JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **34**, No. 1, February 2021
http://dx.doi.org/10.14403/jcms.2021.34.1.27

EXISTENCE OF POSITIVE T-PERIODIC SOLUTIONS OF RATIO-DEPENDENT PREDATOR-PREY SYSTEMS

Kimun Ryu

Abstract. We study the existence of positive T-periodic solutions of ratio-dependent predator-prey systems with time periodic and spatially dependent coefficients. The fixed point theorem by H. Amann is used to obtain necessary and sufficient conditions for the existence of positive T-periodic solutions.

1. Introduction

In this paper, we investigate the existence of positive T-periodic solutions of the following predator-prey reaction-diffusion systems with ratio-dependent functional response

(1.1)

$$\begin{cases} u_t - d_1(x,t)\Delta u = u(a(x,t) - u - \frac{c_1(x,t)v}{u+m(x,t)v}) & \\ v_t - d_2(x,t)\Delta v = v(b(x,t) - e(x,t)v + \frac{c_2(x,t)u}{u+m(x,t)v}) & \text{in } \Omega \times [0,\infty), \\ B_1 u = 0 & \\ B_2 v = 0 & \text{on } \partial\Omega \times [0,\infty), \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, d_i, a, b, c_i, e, m are functions of $(x,t) \in \Omega \times [0,\infty)$ and are T-periodic in the value t for some $T > 0$, $B_i w = \kappa_i \frac{\partial w}{\partial \nu} + \tau_i w$ for nonnegative constants κ_i, τ_i with $\kappa_i^2 + \tau_i^2 \neq 0$ and the outward unit normal $\frac{\partial}{\partial \nu}$ to $\partial\Omega$. Furthermore, d_i, a, c_i, m are positive functions and b is a nonzero function which may change sign. Here, u and v represent the population densities of prey and predator, respectively.

In (1.1), note that $(u,v) = (0,0)$ is a singular point. But since $\lim_{(u,v)\to(0,0)} u(a(x,t) - u - \frac{c_1(x,t)v}{u+m(x,t)v}) = 0$ and $\lim_{(u,v)\to(0,0)} v(b(x,t) -$

Received January 07, 2021; Accepted January 11, 2021.
2010 Mathematics Subject Classification: 35J60.
Key words and phrases: T-periodic positive solution, ratio-dependent, predator-prey.

$e(x,t)v + \frac{c_2(x,t)u}{u+m(x,t)v}) = 0$, we can extend the domain of $u(a(x,t) - u - \frac{c_1(x,t)v}{u+m(x,t)v})$ and $v(b(x,t) - e(x,t)v + \frac{c_2(x,t)u}{u+m(x,t)v})$ to $\{(u,v) : u \geq 0, v \geq 0\}$ by defining $\frac{uv}{u+m(x,t)v} = 0$ at $(u,v) = (0,0)$ so that $(0,0)$ becomes a *trivial solution* of (1.1).

The populations usually live in a temporally periodic environment with spatial variations. In particular, in order to introduce seasonal variations or day-night cycles in the ratio-dependent predator-prey interaction, we assumed that the given coefficient functions in (1.1) are T-periodic with respect to t. Optimal control systems for reaction-diffusion models are actually used agriculture and environmental problems. Because the growth rates of life species change seasonally, it is necessary to study such models whose coefficients are T-periodic. Traditionally, the periodic reaction-diffusion systems have been intensively studied concerning the existence of periodic positive solutions under various boundary conditions. (See [1, 3, 4, 5, 6, 8, 9].) In those articles, the authors applied the fixed point index theory to obtain the desired results and in this application, the principal eigenvalue of the linearized periodic-parabolic problem plays an important role. Especially, in [7], the authors study the existence and asymptotic behavior of positive T-periodic solutions of diffusive Hassell-Varley type predator-prey systems with time periodic and spatially dependent coefficients. In our study, we study the existence of T-periodic positive solutions to (1.1) using the fixed point theorem by H. Amann([2]).

This paper is organized as follows. In Section 2, we present some well-known results which are useful for the latter assertions. In Section 3, we investigate the necessary and sufficient conditions for the existence of T-periodic positive solutions of (1.1).

2. Preliminaries

In this section, we present some well-known results which are useful for the latter assertions.

For $\alpha \in (0,1)$, define

$$F = \{w \in C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega} \times \mathbb{R}) : \ w \text{ is } T\text{-periodic in } t\},$$

$$F_B = \{w \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega} \times \mathbb{R}) : \ Bw = 0 \text{ on } \partial\Omega \times \mathbb{R},\ w \text{ is } T\text{-periodic in } t\},$$

where $Bw = \kappa\frac{\partial w}{\partial \nu} + \tau w$ for nonnegative constants κ, τ with $\kappa^2 + \tau^2 \neq 0$. We denote their usual positive cones for F, F_B by P_F, P_{F_B}, respectively.

For $d, q \in F$ with $d > 0$ in $\overline{\Omega} \times \mathbb{R}$, it is well-known that the T-periodic parabolic eigenvalue problem

$$(2.1) \qquad \begin{cases} u_t - d(x,t)\Delta u + q(x,t)u = \lambda u & \text{in } \Omega \times [0,\infty), \\ Bu = 0 & \text{on } \partial\Omega \times [0,\infty), \\ u \text{ is } T\text{-periodic in } t \end{cases}$$

has the principal eigenvalue and eigenfunction. Denote this principal eigenvalue of (2.1) by $\lambda_1^B(d,q)$ throughout this paper.

The following lemma gives the monotonicity and continuity properties of the eigenvalue problem (2.1). For more details, one can refer [6].

Lemma 2.1. *(i) If $q_1 \le q_2$ and $q_1 \not\equiv q_2$, then $\lambda_1^B(d,q_1) < \lambda_1^B(d,q_2)$.*
(ii) If $q_n \to q \in C(\overline{\Omega} \times [0,T])$, then $\lambda_1^B(d,q_n) \to \lambda_1^B(d,q)$ as $n \to \infty$.

Let $Lu = u_t - d\Delta u + Mu$, where $d \in F$ with $d > 0$ in $\overline{\Omega} \times \mathbb{R}$ and $M > \sup\{|q(x,t)| : (x,t) \in \overline{\Omega} \times [0,T]\}$. Then one can see that $L : F_B \to F$ has a compact strongly positive operator $L^{-1} : F \to F_B$, and so $L^{-1} : F_B \to F_B$ is a compact strongly positive operator since F imbeds compactly into F_B. For this operator L, we can find the following lemma in [5].

Lemma 2.2. *Let $u \in F_B$ with $0 \not\equiv u \ge 0$.*
(i) If $0 \not\equiv Lu \ge 0$, then $\lambda_1^B(d,q) > 0$.
(ii) If $0 \not\equiv Lu \le 0$, then $\lambda_1^B(d,q) < 0$.
(iii) If $Lu \equiv 0$, then $\lambda_1^B(d,q) = 0$.

Denote $r(L)$ by the spectral radius of the linear operator L.

Lemma 2.3. *If $\lambda_1^B(d,q) < 0$, then $r(L^{-1}(M-q)) > 1$.*

Proof. See Lemma 2.1 in [4]. □

Now consider the initial-boundary value problem

$$(2.2) \qquad \begin{cases} u_t - d(x,t)\Delta u = uf(x,t,u) & \text{in } \Omega \times (0,\infty), \\ Bu = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $f(x,t,u) : \overline{\Omega} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a continuous which satisfies the following hypotheses

(H1) f is $C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega} \times [0,T])$ in (x,t) uniformly for u in bounded subsets of $\mathbb{R}$,

(H2) f is C^1-function in u,

(H3) f is T-periodic in t.

Definition 2.4. The function $\underline{u} \in C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{\Omega} \times [0,T]) \cap C^{2,1}(\overline{\Omega} \times (0,T])$ $(0 < \alpha \leq 1)$ is a *lower solution* for the problem (2.2) if

$$\tag{2.3} \begin{cases} \underline{u}_t - d(x,t)\Delta \underline{u} \leq \underline{u} f(x,t,\underline{u}) & \text{in } \Omega \times (0,T], \\ B\underline{u} \leq 0 & \text{on } \partial\Omega \times (0,T], \\ \underline{u}(x,0) \leq \underline{u}(x,T) & \text{in } \Omega. \end{cases}$$

Similarly, an *upper solution* $\overline{u}$ is defined by reversing the inequality signs. The lower and upper solutions $\underline{u}$, $\overline{u}$ are called *$B-$related* if there exists $u^{\#} \in C^{2+\alpha}(\overline{\Omega})$ with $Bu^{\#} = 0$ such that $\underline{u} \leq u^{\#} \leq \overline{u}$.

Lemma 2.5. *Let $\underline{u}$ and $\overline{u}$ be $B-$related lower and upper solutions of the problem (2.2). Then the relation $\underline{u}(x,t) \leq \overline{u}(x,t)$ holds in $\overline{\Omega} \times [0,T]$. Moreover, either $\underline{u}(x,t) < \overline{u}(x,t)$ or $\underline{u}(x,t) \equiv \overline{u}(x,t)$ in $\Omega \times (0,T]$.*

Proof. Let $w = \overline{u} - \underline{u}$, then w satisfies the relation

$$\begin{cases} w_t - d(x,t)\Delta w \geq \overline{u} f(x,t,\overline{u}) - \underline{u} f(x,t,\underline{u}) & \text{in } \Omega \times (0,T], \\ Bw \geq 0 & \text{on } \partial\Omega \times (0,T], \\ w(x,0) \geq 0 & \text{in } \Omega. \end{cases}$$

Using the mean-value theorem,

$$\overline{u} f(x,t,\overline{u}) - \underline{u} f(x,t,\underline{u}) = \frac{\partial}{\partial u}(u f(x,t,u))|_{u=\xi} \cdot (\overline{u} - \underline{u}),$$

where $\xi = \xi(x,t)$ is an intermediate value between $\underline{u}$ and $\overline{u}$. Hence we have

$$w_t - d(x,t)\Delta w + c(x,t)w \geq 0 \text{ in } \Omega \times (0,T],$$

where $c(x,t) = -\frac{\partial}{\partial u}(u f(x,t,u))|_{u=\xi}$. The remaining part of the proof follows from Lemma 2.1 in [9]. □

Theorem 2.6. *If $\underline{u}$ and $\overline{u}$ are $B-$related lower and upper solutions with $\underline{u} < \overline{u}$ for (2.2), then there exists a T-periodic solution u with $\underline{u} < u < \overline{u}$. Moreover, this T-periodic solution u is unique if $\frac{\partial}{\partial \xi}(f(x,t,\xi))$ is a bounded below function in $\Omega \times (0,T] \times \mathbb{R}^+$.*

Proof. The existence of such T-periodic solution follows from Theorem 22.3 in [6]. We prove only the uniqueness of T-periodic solution. Let u_1 and u_2 be T-periodic solutions of (2.2), then u_1 and u_2 are not only lower solutions but also upper solutions of the problem (2.2). Using Lemma 2.5, one can easily see that $u_1 \equiv u_2$. □

Let E be a Banach space and let A be a strongly positive nonlinear compact operator on E such that $A(0) = 0$.

LEMMA 2.7. *Assume $A'(0)$ exists with $r(A'(0)) > 1$ where $A'(0)$ is a Fréchet derivative of F at $u = 0$. If the solution to the equation $u = \theta Au$ has an a priori bound for all $\theta \in (0, 1]$, then A has a positive fixed point u such that $Au = u$ in the positive cone of E.*

Proof. See the proof of Theorem 13.2 in [2]. □

3. Existence of periodic positive solutions

In this section, we investigate the necessary and sufficient conditions for the existence of periodic positive solutions of (1.1).

To begin with, we consider the following first equation in (1.1)

(3.1)
$$\begin{cases} u_t - d_1(x,t)\Delta u = u(a(x,t) - u - \frac{c_1(x,t)v}{u+m(x,t)v}) & \text{in } \Omega \times [0,\infty), \\ B_1 u = 0 & \text{on } \partial\Omega \times [0,\infty), \\ u \text{ is } T\text{-periodic in } t, & \end{cases}$$

where $v \in F_{B_2}$ with $v \geq 0$ in $\Omega \times (0, T]$.

THEOREM 3.1. *(i) If $\lambda_1^{B_1}(d_1, -a) \geq 0$, then (3.1) has no positive T-periodic solution.*

(ii) If $\lambda_1^{B_1}(d_1, -a+\frac{c_1}{m}) < 0$, then (3.1) has a unique positive T-periodic solution.

Proof. (i) Suppose that u is a positive T-periodic solution of (3.1), then $\lambda_1^{B_1}(d_1, -a + u + \frac{c_1 v}{u+mv}) = 0$ by Lemma 2.2 (ii), and so $0 = \lambda_1^{B_1}(d_1, -a + u + \frac{c_1 v}{u+mv}) > \lambda_1^{B_1}(d_1, -a)$ by Lemma 2.1 (i).

(ii) In view of Theorem 2.6, it suffices to show that (3.1) has $B-$related lower and upper solutions. One can easily check that $\overline{u} = a$ is an upper solution for (3.1). To construct a positive lower solution, let $\phi(x, t)$ be the principal eigenfunction corresponding to $\lambda_1^{B_1}(d_1, -a + \frac{c_1}{m})$, that is to say,

$$\begin{cases} \phi_t - d_1\Delta\phi - (a - \frac{c_1}{m(x,t)})\phi = \lambda_1^{B_1}(d_1, -a + \frac{c_1}{m(x,t)})\phi & \text{in } \Omega \times (0,T], \\ B_1\phi = 0 & \text{on } \partial\Omega \times (0,T]. \end{cases}$$

Consider $\underline{u} = \epsilon\phi$ with $\epsilon > 0$. Then $(\epsilon\phi)_t - d_1(x,t)\Delta(\epsilon\phi) - (a(x,t) - \frac{c_1(x,t)}{m(x,t)})(\epsilon\phi) = \lambda_1^{B_1}(d_1, -a + \frac{c_1}{m})(\epsilon\phi) < 0$ in $\Omega \times (0, T]$, and so $(\epsilon\phi)_t - d_1(x,t)\Delta(\epsilon\phi) \leq (\epsilon\phi)(a(x,t) - \epsilon\phi - \frac{c_1(x,t)}{m(x,t)}) \leq (\epsilon\phi)(a(x,t) - \epsilon\phi - \frac{c_1(x,t)v}{\epsilon\phi+m(x,t)v})$ in $\Omega \times (0, T]$ for sufficiently small $\epsilon > 0$ by the continuity. Hence $\underline{u} = \epsilon\phi$ is a positive lower solution for the equation (3.1). Finally, choosing a small constant $\epsilon > 0$ such that $\epsilon\phi < a$, we can see that $\underline{u}$ and $\overline{u}$ are $B-$related lower and upper solutions for the equation (3.1). This completes the proof. $\square$

REMARK 3.2. In Theorem 3.1, note that there is a gap between the necessary condition and the sufficient condition for the existence of positive T-periodic solutions. But if (3.1) has a positive T-periodic solution in such gap range, then it must be unique by Theorem 2.6.

LEMMA 3.3. *If u is a nonnegative T-periodic solution of (3.1), then* $\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times[0,T])} < \infty$.

Proof. In the proof of Theorem 3.1 (ii), since $\overline{u} = a$ is an upper solution of (3.1), $\|u\|_\infty \le a$, where $\|\cdot\|_\infty$ is an $L^\infty-$norm. By Lemma 20.2 in [6], $\|u\|C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{\Omega}\times[T,2T]) \le M(T, \|u\|_{C(\overline{\Omega}\times[T,2T])})$ for some positive constant M, and so using the standard Schauder estimates, we can have the desired result. $\square$

Throughout this paper, denote u_a^* by the unique positive T-periodic solution of the equation

$$(3.2)\qquad \begin{cases} u_t - d_1(x,t)\Delta u = u(a(x,t) - u) & \text{in } \Omega\times(0,T], \\ B_1 u = 0 & \text{on } \partial\Omega\times(0,T], \end{cases}$$

if $\lambda_1^{B_1}(d_1, -a) < 0$. Similarly, v_b^* is denoted by the unique positive T-periodic solution of the equation

$$(3.3)\qquad \begin{cases} v_t - d_2(x,t)\Delta v = v(b(x,t) - e(x,t)v) & \text{in } \Omega\times(0,T], \\ B_2 v = 0 & \text{on } \partial\Omega\times(0,T], \end{cases}$$

if $\lambda_1^{B_2}(d_2, -b) < 0$. The existence of such so called *semi-trivial solutions* u_a^* and v_b^* follow from Theorem 3.1 (ii).

LEMMA 3.4. *Any nonnegative T-periodic solution (u, v) of (1.1) has an a priori bound.*

Proof. By Lemma 3.3, u has already an *a priori* bound. To show that v has an *a priori* bound, consider

$$\overline{v} = \max_{\overline{\Omega}\times[0,T]}\{\frac{bm - ae + \sqrt{(bm-ae)^2 + 4em(ab+ac_2)}}{2em}\}.$$

Note that $\frac{bm-ae+\sqrt{(bm-ae)^2+4em(ab+ac_2)}}{2em}$ is a positive root of $b - ev + \frac{c_2 a}{a+mv} = 0$ with respect to v, and so one can easily see that $\overline{v}$ is an upper solution of the equation

$$\begin{cases} v_t - d_2(x,t)\Delta v = v(b(x,t) - e(x,t)v + \frac{c_2(x,t)u}{u+m(x,t)v}) & \text{in } \Omega \times (0,T], \\ B_2 v = 0 & \text{on } \partial\Omega \times (0,T]. \end{cases}$$

Thus it can be shown that $\|v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times[0,T])} < \infty$ by the similar argument as in the proof of Lemma 3.3. $\square$

Throughout this paper, denote *a priori* bounds for the nonnegative T-periodic solutions u and v by Q and R, respectively, that is, $\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times[0,T])} \leq Q$ and $\|v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times[0,T])} \leq R$ for any nonnegative T-periodic solution (u,v) of (1.1).

Let $\rho = R+1$ and $\overline{P}_\rho = cl\{w \in P_{F_{B_2}} : \|w\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times[0,T])} < \rho\}$. In view of Remark 3.2, we can define the operator $S : \overline{P}_\rho \to P_{F_{B_1}}$ by

$$Sv = \begin{cases} u_v & \text{if there is a unique positive } T\text{-periodic solution } u_v \text{ of (3.1)}, \\ 0 & \text{otherwise.} \end{cases}$$

Then one can show that S is continuous and compact map by the slight modification of Theorem 2.6 in [5]. For $v \in \overline{P}_\rho$, consider the problem (3.4)

$$\begin{cases} v_t - d_2(x,t)\Delta v = v(b(x,t) - e(x,t)v + \frac{c_2(x,t)Sv}{Sv+m(x,t)v}) & \text{in } \Omega \times (0,T], \\ B_2 v = 0 & \text{on } \partial\Omega \times (0,T], \\ v \text{ is } T\text{-periodic in } t. \end{cases}$$

Define a positive compact operator $A : \overline{P}_\rho \to P_{F_{B_2}}$ by

$$A(v) = L^{-1}(v(b(x,t) - e(x,t)v + \frac{c_2(x,t)Sv}{Sv+m(x,t)v}) + Mv),$$

where $Lv = v_t - d_2(x,t)\Delta v + Mv$ and M is a sufficiently large positive constant such that $M > \max_{\overline{\Omega}\times[0,T]}\{b(x,t) + c_2(x,t)\}$. Note that v is a T-periodic fixed point of A if and only if v is a T-periodic solution of (3.4). Also A has a positive T-periodic fixed point v if and only if (Sv, v) is a nonnegative T-periodic solution of (1.1).

Now we give some necessary and sufficient conditions for the existence of positive T-periodic solutions to the system (1.1).

Theorem 3.5. *(i) If $\lambda_1^{B_1}(d_1, -a) \geq 0$, then (1.1) has no positive T-periodic solution, and in addition, if $\lambda_1^{B_2}(d_2, -b) \geq 0$, then (1.1) has no nonnegative nonzero T-periodic solution.*

(ii) Assume that $\lambda_1^{B_2}(d_2, -b) \geq 0$. Then $\lambda_1^{B_1}(d_1, -a) < 0$ and $\lambda_1^{B_2}(d_2, -b - c_2) < 0$ if and only if (1.1) has at least one positive T-periodic solution.

(iii) If $\lambda_1^{B_2}(d_2, -b) < 0$ *and* $\lambda_1^{B_1}(d_1, -a + \frac{c_1}{m}) < 0$*, then (1.1) has at least one positive* T*-periodic solution.*

(iv) If $\lambda_1^{B_2}(d_2, -b) < 0$ *and (1.1) has a positive* T*-periodic solution, then* $\lambda_1^{B_1}(d_1, -a) < 0$ *and* $\lambda_1^{B_1}(d_1, -a + \frac{c_1 v_b^*}{u_a^* + m v_b^*}) < 0$.

Proof. (i) Suppose that (u, v) is a positive T-periodic solution of (1.1), then u satisfies the equation (3.1), and so $\lambda_1^{B_1}(d_1, -a + u - \frac{c_1 v}{u+mv}) = 0$ by Lemma 2.2 (iii). Thus $\lambda_1^{B_1}(d_1, -a) < \lambda_1^{B_1}(d_1, -a + u - \frac{c_1 v}{u+mv}) = 0$ by Lemma 2.1 (i), which derives a contradiction. Next suppose (u, v) is a nonnegative nonzero T-periodic solution of (1.1). Without loss of generality, assume that $u \not\equiv 0$ and $v \equiv 0$, then u satisfies

$$\begin{cases} u_t - d_1(x,t)\Delta u = u(a(x,t) - u) & \text{in } \Omega \times [0, \infty), \\ B_1 u = 0 & \text{on } \partial\Omega \times [0, \infty). \end{cases}$$

Using Lemma 2.2 (ii) and 2.1 (i), one can derive a contradiction, that is, $\lambda_1^{B_1}(d_1, -a) < \lambda_1^{B_1}(d_1, -a + u) = 0$.

(ii) Denote $A_\theta = \theta A$ for some $\theta \in (0, 1]$. Assume that v_θ is a T-periodic fixed point of A_θ, then

$$\begin{cases} (v_\theta)_t - d_2(x,t)\Delta v_\theta = \theta v_\theta(b(x,t) - e(x,t)v_\theta & \\ \qquad\qquad + \frac{c_2(x,t)Sv_\theta}{Sv_\theta + m(x,t)v_\theta}) + M(\theta - 1)v_\theta & \text{in } \Omega \times (0, T], \\ B_2 v_\theta = 0 & \text{on } \partial\Omega \times (0, T]. \end{cases}$$

Take $\hat{v} = \max_{\overline{\Omega} \times [0,T]}\{\frac{b(x,t) + c_2(x,t)}{e(x,t)}\}$, then one can easily check that $\hat{v}$ is an upper solution of the following equation

$$\begin{cases} v_t - d_2(x,t)\Delta v = \theta v(b(x,t) - e(x,t)v & \\ \qquad\qquad + \frac{c_2(x,t)Sv_\theta}{Sv_\theta + m(x,t)v}) + M(\theta - 1)v & \text{in } \Omega \times (0, T], \\ B_2 v = 0 & \text{on } \partial\Omega \times (0, T]. \end{cases}$$

Therefore $v_\theta \le \max_{\overline{\Omega} \times [0,T]}\{\frac{b(x,t) + c_2(x,t)}{e(x,t)}\}$, and so $\|v_\theta\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Omega} \times [0,T])} < \infty$. Note that v_θ does not depend on $\theta \in (0, 1]$. Moreover, since $\lambda_1^{B_2}(d_2, -b - c_2) < 0$ from the assumption, $r(A'(0)) = r(L^{-1}(b + c_2 + M)) > 1$ by Lemma 2.3. Consequently, we can conclude that A has a positive T-periodic fixed point v by Lemma 2.7. Finally, we need $u = Sv > 0$. If $Sv \equiv 0$, then $v \equiv v_b^* > 0$ by the uniqueness of v_b^*, which is impossible since $\lambda_1^{B_2}(d_2, -b) \ge 0$. Hence (Sv, v) is a positive T-periodic solution of (1.1).

Next, let (u, v) be the positive T-periodic solution of (1.1), then we have $\lambda_1^{B_1}(d_1, -a + u + \frac{c_1 v}{u+mv}) = \lambda_1^{B_2}(d_2, -b + ev - \frac{c_2 u}{u+mv}) = 0$ by Lemma 2.2

(iii), and so it can be shown easily $\lambda_1^{B_1}(d_1, -a) < 0$, $\lambda_1^{B_2}(d_2, -b-c_2) < 0$ by Lemma 2.1 (i).

(iii) As in the proof of (ii), we can have a positive T-periodic fixed point v of A since $\lambda_1^{B_1}(d_1, -a) < \lambda_1^{B_1}(d_1, -a+\frac{c_1}{m}) < 0$ and $\lambda_1^{B_2}(d_2, -b-c_2) < \lambda_1^{B_2}(d_2, -b) < 0$. If $u = Sv \equiv 0$, then $v \equiv v_b^* > 0$ by the uniqueness of v_b^*. But since $\lambda_1^{B_1}(d_1, -a+\frac{c_1}{m}) < 0$, $Sv = Sv_b^* > 0$ by Theorem 3.1 (ii), which is a contradiction. Thus (Sv, v) is a positive T-periodic solution of (1.1).

(iv) Let (u, v) be the positive T-periodic solution of (1.1), then $\lambda_1^{B_1}(d_1, -a+u+\frac{c_1 v}{u+mv}) = 0$, and so $\lambda_1^{B_1}(d_1, -a) < 0$. Therefore the semi-trivial solutions u_a^* and v_b^* exist. Using the comparison argument, one can easily show that $u \le u_a^*$ and $v_b^* \le v$, and thus $\lambda_1^{B_1}(d_1, -a+\frac{c_1 v_b^*}{u_a^*+mv_b^*}) < \lambda_1^{B_1}(d_1, -a+u+\frac{c_1 v}{u+mv}) = 0$ by Lemma 2.1 (i). □

References

[1] S. Ahmad and A. C. Lazer, *Asymptotic behavior of solutions of periodic competition diffusion systems*, Nonlinear Anal.-Theory Methods Appl., **13** (1989), 263-283.

[2] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered banach spaces*, SIAM Rev., **18** (1976), no. 4, 620-709.

[3] K. J. Brown and P. Hess, *Positive periodic solutions of predator-prey reaction-diffusion systems*, Nonlinear Anal.-Theory Methods Appl., **16** (1991), Ser. A: Theory Methods, 1147-1158.

[4] Y. Du, *Positive periodic solutions of a competitor-competitor-mutualist model*, Differ. Integral Equ., **9** (1996), no. 5, 1043-1066.

[5] A. Ghoreishi and R. Logan, *Positive solutions to a system of periodic parabolic partial differential equations*, Differ. Integral Equ., **9** (1996), no. 3, 607-618.

[6] P. Hess, *Periodic-parabolic boundary value problems and positivity*, Pitman Research Notes in Math. Ser. 247, Longman Sci. and Tech., Harlow, Essex, UK, 1991.

[7] W. Ko and K. Ryu, *Positive periodic solutions of a Hassell-Varley type predator-prey system*, Indian J. Pure Appl. Math., **44** (2013), no. 6, 865-882.

[8] J. López-Gómez, *Positive periodic solutions of Lotka-Volterra reaction-diffusion systems*, Differ. Integral Equ., **5** (1992), 55-72.

[9] C. V. Pao, *On nonlinear parabolic and elliptic equations*, Plenum Press, New York, 1992.

Department of Mathematics Education
Cheongju University
Cheongju, Chungbuk 28503, Republic of Korea
E-mail: ryukm@cju.ac.kr