a_k\xi|>1$, since $\frac{t}{(\log t)^\beta}$ is increasing in (e^β,∞), we get

$$\min\{1,|a_k\xi|^{-1}|\langle\xi',\theta-\nu\rangle|^{-1}\}\le C\frac{\big(\log e^\beta|\langle\xi',\theta-\nu\rangle|^{-1}\big)^\beta}{\big(\log|a_k\xi|\big)^\beta}.$$

Therefore,

$$|\widehat{\sigma_{k,\Omega,\Gamma,h}}(\xi)|\le C_{\Omega,\gamma,h}\big(\log|a_k\xi|\big)^{-\beta/2},\quad\text{if }|a_k\xi|>1.$$

For (ii), applying the same way as in (i), we have

$$|\widehat{\sigma_{k,\Omega,\Gamma,h}}(\xi)|\le C_{h,\gamma}\Big(\int_{2^k}^{2^{k+1}}\Big|\int_{S^{n-1}}\Omega(\theta)e^{-2\pi i\Gamma(r)\langle\xi,\theta\rangle}d\sigma(\theta)\Big|^2\frac{dr}{r}\Big)^{1/2}$$
$$= C_{h,\gamma}\Big|\int\int_{S^{n-1}\times S^{n-1}}\Omega(\theta)\overline{\Omega}(\nu)$$
$$\times\int_{2^k}^{2^{k+1}}e^{-2\pi i\Gamma(r)\langle\xi,\theta-\nu\rangle}\frac{dr}{r}d\sigma(\theta)d\sigma(\nu)\Big|^{1/2}.$$

Obviously,

$$\Big|\int_{2^k}^{2^{k+1}}e^{-2\pi i\Gamma(r)\langle\xi,\theta-\nu\rangle}\frac{dr}{r}\Big|\le\log 2.$$

On the other hand, using the change of variable $r = \Gamma(s)$, we have

$$\begin{aligned}
&\Big|\int_{2^k}^{2^{k+1}} e^{-2\pi i\Gamma(r)\langle\xi,\theta-\nu\rangle}\frac{dr}{r}\Big| \\
&= \Big|\int_{\Gamma(2^k)}^{\Gamma(2^{k+1})} e^{-2\pi is\langle\xi,\theta-\nu\rangle}\frac{1}{\Gamma^{-1}(s)\Gamma'(\Gamma^{-1}(s))}ds\Big| \\
&\le \Big|\int_{\Gamma(2^k)}^{\Gamma(2^{k+1})} \cos(2\pi s\langle\xi,\theta-\nu\rangle)\frac{1}{\Gamma^{-1}(s)\Gamma'(\Gamma^{-1}(s))}ds\Big| \\
&\quad + \Big|\int_{\Gamma(2^k)}^{\Gamma(2^{k+1})} \sin(2\pi s\langle\xi,\theta-\nu\rangle)\frac{1}{\Gamma^{-1}(s)\Gamma'(\Gamma^{-1}(s))}ds\Big|.
\end{aligned}$$

Here we used the Euler formula in the last inequality.

Since Γ is positive and increasing, and $t\Gamma'(t)$ is increasing, we can derive that $\Gamma^{-1}(s)\Gamma'(\Gamma^{-1}(s))$ is increasing. Thus

$$\begin{aligned}
&\Big|\int_{\Gamma(2^k)}^{\Gamma(2^{k+1})} \cos(2\pi s\langle\xi,\theta-\nu\rangle)\frac{1}{\Gamma^{-1}(s)\Gamma'(\Gamma^{-1}(s))}ds\Big| \\
&\le C\frac{1}{2^k\Gamma'(2^k)}\frac{1}{|\xi||\langle\xi',\theta-\nu\rangle|} \\
&= C\frac{\Gamma(2^k)}{2^k\Gamma'(2^k)}\frac{1}{\Gamma(2^k)|\xi||\langle\xi',\theta-\nu\rangle|} \\
&\le C\frac{1}{|a_k\xi||\langle\xi',\theta-\nu\rangle|}.
\end{aligned}$$

Similarly, we have

$$\Big|\int_{\Gamma(2^k)}^{\Gamma(2^{k+1})} \sin(2\pi s\langle\xi,\theta-\nu\rangle)\frac{1}{\Gamma^{-1}(s)\Gamma'(\Gamma^{-1}(s))}ds\Big| \le C\frac{1}{|a_k\xi||\langle\xi',\theta-\nu\rangle|}.$$

Consequently,

$$\Big|\int_{2^k}^{2^{k+1}} e^{-2\pi i\Gamma(r)\langle\xi,\theta-\nu\rangle}\frac{dr}{r}\Big| \le C\min\{1, |a_k\xi|^{-1}|\langle\xi',\theta-\nu\rangle|^{-1}\}.$$

The same argument as in the proof of (i) deduces that

$$|\widehat{\sigma_{k,\Omega,\Gamma,h}}(\xi)| \le C_{\Omega,\gamma,h}\big(\log|a_k\xi|\big)^{-\beta/2}, \quad \text{if } |a_k\xi| > 1.$$

For (iii), by using a similar approach, we obtain that

$$\begin{aligned}
&|\widehat{\sigma_{k,\Omega,\Gamma,h}}(\xi)| \\
&\le C_{h,\gamma}\Big(\int_{2^k}^{2^{k+1}}\Big|\int_{S^{n-1}}\Omega(\theta)e^{-2\pi i\Gamma(r)\langle\xi,\theta\rangle}d\sigma(\theta)\Big|^2\frac{dr}{r}\Big)^{1/2} \\
&= C_{h,\gamma}\Big(\int_{\Gamma(2^{k+1})}^{\Gamma(2^k)}\Big|\int_{S^{n-1}}\Omega(\theta)e^{-2\pi is\langle\xi,\theta\rangle}d\sigma(\theta)\Big|^2\frac{s}{-\Gamma^{-1}(s)\Gamma'(\Gamma^{-1}(s))}\frac{ds}{s}\Big)^{1/2}
\end{aligned}$$

$$\leq C_{h,\gamma}b\Big|\int\int_{S^{n-1}\times S^{n-1}}\Omega(\theta)\overline{\Omega}(\nu)\int_{\Gamma(2^{k+1})}^{\Gamma(2^k)}e^{-2\pi is\langle\xi,\theta-\nu\rangle}\frac{ds}{s}d\sigma(\theta)d\sigma(\nu)\Big|^{1/2}.$$

It is easy to check that

$$\Big|\int_{\Gamma(2^{k+1})}^{\Gamma(2^k)}e^{-2\pi is\langle\xi,\theta-\nu\rangle}\frac{ds}{s}\Big|\leq\log\frac{\Gamma(2^k)}{\Gamma(2^{k+1})}=\log C_2$$

and

$$\Big|\int_{\Gamma(2^{k+1})}^{\Gamma(2^k)}e^{-2\pi is\langle\xi,\theta-\nu\rangle}\frac{ds}{s}\Big|\leq\frac{2}{|a_{k+1}^{-1}\xi||\langle\xi',\theta-\nu\rangle|}.$$

Therefore,

$$|\widehat{\sigma_{k,\Omega,\Gamma,h}}(\xi)|\leq C_{\Omega,\gamma,h}\big(\log|a_{k+1}^{-1}\xi|\big)^{-\beta/2},\quad\text{if }|a_{k+1}^{-1}\xi|>1.$$

For (iv), we have

$$\begin{aligned}|\widehat{\sigma_{k,\Omega,\Gamma,h}}(\xi)|&\leq C_{h,\gamma}\Big(\int_{2^k}^{2^{k+1}}\Big|\int_{S^{n-1}}\Omega(\theta)e^{-2\pi i\Gamma(r)\langle\xi,\theta\rangle}d\sigma(\theta)\Big|^2\frac{dr}{r}\Big)^{1/2}\\&=C_{h,\gamma}\Big|\int\int_{S^{n-1}\times S^{n-1}}\Omega(\theta)\overline{\Omega}(\nu)\\&\qquad\times\int_{2^k}^{2^{k+1}}e^{-2\pi i\Gamma(r)\langle\xi,\theta-\nu\rangle}\frac{dr}{r}d\sigma(\theta)d\sigma(\nu)\Big|^{1/2}.\end{aligned}$$

Note that

$$\Big|\int_{2^k}^{2^{k+1}}e^{-2\pi i\Gamma(r)\langle\xi,\theta-\nu\rangle}\frac{dr}{r}\Big|\leq\log 2$$

and

$$\int_{2^k}^{2^{k+1}}e^{-2\pi i\Gamma(r)\langle\xi,\theta-\nu\rangle}\frac{dr}{r}=\int_{\Gamma(2^{k+1})}^{\Gamma(2^k)}e^{-2\pi is\langle\xi,\theta-\nu\rangle}\frac{1}{-\Gamma^{-1}(s)\Gamma'(\Gamma^{-1}(s))}ds.$$

Since Γ is positive, decreasing, and convex, we see that $-\Gamma^{'}(s)$ is decreasing. Hence, $-\Gamma^{'}(\Gamma^{-1}(s))$ is positive and increasing. Applying the second mean value integral theorem that there exists η such that $\Gamma(2^{k+1})\leq\eta\leq\Gamma(2^k)$, we have

$$\begin{aligned}&\int_{\Gamma(2^{k+1})}^{\Gamma(2^k)}\cos(2\pi s\langle\xi,\theta-\nu\rangle)\frac{1}{\Gamma^{-1}(s)\Gamma'(\Gamma^{-1}(s))}ds\\&=\frac{1}{-\Gamma'(t)}\int_{\Gamma(2^{k+1})}^{\eta}\cos(2\pi s\langle\xi,\theta-\nu\rangle)\frac{1}{\Gamma^{-1}(s)}ds.\end{aligned}$$

Therefore,

$$\begin{aligned}&\Big|\int_{\Gamma(2^{k+1})}^{\Gamma(2^k)}\cos(2\pi s\langle\xi,\theta-\nu\rangle)\frac{1}{\Gamma^{-1}(s)\Gamma'(\Gamma^{-1}(s))}ds\Big|\\&\leq C\frac{1}{-\Gamma'(2^{k+1})}\frac{1}{\Gamma^{-1}(\eta)}\frac{1}{|\xi||\langle\xi',\theta-\nu\rangle|}\end{aligned}$$

$$\leq C\frac{1}{-\Gamma'(2^{k+1})}\frac{1}{2^k|\xi||\langle\xi',\theta-\nu\rangle|}$$
$$= C\frac{\Gamma(2^{k+1})}{-2^{k+1}\Gamma'(2^{k+1})}\frac{2}{\Gamma(2^{k+1})|\xi||\langle\xi',\theta-\nu\rangle|}$$
$$\leq C\frac{1}{|a_{k+1}^{-1}\xi||\langle\xi',\theta-\nu\rangle|}.$$

Similarly,

$$\Big|\int_{\Gamma(2^k)}^{\Gamma(2^{k+1})}\sin(2\pi s\langle\xi,\theta-\nu\rangle)\frac{1}{\Gamma^{-1}(s)\Gamma'(\Gamma^{-1}(s))}ds\Big|\leq C\frac{1}{|a_{k+1}^{-1}\xi||\langle\xi',\theta-\nu\rangle|}.$$

Consequently,

$$\Big|\int_{2^k}^{2^{k+1}}e^{-2\pi i\Gamma(r)\langle\xi,\theta-\nu\rangle}\frac{dr}{r}\Big|\leq C\min\{1,|a_{k+1}^{-1}\xi|^{-1}|\langle\xi',\theta-\nu\rangle|^{-1}\}.$$

Applying the same argument as in the proof (i) again, we get

$$|\widehat{\sigma_{k,\Omega,\Gamma,h}}(\xi)|\leq C_{\Omega,\gamma,h}\big(\log|a_{k+1}^{-1}\xi|\big)^{-\beta/2},\quad \text{if } |a_{k+1}^{-1}\xi|>1.$$

Lemma 2.3 is proved. □

3. Proofs of main results

This section is concerning with the proofs of Theorems 1.1 and 1.2. Based on the estimates of Lemmas 2.1-2.3, by the Plancherel theorem and interpolation theory, we can reduce Theorems 1.1 and 1.2 to the direct results of the following more general weighted inequalities for the convolution operators. In what follows, we only present the related results in the case that Γ is increasing, since the other case can be given by the same arguments with a slight modification.

Proposition 3.1. *Let $\gamma\in(2,\infty]$, $\beta\in(1,\infty)$ and $\{\sigma_k\}_{k\in\mathbb{Z}}$ be a family of uniformly bounded Borel measures on $\mathbb{R}^n$. Let $\{a_k : k\in\mathbb{Z}\}$ be a family of nonzero numbers and satisfy $\inf a_{k+1}/a_k\geq\lambda>1$. Suppose that there exist constants $C>0$ such that the following conditions hold*:

(i) $\|\sigma_k\|\leq C$ *for any $k\in\mathbb{Z}$ and $\xi\in\mathbb{R}^n$,*
(ii) $|\widehat{\sigma_k}(\xi)|\leq C|a_{k+1}\xi|$ *for $k\in\mathbb{Z}$ and $\xi\in\mathbb{R}^n$,*
(iii) $|\widehat{\sigma_k}(\xi)|\leq C(\log|a_k\xi|)^{-\beta}$ *for $k\in\mathbb{Z}$ and $\xi\in\mathbb{R}^n$,*
(iv) $\omega\in\tilde{A}_{p/\gamma'}(\mathbb{R}^+)$ *and $\|M_\sigma f\|_{L^p(\omega)}\leq C\|f\|_{L^p(\omega)}$ for all $p\in(\gamma',\infty)$, where*

$$M_\sigma f(x)=\sup_{k\in\mathbb{Z}}\big||\sigma_k|*f(x)\big|.$$

Then for any $p\in(\frac{2\gamma'\beta}{\gamma'+2(\beta-1)},\frac{2\gamma'\beta}{\gamma'+2(\beta-1)(\gamma'-1)})$ and $\omega\in\tilde{A}_{p/\gamma'}(\mathbb{R}^+)$, there exists a constant $C>0$ such that

$$\Big\|\sum_{k\in\mathbb{Z}}\sigma_k*f\Big\|_{L^p(\omega)}\leq C\|f\|_{L^p(\omega)}. \tag{3.1}$$

Proof. Let $\{\psi_k\}_{-\infty}^{+\infty}$ be a smooth partition of unity in $(0,\infty)$ adapted to the intervals $E_k=[(a_{k+1})^{-1},(a_{k-1})^{-1}]$. To be precise, we require the following

$$\psi_k\in C^{\infty},\quad 0\le\psi_k\le 1,\quad \sum_{k\in\mathbb{Z}}\psi_k^2(t)=1,$$

$$\operatorname{supp}\psi_k\subseteq E_k,\quad \Big|\frac{d^s\psi_k(t)}{dt^s}\Big|\le\frac{C_s}{t^s}.$$

Define the multiplier operators S_k in $\mathbb{R}^n$ by

$$(\widehat{S_kf})(\xi)=\psi_k(|\xi|)\hat{f}(\xi).$$

It was proved in [21] that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|S_kf|^2\Big)^{1/2}\Big\|_{L^p(w)}\le C\|f\|_{L^p(w)} \tag{3.2}$$

for all $p\in(1,\infty)$ and $w\in A_p$.

We can write

$$\sum_{k\in\mathbb{Z}}\sigma_k*f=\sum_{k\in\mathbb{Z}}\sigma_k*\big(\sum_{j\in\mathbb{Z}}S_{k+j}^2f\big)=\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}S_{k+j}\big(\sigma_k*S_{k+j}\big):=\sum_{j\in\mathbb{Z}}T_jf.$$

By Plancherel's theorem, we have

$$\|T_jf\|_{L^2(\mathbb{R}^n)}^2\le\sum_{k\in\mathbb{Z}}\int_{E_{k+j}}|\hat{f}(\xi)|^2|\hat{\sigma}_k(\xi)|^2d\xi. \tag{3.3}$$

By a simple computation and our assumptions (i)-(iii), we get

$$\|T_jf\|_{L^2(\mathbb{R}^n)}\le C|j|^{-\beta}\|f\|_{L^2(\mathbb{R}^n)},\quad \text{if } j\le -1,$$

$$\|T_jf\|_{L^2(\mathbb{R}^n)}\le C\lambda^{-j}\|f\|_{L^2(\mathbb{R}^n)},\quad \text{if } j\ge 1.$$

In short,

$$\|T_jf\|_{L^2(\mathbb{R}^n)}\le C(1+|j|)^{-\beta}\|f\|_{L^2(\mathbb{R}^n)},\quad \forall\, j\in\mathbb{Z}. \tag{3.4}$$

To obtain the weighted L^p-estimate, we shall prove an auxiliary vector-valued inequality as follows:

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\sigma_k*g_k|^2\Big)^{1/2}\Big\|_{L^p(\omega)}\le C\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^p(\omega)} \tag{3.5}$$

for $1<\gamma'<2$, $p>\gamma'$ and $\omega\in\tilde{A}_{p/\gamma'}(\mathbb{R}^+)$.

Let $r=p/\gamma'$. Since $r>1$, for $\omega\in\tilde{A}_{p/\gamma'}(\mathbb{R}^+)$, we choose a nonnegative function $u\in L^{r'}(\omega)$ with $\|u\|_{L^{r'}(\omega)}\le 1$ such that

$$\begin{aligned}\Big\|\Big(\sum_{k\in\mathbb{Z}}|\sigma_k*g_k|^{\gamma'}\Big)^{1/\gamma'}\Big\|_{L^p(\omega)}^{\gamma'}&=\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\big|\sigma_k*g_k(x)\big|^{\gamma'}u(x)\omega(x)dx\\&\le\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}|g_k(x)|^{\gamma'}M_{\widetilde{\sigma}}(u\omega)(x)dx\end{aligned}$$

$$\leq \Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^{\gamma'}\Big)^{1/\gamma'}\Big\|_{L^p(\omega)}^{\gamma'}\|M_{\tilde{\sigma}}(\mu\omega)\|_{L^{r'}(\omega^{1-r'})}.$$

Recalling $r = p/\gamma'$, it is easy to check that $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$ if and only if $\omega^{1-r'} \in \tilde{A}_{r'}(\mathbb{R}^+)$. Thus, by our hypothesis (iii), we have

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\sigma_k * g_k|^{\gamma'}\Big)^{1/\gamma'}\Big\|_{L^p(\omega)} \leq C\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^{\gamma'}\Big)^{1/\gamma'}\Big\|_{L^p(\omega)}.$$

By applying (iii) again, we obtain that

$$\Big\|\sup_{k\in\mathbb{Z}}|\sigma_k * g_k|\Big\|_{L^p(\omega)} \leq C\Big\|M_\sigma\big(\sup_{k\in\mathbb{Z}}|g_k|\big)\Big\|_{L^p(\omega)} \leq C\Big\|\sup_{k\in\mathbb{Z}}|g_k|\Big\|_{L^p(\omega)}.$$

Therefore, by the vector-valued interpolation [19, p. 482], we obtain (3.5).

We now turn to the proof of the weighted L^p-estimate. For any $p > \gamma'$ and $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$, we have

$$\begin{aligned}\|T_j f\|_{L^p(\omega)} &\leq C\Big\|\Big(\sum_{k\in\mathbb{Z}}|\sigma_k * S_{k+j}f|^2\Big)^{1/2}\Big\|_{L^p(w)}\\ &\leq C\Big\|\Big(\sum_{k\in\mathbb{Z}}|S_{k+j}f|^2\Big)^{1/2}\Big\|_{L^p(w)} \qquad (3.6)\\ &\leq C\|f\|_{L^p(w)},\end{aligned}$$

where the first and the last steps follows from (3.2) since $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+) \subset A_{p/\gamma'}(\mathbb{R}^+)$, whereas the second step follows from (3.5). Hence, by [1, Lemma 2.3], there is an $\varepsilon > 0$ such that

$$\|T_j f\|_{L^p(\omega^{1+\varepsilon})} \leq C\|f\|_{L^p(\omega^{1+\varepsilon})} \qquad (3.7)$$

for any $p > \gamma'$ and $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$.

For $\beta > 1$, $p \in (\frac{2\gamma'\beta}{\gamma'+2(\beta-1)}, \frac{2\gamma'\beta}{\gamma'+2(\beta-1)(\gamma'-1)})$, by interpolation between (3.4) and (3.7) with $\omega = 1$, we find a $\theta_1 > 1$ such that

$$\|T_j f\|_{L^p(\mathbb{R}^n)} \leq C(1+|j|)^{-\theta_1}\|f\|_{L^p(\mathbb{R}^n)}, \quad \forall\, j \in \mathbb{Z}. \qquad (3.8)$$

Applying the Stein and Weiss interpolation theorem with change of measure [26], we interpolate (3.7) and (3.8) to get that,
for any $p \in (\frac{2\gamma'\beta}{\gamma'+2(\beta-1)}, \frac{2\gamma'\beta}{\gamma'+2(\beta-1)(\gamma'-1)})$ and $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$, there is a $\theta_2 > 1$ such that

$$\|T_j f\|_{L^p(\omega)} \leq C(1+|j|)^{-\theta_2}\|f\|_{L^p(\omega)}, \quad \forall\, j \in \mathbb{Z}.$$

Therefore, we get

$$\Big\|\sum_{k\in\mathbb{Z}}\sigma_k * f\Big\|_{L^p(\omega)} \leq C\sum_{j\in\mathbb{Z}}\|T_j f\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}$$

for any $\beta > 1$, $p \in (\frac{2\gamma'\beta}{\gamma'+2(\beta-1)}, \frac{2\gamma'\beta}{\gamma'+2(\beta-1)(\gamma'-1)})$ and $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$. This proves Lemma 2.1. □

To obtain the proof of Theorem 1.2, we need to establish the following proposition.

Proposition 3.2. *Under the same conditions as in Proposition 3.1, for any* $\beta > 3/2$, $p \in (\frac{\gamma'(2\beta-1)}{\gamma'+(2\beta-3)}, \frac{\gamma'(2\beta-1)}{\gamma'+(\gamma'-1)(2\beta-3)})$ *and* $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$, *there exists a constant* $C > 0$ *such that*

$$\Big\| \sup_{k\in\mathbb{Z}} \big| \sum_{j=k} \sigma_j * f \big| \Big\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}. \tag{3.9}$$

Proof. Let ϕ be a $\mathcal{C}^\infty$ function satisfying $\phi(|\xi|) = 1$ when $|\xi| < 1/\lambda$ and $\phi(|\xi|) = 0$ when $|\xi| > \lambda$. Define $\widehat{\psi} = \phi$ and $\psi_k(\xi) = \frac{1}{(a_k)^n}\phi(\frac{\xi}{a_k})$, we make a decomposition as follows:

$$\sum_{j=k}^{+\infty} \sigma_j * f(x) = \psi_k * \sum_{j\in\mathbb{Z}} \sigma_j * f(x) - \psi_k * \sum_{j=-\infty}^{k-1} \sigma_j * f(x) + (\delta - \psi_k) * \sum_{j=k}^{+\infty} \sigma_j * f(x),$$

where δ is the dirac measure at zero. It follows that

$$\begin{aligned}\sup_{k\in\mathbb{Z}} \Big| \sum_{j=k}^{+\infty} \sigma_j * f \Big| \le{}& \sup_{k\in\mathbb{Z}} \Big| \psi_k * \sum_{j\in\mathbb{Z}} \sigma_j * f(x) \Big| + \sup_{k\in\mathbb{Z}} \Big| \psi_k * \sum_{j=-\infty}^{k-1} \sigma_j * f(x) \Big| \\ &+ \sup_{k\in\mathbb{Z}} \Big| (\delta - \psi_k) \sum_{j=k}^{\infty} \sigma_j * f(x) \Big| \\ =:{}& J_1 f + J_2 f + J_3 f.\end{aligned}$$

Below we will estimate $J_1 f$, $J_2 f$ and $J_3 f$, respectively.

For $J_1 f$, it is easy to check that

$$|J_1 f(x)| \le CM\big(\sum_{j\in\mathbb{Z}} \sigma_j * f\big)(x).$$

Then, by Proposition 3.1 and the fact that $\tilde{A}_{p/\gamma'}(\mathbb{R}^+) \subset A_{p/\gamma'}(\mathbb{R}^+)$, we immediately get

$$\|J_1 f\|_{L^p(\omega)} \le \big\| M\big(\sum_{j\in\mathbb{Z}} \sigma_j * f\big)\big\|_{L^p(\omega)} \le C\big\| \sum_{j\in\mathbb{Z}} \sigma_j * f \big\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}$$

for any $\beta > 3/2$, $p \in (\frac{\gamma'(2\beta-1)}{\gamma'+(2\beta-3)}, \frac{\gamma'(2\beta-1)}{\gamma'+(\gamma'-1)(2\beta-3)})$ and $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$.

For $J_2 f$, we can write

$$J_2 f(x) = \sup_{k\in\mathbb{Z}} \Big| \sum_{j=1}^{\infty} \psi_k * \sigma_{k-j} * f(x) \Big| \le \sum_{j=1}^{\infty} \sup_{k\in\mathbb{Z}} |\psi_k * \sigma_{k-j} * f(x)| =: \sum_{j=1}^{\infty} J_2^j f(x).$$

Notice that $|J_2^j f(x)| \le CMM_\sigma f(x)$, invoking our assumptions (iv) yields that

$$\|J_2^j f\|_{L^p(\omega)} \le C\|M_\sigma f\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}$$

for any $p > \gamma'$ and $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$. Hence, there is an $\varepsilon > 0$ such that

$$\|J_2^j f\|_{L^p(\omega^{1+\varepsilon})} \le C\|f\|_{L^p(\omega^{1+\varepsilon})} \tag{3.10}$$

for any $p > \gamma'$ and $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$. On the other hand, by (ii) and Plancherel's theorem, we have

$$\begin{aligned}
\|J_2^j f\|^2_{L^2(\mathbb{R}^n)} &\le \Big\| \Big(\sum_{k\in\mathbb{Z}} |\psi_k * \sigma_{k-j} * f|^2 \Big)^{1/2} \Big\|^2_{L^2(\mathbb{R}^n)} \\
&\le \sum_{k\in\mathbb{Z}} \int_{\{|a_k\xi|\le\lambda\}} |\widehat{\sigma_{k-j}}(\xi)|^2 |\hat{f}(\xi)|^2 d\xi \\
&\le \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |\widehat{\sigma_{k-j}}(\xi)|^2 \chi_{\{|a_k\xi|\le\lambda\}} |\hat{f}(\xi)|^2 d\xi \\
&\le C \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \lambda^{2(k-j)} \chi_{\{|a_k\xi|\le\lambda\}} |\hat{f}(\xi)|^2 d\xi \\
&\le C\lambda^{-2j} \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \lambda^{2k} \chi_{\{|a_k\xi|\le\lambda\}} |\hat{f}(\xi)|^2 d\xi \\
&\le C\lambda^{-2j} \|f\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

That is,

$$\|J_2^j f\|_{L^2(\mathbb{R}^n)} \le C\lambda^{-j} \|f\|_{L^2(\mathbb{R}^n)}. \tag{3.11}$$

Interpolating between (3.11) and (3.10) ($\omega = 1$) gives that

$$\|J_2^j f\|_{L^p(\mathbb{R}^n)} \le C\lambda^{-\theta_3} \|f\|_{L^p(\mathbb{R}^n)} \tag{3.12}$$

for any $p > \gamma'$ and $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$. By Stein and Weiss' interpolation theorem with change of measure between (3.10) and (3.12), we find a positive number θ_3 such that

$$\|J_2^j f\|_{L^p(\omega)} \le C\lambda^{-\theta_3} \|f\|_{L^p(\omega)}$$

for any $p > \gamma'$ and $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$. Hence,

$$\|J_2 f\|_{L^p(\omega)} \le \sum_{j=0}^{\infty} \|J_2^j f\|_{L^p(u)} \le C\|f\|_{L^p(\omega)}$$

for any $p > \gamma'$ and $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$.

Finally, to estimate $J_3 f$, we write

$$\begin{aligned}
J_3 f(x) &= \sup_{k\in\mathbb{Z}} \Big| \sum_{j=0}^{\infty} (\delta - \psi_k) * \sigma_{k+j} * f(x) \Big| \\
&\le \sum_{j=0}^{\infty} \sup_{k\in\mathbb{Z}} |(\delta - \psi_k) * \sigma_{k+j} * f(x)| \\
&=: \sum_{j=0}^{\infty} J_3^j f(x).
\end{aligned}$$

Similarly to the arguments on $J_2^j f$, it is easy to check that

$$\|J_3^j f\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}$$

for any $p > \gamma'$ and $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$.

Also, by (iii) and the Plancherel theorem, we deduce that

$$\begin{aligned}
\|J_3^j f\|^2_{L^2(\mathbb{R}^n)} &\le \Big\|\Big(\sum_{k\in\mathbb{Z}} |(\delta-\psi_k)*\sigma_{k+j}*f|^2\Big)^{1/2}\Big\|^2_{L^2(\mathbb{R}^n)}\\
&\le \sum_{k\in\mathbb{Z}} \int_{\{|a_k\xi|\ge 1/\lambda\}} |\widehat{\sigma_{k+j}}(\xi)|^2 |\hat{f}(\xi)|^2 d\xi\\
&\le \sum_{k\in\mathbb{Z}}\sum_{i=0}^{\infty} \int_{\lambda^{i-1}\le|a_k\xi|<\lambda^i} |\widehat{\sigma_{k+j}}(\xi)|^2 |\hat{f}(\xi)|^2 d\xi\\
&\le C\sum_{k\in\mathbb{Z}}\sum_{i=0}^{\infty} \int_{\lambda^{i-1}\le|a_k\xi|<\lambda^i} |\hat{f}(\xi)|^2 \Big(\log|a_{k+j}\xi|\Big)^{-2\beta} d\xi\\
&\le C\sum_{k\in\mathbb{Z}}\sum_{i=0}^{\infty} \Big(\frac{1}{i+j-1}\Big)^{2\beta} \int_{\lambda^{i-1}\le|a_k\xi|<\lambda^i} |\hat{f}(\xi)|^2 d\xi\\
&\le C\sum_{i=0}^{\infty} \Big(\frac{1}{i+j-1}\Big)^{2\beta} \|f\|^2_{L^2(\mathbb{R}^n)}\\
&\le Cj^{1-2\beta}\|f\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Hence,

$$\|J_3^j f\|_{L^2(\mathbb{R}^n)} \le Cj^{-(\beta-1/2)}\|f\|_{L^2(\mathbb{R}^n)}.$$

An interpolation gives that

$$\|J_3^j f\|_{L^p(\mathbb{R}^n)} \le Cj^{-\theta_3}\|f\|_{L^p(\mathbb{R}^n)}$$

for any $\beta > 3/2$, $p \in (\frac{\gamma'(2\beta-1)}{\gamma'+(2\beta-3)}, \frac{\gamma'(2\beta-1)}{\gamma'+(\gamma'-1)(2\beta-3)})$ and $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$. Stein and Weiss's interpolation theorem with change of measure derives that there is a positive number θ_3 such that

$$\|J_3^j f\|_{L^p(\omega)} \le Cj^{-\theta_3}\|f\|_{L^p(\omega)}$$

for any $\beta > 3/2$, $p \in (\frac{\gamma'(2\beta-1)}{\gamma'+(2\beta-3)}, \frac{\gamma'(2\beta-1)}{\gamma'+(\gamma'-1)(2\beta-3)})$ and $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$. Hence,

$$\|J_3 f\|_{L^p(\omega)} \le \sum_{j=0}^{\infty}\|J_3^j f\|_{L^p(u)} \le C\|f\|_{L^p(\omega)}$$

for any $\beta > 3/2$, $p \in (\frac{\gamma'(2\beta-1)}{\gamma'+(2\beta-3)}, \frac{\gamma'(2\beta-1)}{\gamma'+(\gamma'-1)(2\beta-3)})$ and $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$. Summing up the estimate $J_1 f$, $J_2 f$ and $J_3 f$, we get the desired result and completes the proof of Proposition 3.2. □

Now we prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. According to Lemma 2.1, Lemma 2.2(i)-(ii), Lemma 2.3(i)-(ii), the result of Theorem 1.1 for $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$ directly follows from Proposition 3.1 by substitution $\beta/2$ for β. And the corresponding result for $\omega \in \tilde{A}_{p'/\gamma'}(\mathbb{R}^+)$ follows from the duality arguments and the result for $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$. □

Proof of Theorem 1.2. We note that

$$T^*_{\Omega,\Gamma,h}f(x) = \Big| \int_{\varepsilon<|y|<2^{k+1}} \frac{\Omega(y)h(|y|)}{|y|^n} f(x-\Gamma(|y|)y')dy \Big| + \Big| \sum_{j=k}^{+\infty} \sigma_{k,\Omega,\Gamma,h} * f(x) \Big|$$

for some k satisfying $2^k < \varepsilon < 2^{k+1}$. Therefore

$$T^*_{\Omega,\Gamma,h}f(x) \leq \mathrm{M}_{\sigma,\Omega,\Gamma,h}f(x) + \sup_{k\in\mathbb{Z}} \Big| \sum_{j=k}^{+\infty} \sigma_{k,\Omega,\Gamma,h} * f(x) \Big|.$$

This together with Lemma 2.1 and Proposition 3.2 deduces Theorem 1.2. □

4. Concluding results

In [11], Ding, Xue and Yabuta gave an ingenious way to transfer the effect of the surface to the radial function in the kernel of singular integral along surfaces, and they obtained the following result.

Theorem G. *Suppose that* Γ *is a nonnegative* (*or nonpositive*) *and monotonic* C^1 *function on* $(0,\infty)$ *such that* $\phi(t) := \frac{\Gamma(t)}{t\Gamma'(t)}$ *is bounded. Let* $h \in \triangle_\gamma(\mathbb{R}_+)$ *for some* $\gamma \geq 2$. *If* $\Omega \in H^1(S^{n-1})$ *satisfies* (1.1), *then* $T_{\Omega,\Gamma,h}$ *is bounded on* $L^p(\omega)$ *for* $\gamma' \leq p < \infty$ *and* $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$.

Employing the ideas in [11], we can further weaken the assumptions on Γ and extend our Theorems 1.1 and 1.2 as follows.

Theorem 4.1. *Suppose that* Γ *is a nonnegative* (*or nonpositive*) *and monotonic* C^1 *function on* $(0,\infty)$ *such that* $\phi(t) := \frac{\Gamma(t)}{t\Gamma'(t)}$ *is bounded. Let* $h \in \triangle_\gamma(\mathbb{R}_+)$ *for some* $\gamma \in (2,\infty]$. *If* $\Omega \in \mathcal{WG}_\beta(S^{n-1})$ *for some* $\beta \in (2,\infty)$ *and satisfies* (1.1), *then* $T_{\Omega,\Gamma,h}$ *is bounded on* $L^p(\omega)$ *for* $p \in (\frac{\gamma'\beta}{\gamma'+\beta-2}, \frac{\gamma'\beta}{\gamma'+(\beta-2)(\gamma'-1)})$ *and* $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$ *or* $\tilde{A}_{p'/\gamma'}(\mathbb{R}^+)$.

Theorem 4.2. *Let* Γ, ϕ, r, h *be the same as in Theorem 4.1. If* $\Omega \in \mathcal{WG}_\beta(S^{n-1})$ *for some* $\beta \in (3,\infty)$ *and satisfies* (1.1), *then* $T^*_{\Omega,\Gamma,h}$ *is bounded on* $L^p(\omega)$ *for* $p \in (\frac{\gamma'(\beta-1)}{\gamma'+\beta-3}, \frac{\gamma'(\beta-1)}{\gamma'+(\gamma'-1)(\beta-3)})$ *and* $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$.

To prove the above theorems, we need the following two lemmas in [11].

Lemma 4.3 (cf. [11])**.** *Let* Γ *and* ϕ *be the same as in Theorem 4.1. If* $h \in \triangle_\gamma(\mathbb{R}_+)$ *for some* $\gamma > 1$, *then*

$$\frac{1}{R}\int_0^R |h(|\Gamma^{-1}(t)|)\phi(|\Gamma^{-1}(t)|)|^\gamma dt \leq C_\gamma(\|\phi\|_\infty^{\gamma-1} + \|\phi\|_\infty^\gamma), \quad R > 0.$$

Lemma 4.4 (cf. [11]). *Let Γ, ϕ and h be the same as in Theorem 4.1.*

(1) *If Γ is nonnegative and increasing,* $T_{\Omega,\Gamma,h}f(x)=T_{\Omega,\phi(\Gamma^{-1})h(\Gamma^{-1})}f(x)$.

(2) *If Γ is nonnegative and decreasing,* $T_{\Omega,\Gamma,h}f(x)=-T_{\Omega,\phi(\Gamma^{-1})h(\Gamma^{-1})}f(x)$.

(3) *If Γ is nonpositive and decreasing,* $T_{\Omega,\Gamma,h}f(x)=T_{\tilde{\Omega},\phi(\Gamma^{-1})h(\Gamma^{-1})}f(x)$.

(4) *If Γ is nonpositive and increasing,* $T_{\Omega,\Gamma,h}f(x)=-T_{\tilde{\Omega},\phi(\Gamma^{-1})h(\Gamma^{-1})}f(x)$.

Here $\tilde{\Omega}(x)=\Omega(-x)$.

Proofs of Theorems 4.2 and 4.3. Applying Lemmas 4.3-4.4 and Theorem 1.1 for the special $\Gamma(t)=t$, we get Theorem 4.1. Similarly, Theorem 4.2 directly follows from Lemma 4.3, Lemma 4.4 and Theorem 1.2. □

5. Marcinkiewicz integrals analog surfaces

In this section, we consider the following parametric Marcinkiewicz integrals along surfaces

$$\mathcal{M}^{\rho}_{\Omega,\Gamma,h}(f)(x)=\Big(\int_0^\infty\Big|\frac{1}{t^\rho}\int_{|y|\le t}f(x-\Gamma(|y|)y')\frac{h(|y|)\Omega(y')}{|y|^{n-\rho}}dy\Big|^2\frac{dt}{t}\Big)^{1/2},$$

where $\rho=\tau+i\vartheta$ ($\tau>0$, $\vartheta\in\mathbb{R}$) is a complex number. If $h\equiv 1$, we denote $\mathcal{M}^{\rho}_{\Omega,\Gamma,h}$ by $\mathcal{M}^{\rho}_{\Omega,\Gamma}$. Obviously, if $\Gamma(t)=t$ and $\rho=1$, then the operator $\mathcal{M}^{\rho}_{\Omega,\Gamma}$ reduces to the classical Marcinkiewicz integral operator $\mathcal{M}_\Omega$ introduced by Stein [25].

In [2], Al-Qassem proved the following result.

Theorem H. *Let Γ be in $C^2([0,\infty))$, convex, and an increasing function with $\Gamma(0)=0$. If $\Omega\in\mathcal{G}_\beta(S^{n-1})$ for some $\beta>2$, then there exists $C_p>0$ such that*

$$\|\mathcal{M}^{\rho}_{\Omega,\Gamma}f\|_{L^p(\omega)}\le C_p\|f\|_{L^p(\omega)}$$

for $p\in(\beta',\beta)$ and $\omega\in\tilde{A}_p(\mathbb{R}^+)$.

We remark that the above conditions on Γ automatically implies the condition $\phi(t)\in L^\infty(0,\infty)$ and (ii) in the condition $\mathfrak{F}$. By the similar arguments as in dealing with $T_{\Omega,\Gamma,h}$, we can improve and generalize the above result as follows.

Theorem 5.1. *Let $h\in\triangle_\gamma(\mathbb{R}_+)$ for some $\gamma\in(2,\infty]$, $\Omega\in\mathcal{WG}_\beta(S^{n-1})$ for some $\beta\in(2,\infty)$ and satisfy* (1.1). *Suppose that $\Gamma\in\mathfrak{F}$ and $\omega\in\tilde{A}_{p/\gamma'}(\mathbb{R}^+)$. Then $\mathcal{M}^{\rho}_{\Omega,\Gamma,h}$ is bounded on $L^p(\omega)$ for $p\in(\frac{\gamma'\beta}{\gamma'+\beta-2},\frac{\gamma'\beta}{\gamma'+(\beta-2)(\gamma'-1)})$.*

Proof. As to the operator $\mathcal{M}^{\rho}_{\Omega,\Gamma,h}$, by applying Minkowski's inequality, we can write

$$\begin{aligned}\mathcal{M}^{\rho}_{\Omega,\Gamma,h}f(x)&=\Big(\int_0^\infty\Big|\sum_{k=-\infty}^{0}\frac{1}{t^\rho}\int_{2^{k-1}t<|y|\le 2^kt}f(x-\Gamma(|y|)y')\frac{h(|y|)\Omega(y)}{|y|^{n-\rho}}dy\Big|^2\frac{dt}{t}\Big)^{1/2}\\&\le\sum_{k=-\infty}^{0}\Big(\int_0^\infty\Big|\frac{1}{t^\rho}\int_{2^{k-1}t<|y|\le 2^kt}f(x-\Gamma(|y|)y')\frac{h(|y|)\Omega(y)}{|y|^{n-\rho}}dy\Big|^2\frac{dt}{t}\Big)^{1/2}\end{aligned}$$

$$= \frac{1}{1-2^{-\tau}}\Big(\int_0^\infty \Big|\frac{1}{t^\rho}\int_{t/2<|y|\le t} f(x-\Gamma(|y|)y')\frac{h(|y|)\Omega(y)}{|y|^{n-\rho}}dy\Big|^2\frac{dt}{t}\Big)^{1/2}.$$

We define two sequences of measures $\{\sigma_{t,\rho,\Omega,\Gamma,h} : t \in \mathbb{R}^+\}$ and $\{|\sigma_{t,\rho,\Omega,\Gamma,h}| : t \in \mathbb{R}^+\}$ related to Γ by

$$\int_{\mathbb{R}^n} f d\sigma_{t,\rho,\Omega,\Gamma,h} = \frac{1}{t^\rho}\int_{t/2<|x|\le t} f(\Gamma(|x|)x')\frac{\Omega(x)h(|x|)}{|x|^{n-\rho}}dx,$$

and $|\sigma_{t,\rho,\Omega,\Gamma,h}|$ is defined by the same way as $\sigma_{t,\rho,\Omega,\Gamma,h}$, but with Ω replaced by $|\Omega|$ and h replaced by $|h|$. Thus,

$$\mathcal{M}^\rho_{\Omega,\Gamma,h}f(x) \le \frac{1}{1-2^{-\tau}}\Big(\int_0^\infty \big|\sigma_{t,\rho,\Omega,\Gamma,h} * f(x)\big|^2\frac{dt}{t}\Big)^{1/2}.$$

The maximal operators $M^\rho_{\sigma,\Omega,\Gamma,h}$ on $\mathbb{R}^n$ is defined by

$$M^\rho_{\sigma,\Omega,\Gamma,h}f(x) = \sup_{t\in\mathbb{R}^+}\big||\sigma_{t,\rho,\Omega,\Gamma,h}| * f(x)\big|.$$

Similar to Lemma 2.2 and Lemma 2.3, we can easily obtain the estimates of measures $\sigma_{t,\rho,\Omega,\Gamma,h}$. According to [10, Lemma 3.2], we know that

$$\|M^\rho_{\sigma,\Omega,\Gamma,h}f\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}$$

for all $p \in (\gamma', \infty)$ and $\omega \in \tilde{A}_{p/\gamma'}(\mathbb{R}^+)$. Therefore, it is not difficult to derive Theorem 5.1 by applying the standard method of the same as in Proposition 3.1. We leave the details to the interested reader. □

As applications of the above result, we can also obtain a theorem on the parametric Marcinkiewicz integrals related to the Littlewood-Paley g^*_λ function and the area integral S. We define the operators $\mathcal{M}^{\rho,*}_{\Omega,\Gamma,h,\lambda}$ and $\mathcal{M}^\rho_{\Omega,\Gamma,h,S}$ by

$$\mathcal{M}^{\rho,*}_{\Omega,\Gamma,h,\lambda}(f)(x) = \Big(\int\int_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda}\big|\mathcal{F}^\rho_{\Omega,\Gamma,h}f(y,t)\big|^2\frac{dydt}{t^{n+1}}\Big)^{1/2},$$

where $\lambda > 1$ and $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$,

$$\mathcal{F}^\rho_{\Omega,\Gamma,h}f(x,t) = \frac{1}{t^\rho}\int_{|y|\le t} f(x-\Gamma(|y|)y')\frac{h(|y|)\Omega(y')}{|y|^{n-\rho}}dy,$$

and

$$\mathcal{M}^\rho_{\Omega,\Gamma,h,S}(f)(x) = \Big(\int\int_{V(x)}\big|\mathcal{F}^\rho_{\Omega,\Gamma,h}f(y,t)\big|^2\frac{dydt}{t^{n+1}}\Big)^{1/2},$$

where $V(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$. Then, we have:

Theorem 5.2. *Let $h \in \triangle_\gamma(\mathbb{R}^+)$ for some $\gamma \in (2,\infty]$, $\Omega \in \mathcal{WG}_\beta(S^{n-1})$ for some $\beta \in (2,\infty)$ and satisfy (1.1). Suppose that $\Gamma \in \mathfrak{F}$ and $\omega \in \tilde{A}_{p/2}(\mathbb{R}^+)$. Then $\mathcal{M}^{\rho,*}_{\Omega,\Gamma,h,\lambda}$ is bounded on $L^p(\omega)$ for $p \in [2, \frac{\gamma'\beta}{\gamma'+(\beta-2)(\gamma'-1)})$. The same conclusion holds for $\mathcal{M}^\rho_{\Omega,\Gamma,h,S}$.*

Proof. Following the similar arguments to the proof of in [9, Lemma 5.1], it is not hard to see that

$$\int_{\mathbb{R}^n} \big(\mathcal{M}^{\rho,*}_{\Omega,\Gamma,h,\lambda} f(x)\big)^2 g(x)dx \le C_\lambda \int_{\mathbb{R}^n} \big(\mathcal{M}^{\rho}_{\Omega,\Gamma,h} f(x)\big)^2 Mg(x)dx, \tag{5.1}$$

where M is the classical Hardy-Littlewood maximal operator on $\mathbb{R}^n$.

If $p = 2$, then $\omega \in \tilde{A}_1(\mathbb{R}^+) \subset A_1(\mathbb{R}^+)$. Thus, $M\omega(x) \le C\omega(x)\, a.e.\, x \in \mathbb{R}^n$. This together with (5.1) and Theorem 1.1, we have

$$\int_{\mathbb{R}^n} \big(\mathcal{M}^{\rho,*}_{\Omega,\Gamma,h,\lambda} f(x)\big)^2 \omega(x)dx \le C_\lambda \int_{\mathbb{R}^n} |f(x)|^2 \omega(x)dx.$$

If $p \in (2, \frac{\gamma'\beta}{\gamma'+(\beta-2)(\gamma'-1)})$. Let $q = p/2$ and X denote the set of all $g \in L^{q'}(\omega^{1-q'})$ with $\|g\|_{L^{q'}(\omega^{1-q'})} \le 1$. Then we can write

$$\begin{aligned} \|\mathcal{M}^{\rho,*}_{\Omega,\Gamma,h,\lambda}\|^2_{L^p(\omega)} &= \sup_X \Big| \int_{\mathbb{R}^n} \big(\mathcal{M}^{\rho,*}_{\Omega,\Gamma,h,\lambda} f(x)\big)^2 g(x)dx \Big| \\ &\le C \sup_X \Big| \int_{\mathbb{R}^n} \big(\mathcal{M}^{\rho}_{\Omega,\Gamma,h} f(x)\big)^2 Mg(x)dx \Big| \\ &\le C \|\mathcal{M}^{\rho}_{\Omega,\Gamma,h} f\|^2_{L^p(\omega)} \sup_X \|Mg\|_{L^{q'}(\omega^{1-q'})} \\ &\le C\|f\|^2_{L^p(\omega)}. \end{aligned}$$

On the other hand, it is easy to check that

$$\mathcal{M}^{\rho}_{\Omega,\Gamma,h,S} f(x) \le C_\lambda \mathcal{M}^{\rho,*}_{\Omega,\Gamma,h,\lambda} f(x),$$

which implies Theorem 5.2 and completes our proof. □

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