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ON MIXED PRESSURE-VELOCITY REGULARITY CRITERIA FOR THE 3D MICROPOLAR EQUATIONS IN LORENTZ SPACES

Jae-Myoung Kim* and Jaewoo Kim**

Abstract. In present paper, inspired by the recently paper [1], we give the mixed pressure-velocity regular criteria in view of Lorentz spaces for weak solutions to 3D micropolar equations in a half space. Precisely, if

$$\frac{P}{(e^{-|x|^2}+|u|)^\theta} \in L^p(0,T;L^{q,\infty}(\mathbb{R}^3_+)) , \quad p,q<\infty, \tag{0.1}$$

and

$$\frac{2}{p}+\frac{3}{q}=2-\theta, \quad 0\le\theta\le 1, \tag{0.2}$$

then (u,w) is regular on $(0,T]$.

1. Introduction

This paper is concerned about regularity problem of the weak solutions to the micropolar fluid equations in $\mathbb{R}^3_+$, which are described by

$$\begin{cases} \partial_t u-(\mu+\chi)\Delta u+(u\cdot\nabla)u+\nabla P=2\chi\,\mathrm{rot}\,w, \\ \partial_t w-\kappa\Delta w+(u\cdot\nabla)w+\gamma\nabla\nabla\cdot w+4\chi w=2\chi\,\mathrm{rot}\,u, \ \Omega_T:=\mathbb{R}^3_+\times(0,\,T) \\ \mathrm{div}\ u=0. \end{cases} \tag{1.1}$$

where $u=u(x,t):Q_T\to\mathbb{R}^3$, $w=w(x,t):Q_T\to\mathbb{R}^3$ and $P=P(x,t):Q_T\to\mathbb{R}$ denote the fluid velocity, the angular velocity of the fluid particles and pressure fields, respectively. The constant μ is the

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Correspondence should be addressed to Jaewoo Kim, baseballer@skku.edu.
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kinematic viscosity, χ is the vortex viscosity, κ and γ are spin viscosities. We see the initial and boundary value problem of (1.1), that is,

$$u(x,0)=u_0(x) \quad \text{and} \quad w(x,0)=w_0(x), \quad x\in\mathbb{R}^3_+, \tag{1.2}$$

subject to the boundary conditions

$$u\cdot\nu=0,\ (\nabla\times u)\times\nu=0, \quad \text{and} \quad w=0, \qquad \text{on} \ \ \partial\mathbb{R}^3_+, \tag{1.3}$$

where $\nu:=(0,0,-1)$ is the outward unit normal vector along $\partial\mathbb{R}^3_+$.

The system (1.1) firstly introduced by Eringen [4], represents fluids consisting essentially of randomly oriented particles suspended in a viscous medium if the deformation of fluid particles is ignored. The model can explain a phenomenon appearing in a great amount of complex fluids flows such as suspensions, lubricants and liquid crystals (see e.g. [8] and the references therein). For the existence results for (1.1)–(1.3), we refer to [5], [7], [8]. The uniqueness of weak solutions have been left the question open. For $\Omega\subset\mathbb{R}^3$, it is known that any weak solution becomes unique and smooth in $Q_T:=\Omega\times(0,T)$, provided that the following invariant condition is satisfied:

$$u\in L^p(0,T;L^q(\Omega))\,, \quad \frac{2}{p}+\frac{3}{q}=1\,, \quad q>3\,, \tag{1.4}$$

then (u,w) is a strong solution, which means that (u,w) belong to the H^1-class, that is, $L^\infty(0,T;\,H^1(\Omega)\,)\cap L^2(0,T;\,H^2(\Omega))$ (see e.g. [3] and the references therein).

In view of the regularity conditions in Lorentz space, Yuan [11] proved that a weak solution $(u,\,w)$ for the equations (1.1) becomes smooth under the scaling invariant conditions, namely, if one of the following conditions for u or P is satisfied:

(a) $u(x,t)\in L^q((0,T);L^{p,\infty}(\mathbb{R}^3))$ for $\frac{2}{q}+\frac{3}{p}\le 1$ with $3<p\le\infty$;

(b) $\nabla u(x,t)\in L^q((0,T);L^{p,\infty}(\mathbb{R}^3))$ for $\frac{2}{q}+\frac{3}{p}\le 2$ with $\frac{3}{2}<p\le\infty$;

(c) $P(x,t)\in L^q((0,T);L^{p,\infty}(\mathbb{R}^3))$ for $\frac{2}{q}+\frac{3}{p}\le 2$ with $\frac{3}{2}<p\le\infty$;

(d) $\nabla P(x,t)\in L^q((0,T);L^{p,\infty}(\mathbb{R}^3))$ for $\frac{2}{q}+\frac{3}{p}\le 3$ with $1<p\le\infty$.

After that, Loayza and Rojas-Medar [6] studt regularity results for weak solutions to the micropolar fluid equations in a 3D bounded domain. Precisely, they shown that the weak solution (u,w) is strong on $[0,T]$ if either $u\in L^s(0,T;L^{r,\infty}(\Omega))$ or $\|u\|_{L^{s,\infty}(0,T;L^{r,\infty}(\Omega))}$ is bounded from above with the relation $\frac{3}{r}+\frac{2}{s}=1$ and $r>3$. Recently, Beirão da Veiga and Yang [1] obtained generalized regular criteria for the mixed pressure-velocity in Lorentz spaces for Leray-Hopf weak solutions to 3D Navier-Stokes equations, that is $w=0$ in the equations (1.1) in a bounded domain based on the relation $|P| \simeq |u|^2$. More speaking, they shown that if

$$\frac{P}{(e^{-|x|^2}+|u|)^\theta} \in L^p(0,T;L^{q,\infty}(\Omega))\,, \quad p,q<\infty, \tag{1.5}$$

and

$$\frac{2}{p}+\frac{3}{q}=2-\theta, \quad 0\le\theta\le 1, \tag{1.6}$$

then u is regular on $(0,T]$.

In this respect, inspired by [1], our result is stated as follows.

THEOREM 1.1. *Let* (u,w) *be a weak solution to* (1.1)–(1.3) *with divergence-free initial data* $u_0, w_0 \in L^2(\mathbb{R}^3_+)\cap L^4(\mathbb{R}^3_+)$. *If the condition* (1.5) *with the relation* (1.6) *is satisfied, then* (u,w) *is regular on* $(0,T]\times\mathbb{R}^3_+$.

REMARK 1.2. As mentioned in [1], in case of $\mathbb{T}^3$, the assumption (1.5) is equivalent to

$$\frac{P}{(1+|u|)^\theta} \in L^p(0,T;L^{q,\infty}(\mathbb{T}^3))\,.$$

REMARK 1.3. We can know that Yuan's result in [11] holds for half space $\mathbb{R}^3_+$ due to representation of pressure term(Lemma 2.3 below). Since the proof is same to that in [11], we only mention it.

2. Notations and some auxiliary lemmas

In this section, we introduce the definitions and lemmas used throughout this paper. We also recall some lemmas which are useful to our analysis. For $p\in[1,\infty]$, the notation $L^p(0,T;X)$ stands for the set of measurable functions $f(x,t)$ on the interval $(0,T)$ with values in X and $\|f(\cdot,t)\|_X$ belonging to $L^p(0,T)$.
Next, we remind the definition of a weak solution.

DEFINITION 2.1. (**Weak Solutions**) Let $u_0(x), w_0(x) \in L^2_\sigma(\mathbb{R}^3_+)$. A measurable function $(u(x,t), w(x,t))$ is called a weak solution to the micropolar equations (1.1)–(1.3) on $[0,T)$, if:

(1)

$$u(x,t)\in L^\infty([0,T);L^2_\sigma(\mathbb{R}^3_+))\cap L^2([0,T);H^1_\sigma(\mathbb{R}^3_+)),$$

and

$$w(x,t)\in L^\infty([0,T);L^2(\mathbb{R}^3_+))\cap L^2([0,T);H^1(\mathbb{R}^3_+));$$

(2) $\operatorname{div} u = \operatorname{div} b = 0$ in the sense of distribution;
(3)

$$\int_0^T \{-(u,\partial_\tau\varphi)+(\mu+\chi)(\nabla u,\nabla\varphi)-(u\cdot\nabla\varphi,u)+\chi(\nabla\times\varphi,w)\}\mathrm{d}\tau = (u_0,\varphi(0)),$$

$$\int_0^T \{-(w,\partial_\tau\phi)+\kappa(\nabla w,\nabla\phi)+\gamma(divw,div\phi)+2\chi(w,\phi)$$

$$\int_0^T -(u\cdot\nabla\phi,w)+\chi(\nabla\times\phi,u)\}\mathrm{d}\tau = (w_0,\phi(0)),$$

for any $\varphi(x,t)\in H^1[(0,T);H^1_\sigma(\mathbb{R}^3_+)$ and $\phi(x,t)\in H^1([0,T);H^1(\mathbb{R}^3_+)$ with $\varphi(T)=0$ and $\phi(T)=0$.

Next, we see some facts on Lorentz spaces. For $p,q\in[1,\infty]$, we define

$$\|f\|_{L^{p,q}(\mathbb{R}^3_+)} = \begin{cases} \Big(p\int_0^\infty \alpha^q|\{x\in\mathbb{R}^3_+ : |f(x)|>\alpha\}|^{\frac{q}{p}}\frac{d\alpha}{\alpha}\Big)^{\frac{1}{q}}, & q<\infty,\\ \sup\limits_{\alpha>0}\alpha|\{x\in\mathbb{R}^3_+ : |f(x)|>\alpha\}|^{\frac{1}{p}}, & q=\infty.\end{cases}$$

Furthermore,

$$L^{p,q}(\mathbb{R}^3_+) = \big\{f : f \text{ is a measurable function on } \mathbb{R}^3_+ \text{ and } \|f\|_{L^{p,q}(\mathbb{R}^3_+)}<\infty\big\}.$$

Followed in [10], through the real interpolation methods, Lorentz space $L^{p,q}(\mathbb{R}^3_+)$ is defined by

$$L^{p,q}(\mathbb{R}^3_+) = (L^{p_1}(\mathbb{R}^3_+),\, L^{p_2}(\mathbb{R}^3_+))_{\alpha,q}, \tag{2.1}$$

with

$$\frac{1}{p}=\frac{1-\alpha}{p_1}+\frac{\alpha}{p_2},\quad 1\le p_1<p<p_2\le\infty.$$

We recall the Hölder inequality in Lorentz spaces (see [9]).

Lemma 2.2. *Assume* $1\le p_1,\ p_2\le\infty$, $1\le q_1,\ q_2\le\infty$ *and* $u\in L^{p_1,q_1}(\mathbb{R}^3_+)$, $v\in L^{p_2,q_2}(\mathbb{R}^3_+)$. *Then* $uv\in L^{p_3,q_3}(\mathbb{R}^3_+)$ *with* $\frac{1}{p_3}=\frac{1}{p_1}+\frac{1}{p_2}$ *and* $\frac{1}{q_3}\le\frac{1}{q_1}+\frac{1}{q_2}$, *and the inequality*

$$\|uv\|_{L^{p_3,q_3}(\mathbb{R}^3_+)}\le C\|u\|_{L^{p_1,q_1}(\mathbb{R}^3_+)}\|v\|_{L^{p_2,q_2}(\mathbb{R}^3_+)}$$

is valid.

For the pressure quantity, we need the following pressure representation, see [2, Theorem 2.1].

LEMMA 2.3. *Suppose* (u, w, P) *are a measurable function and a distribution, respectively, satisfying* (1.1)–(1.3) *in the sense of distributions. Then* P *has the following representation; for almost all time* $t \in (0,T)$

$$P(x,t) = \frac{-\delta_{ij}}{3}u_i^* u_j^* + \frac{3}{4\pi}\int_{\mathbb{R}^3_+}\left(\frac{\partial^2}{\partial y_i \partial y_j}\frac{1}{|x-y|}\right)u_i^* u_j^*(y,t)dy$$

in the sense of distributions, where δ_{ij} *is the Kronecker delta function. Here,* $u^*(y) = u(y)$ *for* $y_3 > 0$, *and*

$$u_1^*(y,t) = u_1(y^*,t),\ u_2^*(y,t) = u_2(y^*,t),\ u_3^*(y,t) = -u_3(y^*,t),$$

for $y_3 < 0$, *and* $y^* = (y_1, y_2, -y_3)$.

Lemma 2.3 implies that

$$\|P\|_{L^\alpha(\Omega)} \le C\,\|u\|^2_{L^{2\alpha}(\Omega)}, \qquad 1 < \alpha < \infty.$$

Proof of Theorem 1.1

First, we note that from the L^2-energy inequality

$$\frac{d}{dt}\int_{\mathbb{R}^3_+}\Big(|u|^2 + |w|^2\Big)dx + 2\min\{\mu+\chi,\kappa\}\int_{\mathbb{R}^3_+}\Big(|\nabla u|^2 + |\nabla w|^2\Big)dx \le 0. \tag{2.2}$$

To prove the theorem, it is sufficient to get L^4-estimate. Taking the inner product of the first equation of (1.1) with $|u|^2u$ and the second equation of (1.1) with $|w|^2w$, respectively, and trying the integration by parts, we obtain

$$\begin{aligned}
&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^4_{L^4(\mathbb{R}^3_+)} + \|\omega\|^4_{L^4(\mathbb{R}^3_+)}) + (\mu+\chi)\int_{\mathbb{R}^3_+}|\nabla u|^2|u|^2\mathrm{d}x\\
&+\frac{1}{2}(\mu+\chi)\int_{\mathbb{R}^3_+}|\nabla|u|^2|^2\mathrm{d}x + \gamma\int_{\mathbb{R}^3_+}|\nabla\omega|^2|\omega|^2\mathrm{d}x + \frac{1}{2}\gamma\int_{\mathbb{R}^3_+}|\nabla|\omega|^2|^2\mathrm{d}x\\
&+\kappa\int_{\mathbb{R}^3_+}|div\omega|^2\mathrm{d}x + 2\chi\int_{\mathbb{R}^3_+}|\omega|^4\mathrm{d}x\\
\le& 2\int_{\mathbb{R}^3_+}|P||u|^2|\nabla u|\mathrm{d}x + 3\chi\int_{\mathbb{R}^3_+}|w||u|^2|\nabla u|\mathrm{d}x + 3\chi\int_{\mathbb{R}^3_+}|u||\omega|^2|\nabla\omega|\mathrm{d}x\\
&:= I + II + III,
\end{aligned}$$

Applying Hölder and Young's inequalities for II and III, it follows that

$$II \le \frac{1}{2}\chi\int_\Omega|\nabla u|^2|u|^2\mathrm{d}x + C(\chi)\Big(\|u\|^4_{L^4} + \|\omega\|^4_{L^4}\Big), \tag{2.3}$$

and

$$III \le \frac{1}{2}\gamma \int_\Omega |\nabla\omega|^2|\omega|^2 \mathrm{d}x + C(\gamma)\Big(\|u\|_{L^4}^4 + \|\omega\|_{L^4}^4\Big). \tag{2.4}$$

For I, borrowing the arguments in [1], we let

$$V = e^{-|x|^2} + |u|\,, \quad \tilde{P} = \frac{P}{(e^{-|x|^2} + |u|)^\theta}\,.$$

Due to the definition of V, we observe that

$$\|V^2\|_{L^2}^2 \le C(1 + \|u\|_{L^2}^2 + \||u|^2\|_{L^2}^2)$$

and

$$\|\nabla V^2\|_{L^2}^2 \le C(1 + \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla|u|^2\|_{L^2}).$$

By the interpolation and Sobolev inequalities in Lorentz spaces (2.1), we have

$$\|V^2\|_{L^{(2-\alpha)r_1,2}} \le C\|V^2\|_{L^{2,2}}^{1-\delta_1}\|V^2\|_{L^{6,2}}^{\delta_1} \le C\|V^2\|_{L^2}^{1-\delta_1}\|\nabla V^2\|_{L^2}^{\delta_1} \tag{2.5}$$

and

$$\|V^2\|_{L^{\alpha r_2,2}} \le C\|V^2\|_{L^{2,2}}^{1-\delta_2}\|V^2\|_{L^{6,2}}^{\delta_2} \le C\|V^2\|_{L^2}^{1-\delta_2}\|\nabla V^2\|_{L^2}^{\delta_2}\,, \tag{2.6}$$

where $0 < \delta_1\,, \delta_2 < 1$, and

$$\frac{1}{(2-\alpha)r_1} = \frac{1-\delta_1}{2} + \frac{\delta_1}{6}\,, \quad \frac{1}{\alpha r_2} = \frac{1-\delta_2}{2} + \frac{\delta_2}{6}\,.$$

With the aid of Lemma 2.2 and Lemma 2.3, we get

(2.7)

$$\begin{aligned}
I =& \int_{\mathbb{R}^3_+} \left(\frac{P}{(e^{-|x|^2} + |u|)^\theta}\right)^\alpha |P|^{2-\alpha}(e^{-|x|^2} + |u|)^{\alpha\theta}|u|^2 dx \\
\le& \|\tilde{P}^\alpha\|_{L^{\frac{q}{\alpha},\infty}}\|P^{2-\alpha}\|_{L^{r_1,\frac{2}{2-\alpha}}}\|V^{2\alpha}\|_{L^{r_2,\frac{2}{\alpha}}} \\
=& \|\tilde{P}\|_{L^{q,\infty}}^\alpha\|P\|_{L^{(2-\alpha)r_1,2}}^{2-\alpha}\|V^2\|_{L^{\alpha r_2,2}}^\alpha\,, \quad \frac{\alpha}{q} + \frac{1}{r_1} + \frac{1}{r_2} = 1, \\
&\le C\|\tilde{P}\|_{L^{q,\infty}}^\alpha\||v|^2\|_{L^{(2-\alpha)r_1,2}}^{2-\alpha}\|V^2\|_{L^{\alpha r_2,2}}^\alpha \\
&\le C\|\tilde{P}\|_{L^{q,\infty}}^\alpha\|V^2\|_{L^{(2-\alpha)r_1,2}}^{2-\alpha}\|V^2\|_{L^{\alpha r_2,2}}^\alpha \\
&\le C\|\tilde{P}\|_{L^{q,\infty}}^\alpha\|V^2\|_{L^2}^{(1-\delta_1)(2-\alpha)}\|\ \nabla V^2\|_{L^2}^{\delta_1(2-\alpha)}\|V^2\|_{L^2}^{(1-\delta_2)\alpha}\|\nabla V^2\|_{L^2}^{\delta_2\alpha} \\
&\le C\|\tilde{P}\|_{L^{q,\infty}}^\alpha\|V^2\|_{L^2}^{(1-\delta_1)(2-\alpha)+(1-\delta_2)\alpha}\|\nabla V^2\|_{L^2}^{\delta_1(2-\alpha)+\delta_2\alpha} \\
&\le C\|\tilde{P}\|_{L^{q,\infty}}^{\frac{2\alpha}{2-\delta_1(2-\alpha)-\delta_2\alpha}}\|V^2\|_{L^2}^2 + \frac{\mu}{4}\|\nabla V^2\|_{L^2}^2,
\end{aligned}$$

where we use the inequalities (2.5) and (2.6).

Due to the definition of V, we know from (2.7)

$$I \leq C\|\tilde{P}\|_{L^{q,\infty}}^{\frac{2\alpha}{2-\delta_1(2-\alpha)-\delta_2\alpha}}(1+\|u\|_{L^2}^2+\|u\|_{L^4}^4)$$

$$+C(1+\|u\|_{L^2}^2+\|\nabla u\|_{L^2}^2)+\frac{1}{4}\mu\|\nabla|u|^2\|_{L^2}^2. \tag{2.8}$$

Combining (2.3), (2.4) and (2.8), we finally obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{L^4}^4+\|\omega\|_{L^4}^4)+(\mu+\chi)\int_{\mathbb{R}_+^3}|\nabla u|^2|u|^2\mathrm{d}x+\frac{1}{2}(\mu+\chi)\int_{\mathbb{R}_+^3}|\nabla|u|^2|^2\mathrm{d}x$$

$$+\gamma\int_{\mathbb{R}_+^3}|\nabla\omega|^2|\omega|^2\mathrm{d}x+\frac{1}{2}\gamma\int_{\mathbb{R}_+^3}|\nabla|\omega|^2|^2\mathrm{d}x+4\kappa\int_{\mathbb{R}_+^3}|div\omega|^2\mathrm{d}x+8\chi\int_{\mathbb{R}_+^3}|\omega|^4\mathrm{d}x$$

$$\leq C\Big(\|u\|_{L^4}^4+\|\omega\|_{L^4}^4\Big)+\|\tilde{P}\|_{L^{q,\infty}}^{\frac{2\alpha}{2-\delta_1(2-\alpha)-\delta_2\alpha}}(1+\|u\|_{L^2}^2+\|u\|_{L^4}^4)$$

$$+C(1+\|u\|_{L^2}^2+\|\nabla u\|_{L^2}^2)$$

$$\leq C\Big(\|u\|_{L^4}^4+\|\omega\|_{L^4}^4\Big)+\|\tilde{P}\|_{L^{q,\infty}}^{\frac{2\alpha}{2-\delta_1(2-\alpha)-\delta_2\alpha}}(1+\|u\|_{L^4}^4)+C(1+\|\nabla u\|_{L^2}^2),$$

where we use $u\in L^\infty(0,T;L^2(\mathbb{R}_+^3))$ from (2.2). Using Gronwall's lemma and the criteria (1.4), we obtain that (u,w) is smooth in $\mathbb{R}_+^3\times[0,T]$, provided that

$$\tilde{P}\in L^{\frac{2\alpha}{2-\delta_1(2-\alpha)-\delta_2\alpha}}(0,T;L^{q,\infty}(\mathbb{R}_+^3)),$$

which is completed the proof.

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Department of Mathematics Education
Andong National University
Andong 36729, Republic of Korea
E-mail: jmkim02@anu.ac.kr

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Department of Mathematics
Yonsei University
Seoul 03722, Republic of Korea
E-mail: baseballer@skku.edu