

## VANISHING OF GROMOV-WITTEN INVARIANTS OF PRODUCT OF $\mathbb{P}^1$

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ABSTRACT. We study the Gromov-Witten invariants of  $\underbrace{\mathbb{P}^1 \times \mathbb{P}^1 \times \dots \times \mathbb{P}^1}_n$ .  
Our main result is a proof of some vanishing conditions on the  
Gromov-Witten invariants of  $\mathbb{P}^1 \times \mathbb{P}^1 \times \dots \times \mathbb{P}^1$  in all genera.

### 1. Introduction

#### 1.1. Gromov-Witten invariants

Denote by  $\mathbb{P}[n]$  the product of  $n$  projective lines:

$$\mathbb{P}[n] := \underbrace{\mathbb{P}^1 \times \mathbb{P}^1 \times \dots \times \mathbb{P}^1}_n.$$

Denote by

$$\overline{M}_{g,m}(\mathbb{P}[n], \mathbf{d})$$

the moduli space of stable maps to  $\mathbb{P}[n]$  with degree  $\mathbf{d} = (d_1, d_2, \dots, d_n)$ .  
It has the canonical virtual fundamental class  $[\overline{M}_{g,m}(\mathbb{P}[n], \mathbf{d})]^{vir}$  of dimension  $(1-g)(n-3) + m + 2 \cdot \sum_i d_i$ .

The Gromov-Witten invariants without insertions are defined by

$$GW_{g,0}(\mathbb{P}[n], \mathbf{d}) := \int_{[\overline{M}_{g,0}(\mathbb{P}[n], \mathbf{d})]^{vir}} 1.$$

The Gromov-Witten invariants have been studied more than 20 years, see [6],[7] for an introduction to the subject. By dimensional reason,  $GW_{g,0}(\mathbb{P}[n], \mathbf{d}) = 0$  unless  $(g-1)(n-3) = 2 \sum_k d_k$ . Even though there

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exist many way to calculate the Gromov-Witten invariants of  $\mathbb{P}[n]$ , they are not efficient to get some universal results for higher genus.

In this paper, we prove the following vanishing conditions on the Gromov-Witten invariants of  $\mathbb{P}[n]$  without insertions from a combinatorial observation obtained via torus localization.

**THEOREM 1.** *If  $g$  is even and  $n > 3g - 3$ ,*

$$GW_{g,0}(\mathbb{P}[n], \mathbf{d}) = 0.$$

Theorem 1 can be extended to Gromov-Witten invariants with insertions. To keep the notations simple we state the general theorem in Section 4.

## 1.2. Quasimap invariants and wall-crossing formula

Let  $\mathbf{G} := (\mathbb{C}^*)^n$  act on  $(\mathbb{C}^2)^n$  by the standard diagonal action componentwisly on each component  $\mathbb{C}^2$  so that its associated GIT quotient is

$$(\mathbb{C}^2)^n // \mathbf{G} = \mathbb{P}[n].$$

With this set-up, Ciocan-Fontanine and Kim defined quasimap moduli space

$$Q_{g,m}(\mathbb{P}[n], \mathbf{d})$$

with the canonical virtual fundamental class  $[Q_{g,k}(\mathbb{P}[n], \mathbf{d})]^{vir}$ . See [1], [5], [15] for an introduction. We define quasimap invariants of  $\mathbb{P}[n]$  without insertions by

$$QI_{g,0}(\mathbb{P}[n], \mathbf{d}) := \int_{[Q_{g,0}(\mathbb{P}[n], \mathbf{d})]^{vir}} 1.$$

In [3], authors studied the relationship between quasimap invariants and Gromov-Witten invariants. By applying the general theorem to  $\mathbb{P}[n]$ , We have

**PROPOSITION 2.** ([3])

$$GW_{g,0}(\mathbb{P}[n], \mathbf{d}) = QI_{g,0}(\mathbb{P}[n], \mathbf{d})$$

i.e. quasimap invariants and Gromov-Witten invariants are same for  $\mathbb{P}[n]$ . In section 3, we prove the following theorem which is equivalent to Theorem 1 by Proposition 2.

THEOREM 3. *If  $g$  is even and  $n > 3g - 3$ ,*

$$QI_{g,0}(\mathbb{P}[n], \mathbf{d}) = 0.$$

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## 2. Preliminaries

### 2.1. $\mathbb{T}$ -equivariant theory

Let  $\mathbb{T} := (\mathbb{C}^*)^{2n}$  act on  $\mathbb{P}[n]$  standardly on each component. Denote by

$$\lambda = (\lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2, \dots, \lambda_n, \bar{\lambda}_n)$$

$\mathbb{T}$ -equivariant parameters. We will use following specializations throughout the paper;

$$\lambda_k + \bar{\lambda}_k = 0, \text{ for } 1 \leq k \leq n.$$

First we set the notation for the cohomology basis and its dual basis. Let  $\{p_i\}$  be the set of  $\mathbb{T}$ -fixed points of  $\mathbb{P}[n]$  and let  $\phi_i$  be the basis of  $H_{\mathbb{T}}^*(\mathbb{P}[n])$  defined by satisfying followings:

$$(1) \quad \phi_i|_{p_j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Let  $\phi^i$  be the dual basis with respect to the  $\mathbb{T}$ -equivariant Poincaré pairing, i.e.,

$$\int_Y \phi_i \phi^j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

For each  $\mathbb{T}$ -fixed point  $p_i \in \mathbb{P}[n]$ , Let

$$e_i = e(T_{p_i}(\mathbb{P}[n]))$$

be the equivariant Euler class of the tangent space of  $\mathbb{P}[n]$  at  $p_i$ .

The action of  $\mathbb{T}$  on  $\mathbb{P}[n]$  lifts to  $Q_{g,0}(\mathbb{P}[n], \mathbf{d})$ . The localization formula of [11] applied to the virtual fundamental class  $[Q_{g,0}(\mathbb{P}[n], \mathbf{d})]^{\text{vir}}$  will play a fundamental role in our paper. The  $\mathbb{T}$ -fixed loci are represented in terms of dual graphs, and the contributions of the  $\mathbb{T}$ -fixed loci are given by tautological classes. The formulas are standard, see [12],[15].

## 2.2. Genus zero invariants

In this section, we review the genus zero theory. Integrating along the virtual fundamental class

$$[Q_{0,k}(\mathbb{P}[n], \mathbf{d})]^{\text{vir}}$$

we define correlators  $\langle \dots \rangle_{0,k,\mathbf{d}}^{0+}$  as follows. For  $\gamma_i \in H_{\mathbb{T}}^*(\mathbb{P}[n]) \otimes \mathbb{Q}(\lambda)$ ,

$$\langle \gamma_1 \psi^{a_1}, \dots, \gamma_k \psi^{a_k} \rangle_{0,k,\mathbf{d}}^{0+} = \int_{[Q_{0,k}(\mathbb{P}[n], \mathbf{d})]^{\text{vir}}} \prod_i ev_i^*(\gamma_i) \psi_i^{a_i},$$

where  $\psi_i$  is the psi-class associated to the  $i$ -th marking and  $ev_i$  is the  $i$ -th evaluation map.

Let  $Q_{g,k}(\mathbb{P}[n], \mathbf{d})^{\mathbb{T}, p_i}$  be the fixed loci of  $Q_{g,k}(\mathbb{P}[n], \mathbf{d})$  whose elements have domain components only over  $p_i$ . Integrating along the localized cycle class

$$\frac{[Q_{0,k}(\mathbf{d})^{\mathbb{T}, p_i}]^{\text{vir}}}{e^{\mathbb{T}}(N_{Q_{0,k}(\mathbb{P}[n], \mathbf{d})^{\mathbb{T}, p_i}/Q_{0,k}(\mathbb{P}[n], \mathbf{d})}^{\text{vir}})}$$

we also define local correlators  $\langle \dots \rangle_{0,k,\beta}^{0+, p_i}$  and  $\langle\langle \dots \rangle\rangle_{0,k,\beta}^{0+, p_i}$  as follows:

$$\begin{aligned} \langle \gamma_1 \psi^{a_1}, \dots, \gamma_k \psi^{a_k} \rangle_{0,k,\mathbf{d}}^{0+, p_i} &= \int \frac{[Q_{0,k}(\mathbb{P}[n], \mathbf{d})^{\mathbb{T}, p_i}]^{\text{vir}}}{e^{\mathbb{T}}(N_{Q_{0,k}(\mathbb{P}[n], \mathbf{d})^{\mathbb{T}, p_i}/Q_{0,k}(\mathbb{P}[n], \mathbf{d})}^{\text{vir}})} \prod_i ev_i^*(\gamma_i) \psi_i^{a_i} ; \\ \langle\langle \gamma_1 \psi^{a_1}, \dots, \gamma_k \psi^{a_k} \rangle\rangle_{0,k}^{0+, p_i} &= \sum_{m, \mathbf{d}} \frac{\mathbf{q}^{\mathbf{d}}}{m!} \langle \gamma_1 \psi^{a_1}, \dots, \gamma_k \psi^{a_k}, \mathbf{t}, \dots, \mathbf{t} \rangle_{0, k+m, \mathbf{d}}^{0+, p_i} \end{aligned}$$

where  $\mathbf{q} = (q_1, \dots, q_n)$  are formal Novikov variables and  $\mathbf{t} = \sum_i t_i H_i \in H_{\mathbb{T}}^*(\mathbb{P}[n])$  where  $H_i$  is the pull back of hyperplane class in  $i$ -th  $\mathbb{P}^1$ .

## 2.3. Insertions of 0+ weighted marking

For the explicit calculations of various genus 0 invariants, we need to introduce the notion of 0+ weighted marking. We briefly recall the definitions from [4].

Denote by

$$Q_{g,k|m}^{0+, 0+}(\mathbb{P}[n], \mathbf{d})$$

the moduli space of genus  $g$  (resp. genus zero), degree  $\mathbf{d}$  stable quasimaps to  $\mathbb{P}[n]$  with ordinary  $k$  pointed markings and infinitesimally weighted  $m$  pointed markings, see [4] for more explanations.

Denote by

$$Q_{g,k|m}^{0+,0+}(\mathbb{P}[n], \mathbf{d})^{\mathbb{T}, p_i}$$

the  $\mathbb{T}$ -fixed part of  $Q_{g,k|m}^{0+,0+}(\mathbb{P}[n], \mathbf{d})$ , whose domain components of universal curves are only over  $p_i$ .

Let  $\tilde{H}_i \in H^*([\mathbb{C}^2]^n / \mathbf{G})$  be a lift of  $H_i \in H^*(\mathbb{P}[n])$ , i.e. ,  $\tilde{H}_i|_{\mathbb{P}[n]} = H_i$ . For  $\gamma_i \in H^*(\mathbb{P}[n]) \otimes \mathbb{Q}(\lambda)$ ,  $\mathbf{t} = \sum_i t_i \tilde{H}_i$ ,  $\delta_j \in H^*([\mathbb{C}^2]^n / \mathbf{G}, \mathbb{Q})$ , denote

$$\begin{aligned} & \langle \gamma_1 \psi^{a_1}, \dots, \gamma_k \psi^{a_k}; \delta_1, \dots, \delta_m \rangle_{0,k|m,\mathbf{d}}^{0+,0+} \\ &= \int_{[Q_{0,k|m}^{0+,0+}(\mathbb{P}[n], \mathbf{d})^{\mathbb{T}}]^{\text{vir}}} \prod_i ev_i^*(\gamma_i) \psi_i^{a_i} \prod_j \hat{ev}_j^*(\delta_j) \ ; \\ & \langle \langle \gamma_1 \psi^{a_1}, \dots, \gamma_k \psi^{a_k} \rangle \rangle_{0,k}^{0+,0+} \\ &= \sum_{m,\beta} \frac{\mathbf{q}^{\mathbf{d}}}{m!} \langle \gamma_1 \psi^{a_1}, \dots, \gamma_k \psi^{a_k}; \mathbf{t}, \dots, \mathbf{t} \rangle_{0,k|m,\mathbf{d}}^{0+,0+} \ ; \\ & \langle \gamma_1 \psi^{a_1}, \dots, \gamma_k \psi^{a_k}; \delta_1, \dots, \delta_m \rangle_{0,k|m,\mathbf{d}}^{0+,0+,p_i} \\ &= \int_{\frac{[Q_{0,k|m}^{0+,0+}(\mathbb{P}[n], \mathbf{d})^{\mathbb{T}, p_i}]^{\text{vir}}}{e^{\mathbb{T}(N^{\text{vir}})}_{Q_{0,k|m}^{0+,0+}(\mathbb{P}[n], \mathbf{d})^{\mathbb{T}, p_i} / Q_{0,k|m}^{0+,0+}(\mathbb{P}[n], \mathbf{d})}} \prod_i ev_i^*(\gamma_i) \psi_i^{a_i} \prod_j \hat{ev}_j^*(\delta_j) \ ; \\ & \langle \langle \gamma_1 \psi^{a_1}, \dots, \gamma_k \psi^{a_k} \rangle \rangle_{0,k}^{0+,0+,p_i} \\ &= \sum_{m,\mathbf{d}} \frac{\mathbf{q}^{\mathbf{d}}}{m!} \langle \gamma_1 \psi^{a_1}, \dots, \gamma_k \psi^{a_k}; \mathbf{t}, \dots, \mathbf{t} \rangle_{0,k|m,\mathbf{d}}^{0+,0+,p_i} \ , \end{aligned}$$

where  $\hat{ev}_j$  is the evaluation map to  $[(\mathbb{C}^2)^n / \mathbf{G}]$  at the  $j$ -th infinitesimally weighted marking.

Let  $\{\phi_i\}$  be the equivariant basis of  $H^*(\mathbb{P}[n])$  satisfying (1). Let us define  $\mathbb{U}_i, \mathbb{S}, \mathbb{V}_{ij}$  by

$$\begin{aligned}
\mathbb{U}_{p_i}^{\mathbb{P}[n]} &:= \langle\langle 1, 1 \rangle\rangle_{0,2}^{0+,0+,i} \in \mathbb{R}[[\mathbf{q}, \mathbf{t}]] \quad ; \\
\mathbb{S}^{\mathbb{P}[n]}(\gamma) &:= \sum_{i,d} \phi^i \mathbf{q}^d \langle\langle \frac{\phi_i}{z - \psi}, \gamma \rangle\rangle_{0,2,d}^{0+,0+} \in \mathbb{R}[[z^{-1}, \mathbf{q}, \mathbf{t}]] \quad ; \\
\mathbb{V}_{ij}^{\mathbb{P}[n]}(x, y) &:= \sum_{\mathbf{d}} \mathbf{q}^{\mathbf{d}} \langle\langle \frac{\phi_i}{x - \psi}, \frac{\phi_j}{y - \psi} \rangle\rangle_{0,2,\mathbf{d}}^{0+,0+} \in \mathbb{R}[[x^{-1}, y^{-1}, \mathbf{q}, \mathbf{t}]] \quad ;
\end{aligned}$$

We recall the following WDVV equation from [3],

$$(2) \quad e_i \mathbb{V}_{ij}(x, y) e_j = \frac{\sum_k \mathbb{S}(\phi_k)|_{p_i, z=x} \mathbb{S}(\phi^k)|_{p_j, z=y}}{x + y}$$

To study the properties of  $\mathbb{S}$  and  $\mathbb{V}_{ij}$ , we need to recall infinitesimal  $I$ -function defined in [4]. Denote by  $\mathbb{I}$  the infinitesimal  $I$ -function  $\mathbb{J}^{0+,0+}$  in [4]. Note that  $\mathbb{S}$  coincides with the definition of infinitesimal  $S$ -operator  $\mathbb{S}^{0+,0+}$  in [4]. From (5.1.3) in [4], we know explicit form of  $\mathbb{I}$  of  $\mathbb{P}[n]$

$$(3) \quad \mathbb{I}^{\mathbb{P}[n]} = \prod_{i=1}^n e^{t_i(H_i + d_i z)/z} \frac{q_i^{d_i}}{\prod_{k=1}^{d_i} (H_i - \lambda_i + kz)(H_i - \bar{\lambda}_i + kz)} \in \mathbb{R}[[z, z^{-1}, \mathbf{q}, \mathbf{t}]](\lambda).$$

To find the explicit forms of  $\mathbb{S}$ -operators, we recall following proposition from [12].

**PROPOSITION 4.** (Proposition 2.4 in [12]) There are unique coefficients  $a_i(z, \mathbf{q}) \in \mathbb{Q}(\lambda)[z][[\mathbf{q}]]$  making

$$\sum_i a_i(z, \mathbf{q}) \partial_{\phi_i} \mathbb{I}^{\mathbb{P}[n]} = \gamma + O(1/z).$$

Furthermore LHS coincides with  $\mathbb{S}^{\mathbb{P}[n]}(\gamma)$ .

## 2.4. Ordered graphs.

Let the genus  $g$  and the number of markings  $m$  for the moduli space be in the stable range

$$2g - 2 + m > 0.$$

We can organize the  $\mathbb{T}$ -fixed loci of  $Q_{g,m}(\mathbb{P}[n], \mathbf{d})$  according to decorated graphs. A *decorated graph*  $\Gamma$  consists of the data  $(\mathbf{V}, \mathbf{E}, \mathbf{N}, \mathbf{g})$  where

- (i)  $V$  is the vertex set,
- (ii)  $E$  is the edge set (including possible self-edge),
- (iii)  $N : \{1, 2, \dots, m\} \rightarrow V$  is the marking assignment,
- (iv)  $\mathbf{g} : V \rightarrow \mathbb{Z}_{\geq 0}$  is a genus assignment satisfying

$$g = \sum_{v \in V} \mathbf{g}(v) + h^1(\Gamma).$$

Let  $\mathbf{G}_{g,m}$  be the set of decorated graph as defined as above. The *flags* of  $\Gamma$  are the half-edges<sup>1</sup>. Let  $F$  be the set of flags. For each  $\Gamma \in \mathbf{G}_{g,m}$ , we choose an ordering

$$(4) \quad \nu_\Gamma : V \rightarrow \{1, 2, \dots, |V|\}.$$

Let  $\mathbf{G}_{g,m}^{\text{ord}}$  be the set of decorated graph with fixed choice of orderings on vertices. We will sometimes identify  $V$  with  $\{1, 2, \dots, |V|\}$  by (4).

## 2.5. Universal ring

Denote by

$$\mathbf{R}$$

the ring generated by

$$\lambda_{1,j}^{\pm 1}, \lambda_{2,j}^{\pm 1}, \dots, \lambda_{n,j}^{\pm 1}, \quad j \in \mathbb{N}$$

with relations

$$(5) \quad \lambda_{k,j}^2 = 1, \text{ for } 1 \leq k \leq n, j \in \mathbb{N}.$$

- A monomial in  $\mathbf{R}$  is called *canonical* if degree of  $\lambda_{i,j}$  are 1 or 0 for all  $1 \leq i \leq n, j \in \mathbb{N}$ .
- *The length* of canonical monomial is the number of  $\lambda_{i,j}$  whose degree is 1.
- For a monomial  $m \in \mathbf{R}$ , *the length* of  $m$  is the length of unique canonical monomial equal to  $m$  in  $\mathbf{R}$ .

The ring  $\mathbf{R}$  will play a fundamental role in the proof of the main theorem.

**DEFINITION 5.** For monomial  $m \in \mathbf{R}$ , we say  $m$  has type  $(a, b) \in \mathbb{N}^2$  if  $m$  has following form;

$$m = \prod_j \prod_i \lambda_{i,j}^{a_{i,j}} f$$

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<sup>1</sup>Flags are either half-edges or markings.

where  $\sum_j a_{i,j} = a$  and  $f$  is a monomial whose length is less than or equal to  $b$ .

DEFINITION 6. For a polynomial  $f \in \mathbb{R}$ , we say  $f$  has type  $(a, b)$  if  $f$  is a sum of monomials of type  $(a, b)$ .

From the above definition, we can easily check the following two lemmas.

LEMMA 7. If  $m_1$  has type  $(a_1, b_1)$  and  $m_2$  has type  $(a_2, b_2)$ , then  $m_1 m_2$  has type  $(a_1 + a_2, b_1 + b_2)$ .

LEMMA 8. If  $f \in \mathbb{R}$  has type  $(a, b)$  with odd  $a$  and  $n > b$ ,

$$\sum_{\lambda_{i,j}=\pm 1} f = 0.$$

The notation  $\sum_{\lambda_{i,j}=\pm 1} f$  in Lemma 8 denotes the sum of evaluations of  $f \in \mathbb{R}$  with all possible choices of  $\lambda_{i,j} = 1$  or  $-1$ .

## 2.6. Universal polynomial

We review here the definition of universal polynomial  $\mathbf{P}$  from [14]. The universal polynomial play the essential role in calculating higher genus invariants from genus 0 invariants, see [9], [13], [14] for more explanations.

Let  $t_0, t_1, t_2, \dots$  be formal variables. The series

$$T(c) = t_0 + t_1 c + t_2 c^2 + \dots$$

in the additional variable  $c$  plays a basic role. The variable  $c$  will later be replaced by the first Chern class  $\psi_i$  of a cotangent line over  $\overline{M}_{g,n}$ ,

$$T(\psi_i) = t_0 + t_1 \psi_i + t_2 \psi_i^2 + \dots,$$

with the index  $i$  depending on the position of the series  $T$  in the correlator.

Let  $2g - 2 + n > 0$ . For  $a_i \in \mathbb{Z}_{\geq 0}$  and  $\gamma \in H^*(\overline{M}_{g,n})$ , define the correlator

$$\langle\langle \psi^{a_1}, \dots, \psi^{a_n} | \gamma \rangle\rangle_{g,n} = \sum_{k \geq 0} \frac{1}{k!} \int_{\overline{M}_{g,n+k}} \gamma \psi_1^{a_1} \dots \psi_n^{a_n} \prod_{i=1}^k T(\psi_{n+i}).$$

We consider  $\mathbb{C}(t_1)[t_2, t_3, \dots]$  as  $\mathbb{Z}$ -graded ring over  $\mathbb{C}(t_1)$  with

$$\deg(t_i) = i - 1 \quad \text{for } i \geq 2.$$



Define a subring of homogeneous elements by

$$\mathbb{C} \left[ \frac{1}{1-t_1} \right] [t_2, t_3, \dots]_{\text{Hom}} \subset \mathbb{C}(t_1)[t_2, t_3, \dots].$$

We easily see

$$\langle\langle \psi^{a_1}, \dots, \psi^{a_n} | \gamma \rangle\rangle_{g,n} |_{t_0=0} \in \mathbb{C} \left[ \frac{1}{1-t_1} \right] [t_2, t_3, \dots]_{\text{Hom}}.$$

Using the leading terms (of lowest degree in  $\frac{1}{(1-t_1)}$ ), we obtain the following result.

LEMMA 9. *The set of genus 0 correlators*

$$\left\{ \langle\langle 1, \dots, 1 \rangle\rangle_{0,n} |_{t_0=0} \right\}_{n \geq 4}$$

freely generate the ring  $\mathbb{C}(t_1)[t_2, t_3, \dots]$  over  $\mathbb{C}(t_1)$ .

By Lemma 9, we can find a unique representation of  $\langle\langle \psi^{a_1}, \dots, \psi^{a_n} \rangle\rangle_{g,n} |_{t_0=0}$  in the variables

$$(6) \quad \left\{ \langle\langle 1, \dots, 1 \rangle\rangle_{0,n} |_{t_0=0} \right\}_{n \geq 3}.$$

The  $n = 3$  correlator is included in the set (6) to capture the variable  $t_1$ . For example, in  $g = 1$ ,

$$\langle\langle 1, 1 \rangle\rangle_{1,2} |_{t_0=0} = \frac{1}{24} \left( \frac{\langle\langle 1, 1, 1, 1, 1 \rangle\rangle_{0,5} |_{t_0=0}}{\langle\langle 1, 1, 1 \rangle\rangle_{0,3} |_{t_0=0}} - \frac{\langle\langle 1, 1, 1, 1 \rangle\rangle_{0,4}^2 |_{t_0=0}}{\langle\langle 1, 1, 1 \rangle\rangle_{0,3}^2 |_{t_0=0}} \right),$$

$$\langle\langle 1 \rangle\rangle_{1,1} |_{t_0=0} = \frac{1}{24} \frac{\langle\langle 1, 1, 1, 1 \rangle\rangle_{0,4} |_{t_0=0}}{\langle\langle 1, 1, 1 \rangle\rangle_{0,3} |_{t_0=0}}$$

A more complicated example in  $g = 2$  is

$$\begin{aligned} \langle\langle \rangle\rangle_{2,0} |_{t_0=0} &= \frac{1}{1152} \frac{\langle\langle 1, 1, 1, 1, 1, 1 \rangle\rangle_{0,6} |_{t_0=0}}{\langle\langle 1, 1, 1 \rangle\rangle_{0,3}^2 |_{t_0=0}} \\ &\quad - \frac{7}{1920} \frac{\langle\langle 1, 1, 1, 1, 1 \rangle\rangle_{0,5} |_{t_0=0} \langle\langle 1, 1, 1 \rangle\rangle_{0,4} |_{t_0=0}}{\langle\langle 1, 1, 1 \rangle\rangle_{0,3}^3 |_{t_0=0}} \\ &\quad + \frac{1}{360} \frac{\langle\langle 1, 1, 1, 1 \rangle\rangle_{0,4}^3 |_{t_0=0}}{\langle\langle 1, 1, 1 \rangle\rangle_{0,3}^4 |_{t_0=0}}. \end{aligned}$$

DEFINITION 10. For  $\gamma \in H^*(\overline{M}_{g,k})$ , let

$$\mathbf{P}_{g,n}^{a_1, \dots, a_n, \gamma}(s_0, s_1, s_2, \dots) \in \mathbb{Q}(s_0, s_1, \dots)$$

be the unique rational function satisfying the condition

$$\langle\langle \psi^{a_1}, \dots, \psi^{a_n} | \gamma \rangle\rangle_{g,n} |_{t_0=0} = \mathbf{P}_{g,n}^{a_1, a_2, \dots, a_n, \gamma} |_{s_i = \langle\langle 1, \dots, 1 \rangle\rangle_{0, i+3} |_{t_0=0}}.$$

We use the following notation.

$$\mathbb{P} \left[ \psi_1^{k_1}, \dots, \psi_n^{k_n} \mid \mathbf{H}_h^{p_i} \right]_{h,n}^{0+,p_i} := \mathbb{P}_{h,1}^{k_1, \dots, k_n, \mathbf{H}_h^{p_i}} \left( \langle \langle 1, 1, 1 \rangle \rangle_{0,3}^{0+,p_i}, \langle \langle 1, 1, 1, 1 \rangle \rangle_{0,4}^{0+,p_i}, \dots \right).$$

### 3. Higher genus series on $\mathbb{P}[n]$

We review here the standard method first used by Givental to express genus  $g$  descendent correlators in terms of genus 0 data. See [10],[13],[12] for more explanations.

DEFINITION 11. *Let us define equivariant genus  $g$  generating function  $F^g(\mathbf{q}) \in \mathbb{C}[[\mathbf{q}]](\lambda)$  of  $\mathbb{P}[n]$  as following;*

$$\begin{aligned} F^g(\mathbf{q}) &:= \sum_{d_1, d_2, \dots, d_n \geq 0} q_1^{d_1} q_2^{d_2} \dots q_n^{d_n} \int_{[Q_{g,0}(\mathbb{P}[n], \mathbf{d})]^{\text{vir}}} 1 \\ &= \sum_{\mathbf{d} \geq 0} \mathbf{q}^{\mathbf{d}} \int_{[Q_{g,0}(\mathbb{P}[n], \mathbf{d})]^{\text{vir}}} 1 \in \mathbb{C}[[\mathbf{q}]](\lambda). \end{aligned}$$

We apply the localization strategy introduced first by Givental for Gromov-Witten theory to the quasimap invariants of  $\mathbb{P}[n]$ . We write the localization formula as

$$F^g|_{\lambda_i=1, \bar{\lambda}_i=-1} = \sum_{\lambda_i, j=\pm 1} \sum_{\Gamma \in \mathbf{G}_{g,0}^{\text{ord}}} \text{Cont}_{\Gamma}$$

where  $\text{Cont}_{\Gamma}$  is the contribution to  $F^g$  of the  $T$ -fixed loci associated  $\Gamma$ . We have the following formula for  $\text{Cont}_{\Gamma}$ .

$$\text{Cont}_{\Gamma} = \frac{1}{\text{Aut}(\Gamma)} \sum_{\mathbf{A} \in \mathbb{Z}_{\geq 1}^{F(\Gamma)}} \prod_v \prod_e \text{Cont}_{\Gamma}^{\mathbf{A}}(v) \text{Cont}_{\Gamma}^{\mathbf{A}}(e).$$

We need explanations for the  $\text{Cont}_{\Gamma}^{\mathbf{A}}(v)$  and  $\text{Cont}_{\Gamma}^{\mathbf{A}}(e)$ . Let  $v \in V(\Gamma)$  be a vertex of genus  $\mathbf{g}(v)$ .

$$\text{Cont}_{\Gamma}^{\mathbf{A}}(v) = \mathbb{P}[\psi^{a_1-1}, \psi^{a_2-1}, \dots, \psi^{a_k-1} \mid \mathbf{H}_{\mathbf{g}(v)}^v]_{\mathbf{g}(v),k}^{0+,v}$$

where,

- $H_{\mathbf{g}(v)}^v = \prod_{k=1}^n \frac{\prod_{j=1}^{\mathbf{g}(v)} ((\lambda_{k,v} - \bar{\lambda}_{k,v}) - c_j)}{(\lambda_{k,v} - \bar{\lambda}_{k,v})}$
- $k = \text{val}(v)$  is valency at  $v$  and  $(a_1, a_2, \dots, a_k)$  are component of  $\mathbf{A} \in \mathbb{Z}_{\geq 1}^{F(\Gamma)}$  corresponding to flag of  $v$ .
- $\lambda_{k,v}$  is the weight of  $k$ -th  $\mathbb{C}^*$  at  $\nu(v)$ .
- $\{c_j | 1 \leq j \leq \mathbf{g}(v)\}$  are chern roots of Hodge bundle on  $\overline{M}_{\mathbf{g}(v),k}$ .

Let  $e \in E(\Gamma)$  be an edge between  $v_i$  and  $v_j$ .

$$\text{Cont}_{\Gamma}^{\mathbf{A}}(e) = \left[ e^{-\frac{\bar{U}_i^{\mathbb{P}[n]}}{x} - \frac{\bar{U}_j^{\mathbb{P}[n]}}{y}} e_i \bar{\mathbb{V}}_{ij}^{\mathbb{P}[n]}(x, y) e_j \right]_{x^{a_i-1} y^{a_j-1}}$$

where  $(a_i, a_j)$  are component of  $\mathbf{A} \in \mathbb{Z}_{\geq 1}^{F(\Gamma)}$  associated to flags  $(v_i, e)$  and  $(v_j, e)$ . The notation  $[\dots]_{x^{a_i-1} y^{a_j-1}}$  above denotes the coefficient of  $x^{a_i-1} y^{a_j-1}$  in the series expansion of the argument. The bar above  $\mathbb{U}_i$  and  $\mathbb{V}_{ij}$  means evaluation  $t = 0$ .

REMARK 12. By (4),  $\text{Cont}_{\Gamma}^{\mathbf{A}}(v)$  and  $\text{Cont}_{\Gamma}^{\mathbf{A}}(e)$  can be considered as elements in  $\mathbb{R}[[q, x^{-1}, y^{-1}]]$ .

Finally we obtain the following result.

PROPOSITION 13. For  $\mathbb{P}[n]$ , We have

$$F^g|_{\lambda_i=1, \bar{\lambda}_i=-1} = \sum_{\lambda_{i,j}=\pm 1} \sum_{\Gamma \in \mathbb{G}_{g,0}^{\text{ord}}} \frac{1}{\text{Aut}(\Gamma)} \sum_{\mathbf{A} \in \mathbb{Z}_{\geq 1}^{F(\Gamma)}} \prod_v \prod_e \text{Cont}_{\Gamma}^{\mathbf{A}}(v) \text{Cont}_{\Gamma}^{\mathbf{A}}(e),$$

where  $\text{Cont}_{\Gamma}^{\mathbf{A}}(v), \text{Cont}_{\Gamma}^{\mathbf{A}}(e)$  are defined as above.

## 4. Proof of main theorem

### 4.1. Overview

Using (3) and Prop 4, we can explicitly calculate the  $\mathbb{S}$ -operator of  $\mathbb{P}[n]$ .

PROPOSITION 14. For  $\{i_1, i_2, \dots, i_n\} = \{0, 1\}$ ,

$$\mathbb{S}^{\mathbb{P}[n]}(H_1^{i_1}, H_2^{i_2}, \dots, H_n^{i_n}) = \left(z \frac{d}{dt_1}\right)^{i_1} \left(z \frac{d}{dt_2}\right)^{i_2} \dots \left(z \frac{d}{dt_n}\right)^{i_n} \mathbb{I}^{\mathbb{P}[n]}.$$

If we restrict  $\mathbb{S}$ -operator to the fixed points, it admits Birkhoff factorizations, see [2].

DEFINITION 15. For fixed point  $p \in (\mathbb{P}[n])^T$ , let us define  $\mathbb{R}_{p,k}^{i_1, i_2, \dots, i_n}$  by following;

$$\begin{aligned} \mathbb{S}^{\mathbb{P}[n]}(H_1^{i_1} H_2^{i_2} \dots H_n^{i_n})|_p = \\ e^{\frac{U_p^{\mathbb{P}[n]}}{z}} \lambda_{1,p}^{\delta_{1i_1}} \lambda_{2,p}^{\delta_{1i_2}} \dots \lambda_{n,p}^{\delta_{1i_n}} \left( \sum_k \mathbb{R}_{p,k}^{i_1, i_2, \dots, i_n} z^k \right) \in \mathbb{R}[[z, z^{-1}, \mathfrak{q}, \mathfrak{t}]](\lambda). \end{aligned}$$

Here  $\lambda_{k,p}$  is the weight of  $k$ -th  $\mathbb{C}^*$  at  $p$ . We consider  $\lambda_{k,p}$  as an element of  $\mathbb{R}$  by identifying  $\lambda_{k,p}$  with  $\lambda_{k,1}$ .

PROPOSITION 16.  $\mathbb{R}_{p,k}^{i_1, i_2, \dots, i_n}$  has type  $(0, k)$ .

*Proof.* Applying Prop 14 to (3), the proof follows from following lemma.  $\square$

LEMMA 17. We have following assymtotic expansion of local  $\mathbb{S}$ -operators of  $\mathbb{P}^1$  for fixed point  $p \in (\mathbb{P}^1)^T$ .

$$\begin{aligned} \mathbb{S}^{\mathbb{P}^1}(1)|_p &= e^{\frac{U_p^{\mathbb{P}^1}}{z}} \left( \sum_k \mathbb{Q}_{0k} \lambda_p^{\delta(k)} z^k \right), \\ \mathbb{S}^{\mathbb{P}^1}(H)|_p &= e^{\frac{U_p^{\mathbb{P}^1}}{z}} \lambda_p \left( \sum_k \mathbb{Q}_{1k} \lambda_p^{\delta(k)} z^k \right) \end{aligned}$$

where  $\delta(k) = 0$  if  $k$  is even,  $\delta(k) = 1$  if  $k$  is odd. Here  $\mathbb{Q}_{ik}$  are constant with respect to  $\lambda_p$ .

REMARK 18. We can calculate  $\mathbb{Q}_{ik}$  in closed form using Picard-Fuchs equation of  $I$ -function. Since this is not needed in our paper, we leave the details to the readers.

PROPOSITION 19. Let  $\Gamma \in \mathbb{G}_{g,0}^{\text{ord}}$  be a decorated graph.

1.  $\text{Cont}_{\Gamma}^{\mathbf{A}}(v)$  has type  $(\mathfrak{g}(v) - 1, 3\mathfrak{g}(v) - 3 + N_v(\mathbf{A}))$ , where  $N_v(\mathbf{A}) = \sum_{i=1}^k (2 - a_i)$ . Here  $(a_1, \dots, a_k)$  are components of  $\mathbf{A}$  associated to  $v$ .
2.  $\text{Cont}_{\Gamma}$  has type  $(g - 1, 3g - 3)$ .

*Proof.* For the first parts, if we expand

$$\begin{aligned} \text{Cont}_{\Gamma}^{\mathbf{A}}(v) &= [\psi^{a_1-1}, \psi^{a_2-1}, \dots, \psi^{a_k-1} | \mathbf{H}_{\mathfrak{g}(v)}^v]_{\mathfrak{g}(v),k}^{0+,v} \\ &= \sum_{l_1, l_2, \dots, l_{\mathfrak{g}(v)} \geq 0} b_{l_1, l_2, \dots, l_{\mathfrak{g}(v)}} [\psi^{a_1-1}, \psi^{a_2-1}, \dots, \psi^{a_k-1} | c_1^{l_1} c_2^{l_2} \dots c_{\mathfrak{g}(v)}^{l_{\mathfrak{g}(v)}}]_{\mathfrak{g}(v),k}^{0+,v} \end{aligned}$$

where  $a_1, a_2, \dots, a_k$  are components of  $\mathbf{A}$  associated to  $v$ . Now we can check

- $b_{l_1, l_2, \dots, l_{\mathbf{g}(v)}}$  has type  $(\mathbf{g}(v) - 1, \sum_{i=1}^{\mathbf{g}(v)} l_i)$ ,
- $[\psi^{a_1-1}, \dots, \psi^{a_k-1} | c_1^{l_1} c_2^{l_2} \dots c_g^{l_g}]_{\mathbf{g}(v), k}^v$  has type  $(0, 3\mathbf{g}(v)_v - 3 - \sum_{i=1}^{\mathbf{g}(v)} l_i + \sum_{i=1}^k (2 - a_i))$ .

The proof of first part follows from Lemma 7.

For the second parts, let  $e$  be an edge connecting  $v_i$  and  $v_j$ . Then, by Prop 16 and (2), we can check that

$$\begin{aligned} \text{Cont}_{\Gamma}^{\mathbf{A}}(e) &= \left[ (-1)^{k+l} e^{-\frac{\bar{U}_i^{\mathbb{P}[n]}}{x} - \frac{\bar{U}_j^{\mathbb{P}[n]}}{y}} e_i \bar{\mathbb{V}}_{ij}^{\mathbb{P}[n]}(x, y) e_j \right]_{x^{a_i-1} y^{a_j-1}} \\ &= \left[ (-1)^{k+l} e^{-\frac{\bar{U}_i^{\mathbb{P}[n]}}{x} - \frac{\bar{U}_j^{\mathbb{P}[n]}}{y}} \sum_k \bar{\mathbb{S}}^{\mathbb{P}[n]}(\phi_k)|_{v_i, z=x} \bar{\mathbb{S}}^{\mathbb{P}[n]}(\phi_k)|_{v_j, z=y} \right]_{x^k y^{l-1} - x^{k+1} y^{l-2} + \dots + (-1)^{k-1} x^{k+l-1}} \end{aligned}$$

has type  $(1, 1 + (a_i - 1) + (a_j - 1))$ . Here  $(a_i, a_j)$  are components of  $\mathbf{A}$  associated to flags  $(v_i, e)$  and  $(v_j, e)$ . By multiplying all contributions of  $\Gamma$ , the proof of second part follows from Lemma 7.  $\square$

#### 4.2. Proof of Theorem 3

Consider the following decomposition of  $F^g$ ,

$$F^g|_{\lambda_i=1, \bar{\lambda}_i=-1} = \sum_{\lambda_{i,j}=\pm 1} \sum_{\Gamma \in \mathbf{G}_{g,0}^{\text{ord}}} \text{Cont}_{\Gamma}.$$

Each  $\text{Cont}_{\Gamma}$  has type  $(g-1, 3g-3)$  by Prop 19. Using Lemma 8, we obtain the following result for  $n > 3g-3$  and even  $g$ ,

$$\sum_{\lambda_{i,j}=\pm 1} \text{Cont}_{\Gamma} = 0.$$

Therefore we conclude

$$F^g|_{\lambda_i=1, \bar{\lambda}_i=-1} = \sum_{\lambda_{i,j}=\pm 1} \sum_{\Gamma \in \mathbf{G}_{g,0}^{\text{ord}}} \text{Cont}_{\Gamma} = 0,$$

for  $n > 3g-3$  and even  $g$ . The  $\mathbf{q}$ -coefficients in  $F^g$  of degree  $\mathbf{d}$  with virtual dimension 0 are independent of  $\lambda$ . Theorem 3 is immediate consequence.

## 5. Gromov-Witten invariants with insertions

Here we extend the main result to the Gromov-Witten invariants with insertions. Let  $\pi$  be morphism to the moduli space of stable curves determined by the domain,

$$\pi : Q_{g,n}(\mathbb{P}[n], \mathbf{d}) \rightarrow \overline{M}_{g,n}.$$

For  $\mathbf{c}_k = (c_{1,k}, c_{2,k}, \dots, c_{m,k}) \in \mathbb{Z}_2^m$ , Gromov-Witten invariants is defined by

$$GW_{g,m,\mathbf{d}}^{\mathbb{P}[n]}[\tau_{k_1} \mathbf{c}_1, \dots, \tau_{k_m} \mathbf{c}_m] := \int_{[\overline{M}_{g,m}(\mathbb{P}[n], \mathbf{d})]^{vir}} \prod_{i=1}^m \pi^*(\psi_i)^{k_i} \text{ev}_i^*(\otimes_{j=1}^n H_j^{c_{j,i}}).$$

Define the equivariant quasimap series

$$F_m^g[\tau_{k_1} \mathbf{c}_1, \dots, \tau_{k_m} \mathbf{c}_m] := \sum_{\mathbf{d}=0}^{\infty} \mathbf{q}^{\mathbf{d}} \int_{[Q_{g,m}(\mathbb{P}[n], \mathbf{d})]^{vir}} \prod_{i=1}^m \pi^*(\psi_i)^{k_i} \text{ev}_i^*(\otimes_{j=1}^n H_j^{c_{j,i}}).$$

The methods of proof of Theorem 3 immediately yield the extended results for the equivariant quasimap series with insertions.

**THEOREM 20.** *We have*

$$F_m^g[\tau_{k_1} \mathbf{c}_1, \dots, \tau_{k_m} \mathbf{c}_m] \Big|_{\lambda_i=1, \bar{\lambda}_i=-1} = 0,$$

*if the following conditions hold.*

- (i)  $(g - 1 + \sum_{k=1}^m c_{ik})$  is odd for  $1 \leq i \leq n$ ,
- (ii)  $3g - 3 + m < n$ .

As a consequence, we can extend the Theorem 1 to the following result.

**THEOREM 21.** *We have*

$$GW_{g,m,\mathbf{d}}^{\mathbb{P}[n]}[\tau_{k_1} \mathbf{c}_1, \dots, \tau_{k_m} \mathbf{c}_m] = 0,$$

*if the following conditions hold.*

- (i)  $(g - 1 + \sum_{k=1}^m c_{ik})$  is odd for  $1 \leq i \leq n$ ,
- (ii)  $3g - 3 + m < n$ .

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