Korean J. Math. **29** (2021), No. 4, pp. 687–704
http://dx.doi.org/10.11568/kjm.2021.29.4.687

RIEMANN-LIOUVILLE FRACTIONAL VERSIONS OF HADAMARD INEQUALITY FOR STRONGLY (α, m)-CONVEX FUNCTIONS

Ghulam Farid, Saira Bano Akbar, Laxmi Rathour, and Lakshmi Narayan Mishra*

Abstract. The refinement of an inequality provides better convergence of one quantity towards the other one. We have established the refinements of Hadamard inequalities for Riemann-Liouville fractional integrals via strongly (α, m)-convex functions. In particular, we obtain two refinements of the classical Hadamard inequality. By using some known integral identities we also give refinements of error bounds of some fractional Hadamard inequalities.

1. Introduction

Convex functions are useful in the formation of new notions and concepts. Convex functions and their related concepts are further utilized in different aspects to study generalizations, extensions and refinements of inequalities given in mathematical analysis, mathematical statistics, optimization theory etc. A lot of integral inequalities have been established due to convex and related functions in literature.

Definition 1.1. [24] A function $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$ is called convex function, if the following inequality holds:

$$f(xt + (1-t)y) \leq tf(x) + (1-t)f(y), \quad t \in [0,1], \quad x, y \in I. \tag{1}$$

If the inequality (1) holds in reverse order, then f is called concave function.

A well-known inequality named Hadamard inequality is another interpretation of convex function. It is stated as follows:

Theorem 1.2. *Let $f : I \to \mathbb{R}$ be a convex function on interval $I \subset \mathbb{R}$ and $x, y \in I$ where $x < y$. Then the following inequalities hold:*

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(u)du \leq \frac{f(x)+f(y)}{2}. \tag{2}$$

Received June 7, 2021. Revised October 12, 2021. Accepted October 22, 2021.
2010 Mathematics Subject Classification: 26A51, 26A33, 33E12.
Key words and phrases: Convex function, Strongly (α, m)-convex function, Hadamard inequality, Riemann-Liouville fractional integrals.
* Corresponding author.

If the order of inequalities in (2) is reversed, then they hold for concave function. Recently, many authors have published refinements, extensions and generalizations of the Hadamard inequality. For further detail see [2–4,6,7,9,13–15,25–27] and references therein.
Next we give the definition of strongly convex function related to convex function.

Definition 1.3. [23] (see also, [29]) Let D be a convex subset of normed space $(\mathbb{X}, ||.||)$. A function $f : D \to \mathbb{R}$ is called strongly convex function with modulus C, if

$$f(xt + (1-t)y) \leq tf(x) + (1-t)f(y) - Ct(1-t)||x-y||^2 \tag{3}$$

holds for every $x, y \in D, t \in [0,1]$ and $C \geq 0$.

Remark 1.4. If we take $C = 0$ in (3), then we get inequality (1).

Many authors have investigated the properties and applications of strongly convex functions, see [19, 21, 29].

Definition 1.5. [28] A function $f : [0,b] \to \mathbb{R}, b > 0$ is called m-convex function, if

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y), \tag{4}$$

holds for every $x, y \in [0,b]$ and $t, m \in [0,1]$.

Definition 1.6. [12] A function $f : [0,b] \to \mathbb{R}$ is called strongly m-convex function, if

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y) - Cmt(1-t)|y-x|^2, \tag{5}$$

holds for every $x, y \in [0,b], b > 0, C \geq 0$ and $t, m \in [0,1]$.

Definition 1.7. [18] A function $f : [0,b] \to \mathbb{R}$, $b > 0$ is called (α, m)-convex function, if

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y), \tag{6}$$

holds for every $x, y \in [0,b], b > 0, (\alpha, m) \in [0,1]^2$ and $t \in [0,1]$.

Definition 1.8. [5] A function $f : [0,b] \to \mathbb{R}$, $b > 0$ is called strongly (α, m)-convex function, if

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y) - Cmt^\alpha(1-t^\alpha)|y-x|^2, \tag{7}$$

holds for every $x, y \in [0,b], b > 0, C \geq 0$ and $t \in [0,1]$.

The inequality (7) provides the definition of (α, m)-convex function for $C = 0$; strongly m-convex function for $\alpha = 1$; m-convex function for $\alpha = 1$, $C = 0$; strongly convex function for $\alpha = m = 1$; convex function for $\alpha = m = 1$, $C = 0$.
Fractional integral operators are very useful in the advance of mathematical analysis [16, 17]. The first formulation of fractional integral operators is the Riemann-Liouville integral operators defined as follows:

Definition 1.9. [10] Let $f \in L_1[x,y]$. Then Riemann-Liouville fractional integral operators of order $\lambda > 0$ are defined as follows:

$$J^\lambda_{x^+} f(u) = \frac{1}{\Gamma(\lambda)} \int_x^u (u-t)^{\lambda-1} f(t)dt, u > x \tag{8}$$

$$J^\lambda_{y^-} f(u) = \frac{1}{\Gamma(\lambda)} \int_u^y (t-u)^{\lambda-1} f(t)dt, u < y. \tag{9}$$

The following special functions are also involved in the findings of this paper.

Definition 1.10. [20] The beta function, also referred to as first-type of Euler integral is defined by:

$$\beta(\alpha, s) = \int_0^1 t^{\alpha-1}(1-t)^{s-1}dt, \tag{10}$$

where $Re(\alpha),\ Re(s) > 0$.

A generalization of the beta function called incomplete beta function [20] is defined by:

$$\beta(b; \alpha, s) = \int_0^b t^{\alpha-1}(1-t)^{s-1}dt, \tag{11}$$

where $Re(\alpha),\ Re(s) > 0$ with $0 < b < 1$. The incomplete beta function $\beta(b; \alpha, s)$ reduces to the ordinary $\beta(\alpha, s)$ (beta function) by setting $b = 1$.
The following Hadamard inequalities have been established for Riemann-Liouville fractional integrals via convex functions.

Theorem 1.11. [26] *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L_1[a, b]$. If f is convex function on $[a, b]$, then the following inequality for fractional integrals holds:*

$$f\left(\frac{x+y}{2}\right) \le \frac{\Gamma(\lambda+1)}{2(y-x)^\lambda}\left[(J^\lambda_{x^+}f)(y) + (J^\lambda_{y^-}f)(x)\right] \le \frac{f(x)+f(y)}{2} \tag{12}$$

with $\lambda > 0$.

Theorem 1.12. [27] *Under the assumptions of Theorem 1.11, the following fractional integral inequality holds:*

$$f\left(\frac{a+b}{2}\right) \le \frac{2^{\lambda-1}\Gamma(\lambda+1)}{(b-a)^\alpha}\left[I^\lambda_{\left(\frac{a+b}{2}\right)^+}f(b) + I^\lambda_{\left(\frac{a+b}{2}\right)^-}f(a)\right] \le \frac{f(a)+f(b)}{2}, \tag{13}$$

with $\lambda > 0$.

The aim of this paper is to establish the Hadamard type inequalities for Riemann-Liouville fractional integral operators. To prove these inequalities we will utilize strongly (α, m)-convex functions. Further, we find the error bounds of the Hadamard inequalities for strongly (α, m)-convex functions. We also, give the Hadamard inequalities for strongly α-convex functions and strongly m-convex functions.
we organized our paper as follows: In Section 2, two Hadamard inequalities for strongly (α, m)-convex functions are proved by using Riemann-Liouville fractional integrals. In Section 3, the error estimations of Hadamard inequalities for Riemann-Liouville fractional integrals are given for differentiable strongly (α, m)-convex functions.

2. Main Results

Theorem 2.1. *Let $f : [x, y] \to \mathbb{R}$ with $0 \le x < y$ and $f \in L_1[x, y]$ be a positive function. If f is strongly (α, m)-convex function on $[x, my]$, with modulus C, $m \neq$*

0, $\alpha \in (0,1]$, *then the following fractional integral inequality holds:*

$$f\left(\frac{x+my}{2}\right)+\frac{Cm\lambda(2^\alpha-1)}{2^{2\alpha}(\lambda+2)}\left((x-y)^2+\frac{2(my-\frac{x}{m})^2}{\lambda(\lambda+1)}+\frac{2(x-y)(my-\frac{x}{m})}{(\lambda+1)}\right) \tag{14}$$
$$\leq \frac{\Gamma(\lambda+1)}{2^\alpha(my-x)^\lambda}\left[J^\lambda_{x^+}f(my)+m^{\lambda+1}(2^\alpha-1)J^\lambda_{y^-}f\left(\frac{x}{m}\right)\right]$$
$$\leq \frac{\lambda\left(f(x)-m^2(2^\alpha-1)f\left(\frac{x}{m^2}\right)\right)}{2^\alpha(\alpha+\lambda)}+\frac{m\left[f(y)\left(1+\frac{\lambda(2^\alpha-2)}{\alpha+\lambda}\right)+m(2^\alpha-1)f\left(\frac{x}{m^2}\right)\right]}{2^\alpha}$$
$$-\frac{Cm\lambda\alpha\left((y-x)^2+m(2^\alpha-1)(y-\frac{x}{m^2})^2\right)}{2^\alpha(\alpha+\lambda)(2\alpha+\lambda)}.$$

with $\lambda>0$.

Proof. Since f is strongly (α,m)-convex function, for $u,v\in[x,y]$ we have

$$f\left(\frac{u+mv}{2}\right)\leq\frac{f(u)+m(2^\alpha-1)f(v)}{2^\alpha}-\frac{Cm}{2^{2\alpha}}(2^\alpha-1)|u-v|^2. \tag{15}$$

By setting $u=xt+m(1-t)y$ and $v=yt+(1-t)\frac{x}{m}$, we get

$$2^\alpha f\left(\frac{x+my}{2}\right)\leq f(xt+m(1-t)y)+m(2^\alpha-1)f\left(yt+(1-t)\frac{x}{m}\right) \tag{16}$$
$$-\frac{Cm}{2^\alpha}(2^\alpha-1)\left|t(y-x)+(1-t)\left(\frac{x}{m}-my\right)\right|^2.$$

Multiplying both sides of (16) with $t^{\lambda-1}$ and then integrating over the interval $[0,1]$, we get

(17)
$$f\left(\frac{x+my}{2}\right)\int_0^1 t^{\lambda-1}dt\leq\frac{1}{2^\alpha}\int_0^1 f(xt+m(1-t)y)t^{\lambda-1}dt+\frac{m}{2^\alpha}(2^\alpha-1)$$
$$\int_0^1 f\left(yt+(1-t)\frac{x}{m}\right)t^{\lambda-1}dt-\frac{Cm}{2^{2\alpha}}(2^\alpha-1)\int_0^1\left|t(y-x)+(1-t)\left(\frac{x}{m}-my\right)\right|^2 t^{\lambda-1}dt.$$

By change of variables we have

(18)
$$\frac{1}{\lambda}f\left(\frac{x+my}{2}\right)\leq\frac{\Gamma(\lambda)}{2^\alpha(my-x)^\lambda}\left[\frac{1}{\Gamma(\lambda)}\int_{my}^x(my-u)^{\lambda-1}f(u)du+\frac{m^{\lambda+1}(2^\alpha-1)}{\Gamma(\lambda)}\right.$$
$$\left.\int_{\frac{x}{m}}^y\left(v-\frac{x}{m}\right)^{\lambda-1}f(v)dv\right]-\frac{Cm(2^\alpha-1)}{2^{2\alpha}}\left(\frac{(x-y)^2}{\lambda+2}+\frac{2(my-\frac{x}{m})^2}{\lambda(\lambda+1)(\lambda+2)}+\frac{2(x-y)(my-\frac{x}{m})}{(\lambda+1)(\lambda+2)}\right).$$

The above inequality become as follows:

$$f\left(\frac{x+my}{2}\right)\leq\frac{\Gamma(\lambda+1)}{2^\alpha(my-x)^\lambda}\left[J^\lambda_{x^+}f(my)+m^{\lambda+1}(2^\alpha-1)J^\lambda_{y^-}f\left(\frac{x}{m}\right)\right] \tag{19}$$
$$-\frac{Cm\lambda}{2^{2\alpha}}(2^\alpha-1)\left(\frac{(x-y)^2}{\lambda+2}+\frac{2(my-\frac{x}{m})^2}{\lambda(\lambda+1)(\lambda+2)}+\frac{2(x-y)(my-\frac{x}{m})}{(\lambda+1)(\lambda+2)}\right).$$

Again for strongly (α, m)-convex function f with modulus C, for $t \in [0,1]$ we have the following inequality:

(20)

$$f(tx+m(1-t)y)+m(2^\alpha-1)f\left(yt+(1-t)\frac{x}{m}\right) \leq \left[f(x)-m^2(2^\alpha-1)f\left(\frac{x}{m^2}\right)\right]t^\alpha+mf(y)$$
$$[1+(2^\alpha-2)t^\alpha]+m^2(2^\alpha-1)f\left(\frac{x}{m^2}\right)-Cm\left((y-x)^2+m(2^\alpha-1)\left(y-\frac{x}{m^2}\right)^2\right)t^\alpha(1-t^\alpha).$$

Multiplying both sides of (20) with $t^{\lambda-1}$ and then integrating over the interval $[0,1]$, we get

(21)

$$\int_0^1 f(tx+m(1-t)y)t^{\lambda-1}dt+m(2^\alpha-1)\int_0^1 f\left(yt+(1-t)\frac{x}{m}\right)t^{\lambda-1}dt$$
$$\leq \left[f(x)-m^2(2^\alpha-1)f\left(\frac{x}{m^2}\right)\right]\int_0^1 t^{\alpha+\lambda-1}dt+mf(y)[1+(2^\alpha-2)t^\alpha]\int_0^1 t^{\lambda-1}dt+m^2(2^\alpha-1)$$
$$f\left(\frac{x}{m^2}\right)\int_0^1 t^{\lambda-1}dt-Cm\left((y-x)^2+m(2^\alpha-1)\left(y-\frac{x}{m^2}\right)^2\right)\int_0^1(t^{\alpha+\lambda-1}-t^{2\alpha+\lambda-1})dt.$$

By change of variables we have

$$(22)\quad \frac{\Gamma(\lambda)}{(my-x)^\lambda}\left[\frac{1}{\Gamma(\lambda)}\int_{my}^{x}(my-u)^{\lambda-1}f(u)du+\frac{m^{\lambda+1}(2^\alpha-1)}{\Gamma(\lambda)}\int_{\frac{x}{m}}^{y}\left(v-\frac{x}{m}\right)^{\lambda-1}f(v)dv\right]$$
$$\leq \frac{f(x)-m^2(2^\alpha-1)f\left(\frac{x}{m^2}\right)}{\alpha+\lambda}+mf(y)\left(\frac{1}{\lambda}+\frac{2^\alpha-2}{\alpha+\lambda}\right)+\frac{m^2}{\lambda}(2^\alpha-1)f\left(\frac{x}{m^2}\right)$$
$$-\frac{Cm\alpha\left((y-x)^2+m(2^\alpha-1)(y-\frac{x}{m^2})^2\right)}{(\alpha+\lambda)(2\alpha+\lambda)}.$$

Consequently, the above inequality takes the following form

$$(23)\quad \frac{\Gamma(\lambda+1)}{2^\alpha(my-x)^\lambda}\left[J_{x^+}^\lambda f(my)+m^{\lambda+1}(2^\alpha-1)J_{y^-}^\lambda f\left(\frac{x}{m}\right)\right]$$
$$\leq \frac{\lambda\left(f(x)-m^2(2^\alpha-1)f\left(\frac{x}{m^2}\right)\right)}{2^\alpha(\alpha+\lambda)}+\frac{m}{2^\alpha}\left[f(y)\left(1+\frac{\lambda(2^\alpha-2)}{\alpha+\lambda}\right)+m(2^\alpha-1)f\left(\frac{x}{m^2}\right)\right]$$
$$-\frac{Cm\lambda\alpha\left((y-x)^2+m(2^\alpha-1)(y-\frac{x}{m^2})^2\right)}{2^\alpha(\alpha+\lambda)(2\alpha+\lambda)}.$$

From inequalities (19) and (23), one can get inequality (14). □

Corollary 2.2. *For $\alpha = 1$ in (14), we have the result for Riemann-Liouville fractional integrals of strongly m-convex functions*

$$(24)\quad f\left(\frac{x+my}{2}\right)+\frac{Cm\lambda}{4}\left(\frac{(x-y)^2}{\lambda+2}+\frac{2(my-\frac{x}{m})^2}{\lambda(\lambda+1)(\lambda+2)}+\frac{2(x-y)(my-\frac{x}{m})}{(\lambda+1)(\lambda+2)}\right)$$
$$\leq \frac{\Gamma(\lambda+1)}{2(my-x)^\lambda}\left[J_{x^+}^\lambda f(my)+m^{\lambda+1}J_{y^-}^\lambda f\left(\frac{x}{m}\right)\right] \leq \frac{\lambda\left(f(x)-m^2f\left(\frac{x}{m^2}\right)\right)}{2(\lambda+1)}$$
$$+\frac{m}{2}\left(f(y)+mf\left(\frac{x}{m^2}\right)\right)-\frac{Cm\lambda\left((y-x)^2+m(y-\frac{x}{m^2})^2\right)}{2(\lambda+1)(\lambda+2)}.$$

with $\lambda > 0$.

Corollary 2.3. *For $m = 1$ in (14), we have the result for Riemann-Liouville fractional integrals of strongly α-convex functions*

(25)

$$f\left(\frac{x+y}{2}\right)+\frac{C(2^\alpha-1)(y-x)^2\left(\lambda^2-\lambda+2\right)}{2^{2\alpha}(\lambda+1)(\lambda+2)}\leq\frac{\Gamma(\lambda+1)}{2^\alpha(y-x)^\lambda}\left[J^\lambda_{x^+}f(y)+(2^\alpha-1)J^\lambda_{y^-}f(x)\right]$$
$$\leq\frac{\lambda f(x)\left(2-2^\alpha\right)}{2^\alpha(\alpha+\lambda)}+\frac{f(y)\left(1+\frac{\lambda(2^\alpha-2)}{\alpha+\lambda}\right)+(2^\alpha-1)f(x)}{2^\alpha}-\frac{C\alpha\lambda(y-x)^2}{(\alpha+\lambda)(2\alpha+\lambda)},$$

with $\lambda > 0$.

Remark 2.4. (i) If we take $\alpha = 1$ and $C = 0$ in (14), then we get [3, Theorem 2.1].
(ii) If we take $\alpha = 1$ and $m = 1$ in (14), then we get result for strongly convex function.
(iii) If we take $\alpha = 1$, $m = 1$ and $C = 0$ in (14), then we get [26, Theorem 2].
(iv) If we take $\alpha = 1$, $m = 1$, $\lambda = 1$ and $C = 0$ in (14), then we get (1.2).

The upcoming result is another version of the Hadamard inequality for Riemann-Liouville fractional integrals of strongly (α, m)-convex functions.

Theorem 2.5. *Let $f : [x, y] \to \mathbb{R}$ with $0 \leq x < y$ and $f \in L_1[x, y]$ be a positive function. If f is strongly (α, m)-convex function on $[x, my]$ with modulus C, $m \neq 0$, $\alpha \in (0, 1]$, then the following fractional integral inequality holds:*

(26)

$$f\left(\frac{x+my}{2}\right)$$
$$+\frac{Cm\lambda(2^\alpha-1)}{2^{2\alpha}}\left[\frac{(x-y)^2}{4(\lambda+2)}+\frac{(my-\frac{x}{m})^2(\lambda^2+5\lambda+8)}{4\lambda(\lambda+1)(\lambda+2)}+\frac{(x-y)(my-\frac{x}{m})(\lambda+3)}{2(\lambda+1)(\lambda+2)}\right]$$
$$\leq\frac{2^{\lambda-\alpha}\Gamma(\lambda+1)}{(my-x)^\lambda}\left[J^\lambda_{(\frac{x+my}{2})^+}f(ym)+m^{\lambda+1}(2^\alpha-1)J^\lambda_{(\frac{x+ym}{2m})^-}f\left(\frac{x}{m}\right)\right]$$
$$\leq\frac{\lambda\left(f(x)-m^2(2^\alpha-1)f\left(\frac{x}{m^2}\right)\right)}{2^{2\alpha}(\lambda+\alpha)}+\frac{mf(y)}{2^\alpha}\left(1+\frac{(2^\alpha-2)\lambda}{2^\alpha(\alpha+\lambda)}\right)+\frac{m^2}{2^\alpha}(2^\alpha-1)f\left(\frac{x}{m^2}\right)$$
$$-\frac{Cm\lambda((y-x)^2+m(2^\alpha-1)(y-\frac{x}{m^2})^2)(\lambda+3\alpha)}{2^{3\alpha}(\lambda+\alpha)(\lambda+2\alpha)}.$$

with $\lambda > 0$.

Proof. For $t \in [0, 1]$ using strongly (α, m)-convexity of function f, let $u = x\frac{t}{2} + m(\frac{2-t}{2})y$ and $v = (\frac{2-t}{2})\frac{x}{m} + y\frac{t}{2}$ in inequality (15), we have

(27)
$$f\left(\frac{x+my}{2}\right)\leq\frac{1}{2^\alpha}f\left(x\frac{t}{2}+m\left(\frac{2-t}{2}\right)y\right)+\frac{m(2^\alpha-1)}{2^\alpha}f\left(\left(\frac{2-t}{2}\right)\frac{x}{m}+y\frac{t}{2}\right)$$
$$-\frac{Cm(2^\alpha-1)}{2^{2\alpha}}\left|\frac{t}{2}(x-y)+\frac{2-t}{2}\left(my-\frac{x}{m}\right)\right|^2.$$

Multiplying both sides of (27) with $t^{\lambda-1}$ and then integrating over the interval $[0,1]$, we get

(28)
$$f\left(\frac{x+my}{2}\right)\int_0^1 t^{\lambda-1}dt \leq \frac{1}{2^\alpha}\int_0^1 f\left(x\frac{t}{2}+m\left(\frac{2-t}{2}\right)y\right)t^{\lambda-1}dt+\frac{m(2^\alpha-1)}{2^\alpha}$$
$$\int_0^1 f\left(\left(\frac{2-t}{2}\right)\frac{x}{m}+y\frac{t}{2}\right)t^{\lambda-1}dt-\frac{Cm(2^\alpha-1)}{2^{2\alpha}}\int_0^1\left|\frac{t}{2}(x-y)+\frac{2-t}{2}\left(my-\frac{x}{m}\right)\right|^2 t^{\lambda-1}dt.$$

By using change of variables we have

(29)
$$\frac{1}{\lambda}f\left(\frac{x+my}{2}\right)$$
$$\leq \frac{2^\lambda\Gamma(\lambda)}{2^\alpha(my-x)^\lambda}\left[\frac{1}{\Gamma(\lambda)}\int_{my}^{\frac{x+my}{2}}(my-u)^{\lambda-1}f(u)du+\frac{m^{\lambda+1}(2^\alpha-1)}{\Gamma(\lambda)}\int_{\frac{x}{m}}^{\frac{ym+x}{2m}}\left(v-\frac{x}{m}\right)^{\lambda-1}f(v)dv\right]$$
$$-\frac{Cm(2^\alpha-1)}{2^{2\alpha}}\left[\frac{(x-y)^2}{4(\lambda+2)}+\frac{(my-\frac{x}{m})^2(\lambda^2+5\lambda+8)}{4\lambda(\lambda+1)(\lambda+2)}+\frac{(x-y)(my-\frac{x}{m})(\lambda+3)}{2(\lambda+1)(\lambda+2)}\right].$$

Consequently, the above inequality takes the following form

(30)
$$f\left(\frac{x+my}{2}\right)\leq\frac{2^{\lambda-\alpha}\Gamma(\lambda+1)}{(my-x)^\lambda}\left[\left(J^\lambda_{(\frac{x+my}{2})^+}f\right)(ym)+m^{\lambda+1}(2^\alpha-1)\left(J^\lambda_{(\frac{ym+x}{2m})^-}f\right)\left(\frac{x}{m}\right)\right]$$
$$-\frac{Cm\lambda(2^\alpha-1)}{2^{2\alpha}}\left[\frac{(x-y)^2}{4(\lambda+2)}+\frac{(my-\frac{x}{m})^2(\lambda^2+5\lambda+8)}{4\lambda(\lambda+1)(\lambda+2)}+\frac{(x-y)(my-\frac{x}{m})(\lambda+3)}{2(\lambda+1)(\lambda+2)}\right].$$

The first inequality of (26) is obtained in (30).
Now for proving the second inequality of (26), using that f is strongly (α,m)-convex function and $t\in[0,1]$, we have the following inequality

$$f\left(x\frac{t}{2}+m\left(\frac{2-t}{2}\right)y\right)+m(2^\alpha-1)f\left(\left(\frac{2-t}{2}\right)\frac{x}{m}+y\frac{t}{2}\right)\tag{31}$$
$$\leq\left(f(x)-m^2(2^\alpha-1)f\left(\frac{x}{m^2}\right)\right)\left(\frac{t}{2}\right)^\alpha+mf(y)\left(1+\frac{(2^\alpha-2)t^\alpha}{2^\alpha}\right)$$
$$+m^2(2^\alpha-1)f\left(\frac{x}{m^2}\right)-\frac{Cm}{2^{2\alpha}}\left[(y-x)^2+m(2^\alpha-1)\left(y-\frac{x}{m^2}\right)^2\right]t^\alpha(2-t^\alpha).$$

Multiplying inequality (31) with $t^{\lambda-1}$ on both sides and then integrating over the interval $[0,1]$, we get

$$\int_0^1 f\left(x\frac{t}{2}+m\left(\frac{2-t}{2}\right)y\right)t^{\lambda-1}dt+m(2^\alpha-1)\int_0^1 f\left(\left(\frac{2-t}{2}\right)\frac{x}{m}+y\frac{t}{2}\right)t^{\lambda-1}dt\tag{32}$$
$$\leq\frac{1}{2^\alpha}\left(f(x)-m^2(2^\alpha-1)f\left(\frac{x}{m^2}\right)\right)\int_0^1 t^{\alpha+\lambda-1}dt$$
$$+mf(y)\left(1+\frac{(2^\alpha-2)t^\alpha}{2^\alpha}\right)\int_0^1 t^{\lambda-1}dt+m^2(2^\alpha-1)f\left(\frac{x}{m^2}\right)\int_0^1 t^{\lambda-1}dt$$
$$-\frac{Cm}{2^{2\alpha}}\left[(y-x)^2+m(2^\alpha-1)\left(y-\frac{x}{m^2}\right)^2\right]t^{\alpha+\lambda-1}(2-t^\alpha)dt.$$

By using the the change of variables we have

(33)
$$\frac{2^{\lambda}\Gamma(\lambda)}{(my-x)^{\lambda}}\left[\frac{1}{\Gamma(\lambda)}\int_{my}^{\frac{x+my}{2}}(my-u)^{\lambda-1}f(u)du+\frac{m^{\lambda+1}(2^{\alpha}-1)}{\Gamma(\lambda)}\int_{\frac{x}{m}}^{\frac{my+x}{2m}}\left(v-\frac{x}{m}\right)^{\lambda-1}f(v)dv\right]$$
$$\leq\frac{\left(f(x)-m^2(2^{\alpha}-1)f\left(\frac{x}{m^2}\right)\right)}{2^{\alpha}(\alpha+\lambda)}+mf(y)\left(\frac{1}{\lambda}+\frac{2^{\alpha}-2}{2^{\alpha}(\alpha+\lambda)}\right)+\frac{m^2}{\lambda}(2^{\alpha}-1)f\left(\frac{x}{m^2}\right)$$
$$-\frac{Cm((y-x)^2+m(2^{\alpha}-1)(y-\frac{x}{m^2})^2)(\lambda+3\alpha)}{2^{2\alpha}(\lambda+\alpha)(\lambda+2\alpha)}.$$

Consequently, the above inequality takes the following form

$$(34)\quad \frac{2^{\lambda-\alpha}\Gamma(\lambda+1)}{(my-x)^{\lambda}}\left[J^{\lambda}_{(\frac{x+my}{2})^+}f(ym)+m^{\lambda+1}(2^{\alpha}-1)J^{\lambda}_{(\frac{x+ym}{2m})^-}f\left(\frac{x}{m}\right)\right]$$
$$\leq\frac{\lambda\left(f(x)-m^2(2^{\alpha}-1)f\left(\frac{x}{m^2}\right)\right)}{2^{2\alpha}(\lambda+\alpha)}+\frac{mf(y)}{2^{\alpha}}\left(1+\frac{(2^{\alpha}-2)\lambda}{2^{\alpha}(\alpha+\lambda)}\right)+\frac{m^2(2^{\alpha}-1)}{2^{\alpha}}f\left(\frac{x}{m^2}\right)$$
$$-\frac{Cm\lambda((y-x)^2+m(2^{\alpha}-1)(y-\frac{x}{m^2})^2)(\lambda+3\alpha)}{2^{3\alpha}(\lambda+\alpha)(\lambda+2\alpha)}.$$

From inequalities (30) and (34), we get the inequality (26). □

COROLLARY 2.6. *For $\alpha=1$ in (26), we get the following inequality for strongly m-convex function*

$$(35)\quad f\left(\frac{x+my}{2}\right)$$
$$+\frac{Cm\lambda}{4}\left[\frac{(x-y)^2}{4(\lambda+2)}+\frac{(my-\frac{x}{m})^2(\lambda^2+5\lambda+8)}{4\lambda(\lambda+1)(\lambda+2)}+\frac{(x-y)(my-\frac{x}{m})(\lambda+3)}{2(\lambda+1)(\lambda+2)}\right]$$
$$\leq\frac{2^{\lambda-1}\Gamma(\lambda+1)}{(my-x)^{\lambda}}\left[J^{\lambda}_{(\frac{x+my}{2})^+}f(ym)+m^{\lambda+1}J^{\lambda}_{(\frac{x+ym}{2m})^-}f\left(\frac{x}{m}\right)\right]$$
$$\leq\frac{\lambda\left(f(x)-m^2f\left(\frac{x}{m^2}\right)\right)}{4(\lambda+1)}+\frac{m}{2}\left(f(y)+mf\left(\frac{x}{m^2}\right)\right)$$
$$-\frac{Cm\lambda((y-x)^2+m(y-\frac{x}{m^2})^2)(\lambda+3)}{8(\lambda+1)(\lambda+2)}$$

with $\lambda>0$.

COROLLARY 2.7. *For $m=1$ in (26), we get the following inequality for α-convex functions*

(36)
$$f\left(\frac{x+y}{2}\right)+\frac{C(2^{\alpha}-1)(y-x)^2}{2^{2\alpha-1}(\lambda+1)(\lambda+2)}\leq\frac{2^{\lambda-\alpha}\Gamma(\lambda+1)}{(y-x)^{\lambda}}\left[J^{\lambda}_{(\frac{x+y}{2})^+}f(y)+(2^{\alpha}-1)J^{\lambda}_{(\frac{x+y}{2})^-}f(x)\right]$$
$$\leq f(x)\left[\frac{\lambda(2-2^{\alpha})+2^{\alpha}(2^{\alpha}-1)(\lambda+\alpha)}{2^{2\alpha}(\lambda+\alpha)}\right]+f(y)\left(\frac{2^{\alpha}(\lambda+\alpha)+\lambda(2^{\alpha}-2)}{2^{2\alpha}(\lambda+\alpha)}\right)$$
$$-\frac{C\lambda(y-x)^2(\lambda+3\alpha)}{2^{2\alpha}(\lambda+\alpha)(\lambda+2\alpha)}.$$

with $\lambda>0$.

Corollary 2.8. *For $\alpha = 1, m = 1$ in (26), we get the following inequality for strongly convex function.*

$$f\left(\frac{x+y}{2}\right) + \frac{C(y-x)^2}{2(\lambda+1)(\lambda+2)} \leq \frac{2^{\lambda-1}\Gamma(\lambda+1)}{(y-x)^\lambda}\left[J^\lambda_{(\frac{x+y}{2})^+}f(y) + J^\lambda_{(\frac{x+y}{2})^-}f(x)\right] \leq \frac{f(x)+f(y)}{2} - \frac{C\lambda(y-x)^2(\lambda+3)}{4(\lambda+1)(\lambda+2)}. \tag{37}$$

with $\lambda > 0$.

Remark 2.9. (i) If we take $\alpha = 1$ and $C = 0$ in (26), then we get [4, Theorem 2.1].
(ii) If we take $\alpha = 1$, $m = 1$ and $C = 0$ in (26), then we get [27, Theorem:4].
(iii) If we take $\alpha = 1$, $m = 1$, $\lambda = 1$ and $C = 0$ in (26), then we get Theorem 1.2.

3. Error estimations for strongly (α, m)-convex functions

In this section we give refinements of the error bounds of fractional Hadamard inequalities for Riemann-Liouville fractional integrals. The following two lemmas are useful to establish the next results.

Lemma 3.1. [26] *Let $f : [x, y] \to \mathbb{R}$ be a differentiable mapping on (x, y) with $x < y$. If $f' \in L[x, y]$, then the following fractional integrals equality holds:*

$$\frac{f(x)+f(y)}{2} - \frac{\Gamma(\lambda+1)}{2(y-x)^\lambda}\left[(J^\lambda_{x^+}f)(y) + (J^\lambda_{y^-}f)(x)\right] = \frac{y-x}{2}\int_0^1 \left[(1-t)^\lambda - t^\lambda\right] f'(tx+(1-t)y)\,dt. \tag{38}$$

Lemma 3.2. [4] *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f' \in L[a, b]$, then the following equality for fractional integrals holds:*

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^\alpha}\left[J^\alpha_{(\frac{a+mb}{2})^+}f(mb) + m^{\alpha+1}J^\alpha_{(\frac{a+mb}{2m})^-}f\left(\frac{a}{m}\right)\right] - \frac{1}{2}\left[f\left(\frac{a+mb}{2}\right) + mf\left(\frac{a+mb}{2m}\right)\right] = \frac{mb-a}{4}\left[\int_0^1 t^\alpha f'\left(\frac{t}{2}a + m\frac{2-t}{2}b\right)dt - \int_0^1 t^\alpha f'\left(\frac{2-t}{2m}a + \frac{t}{2}b\right)dt\right]. \tag{39}$$

Theorem 3.3. *Let $f : [x, y] \to \mathbb{R}$ be a differentiable mapping on (x, y) with $x < y$. If $|f'|$ is a strongly (α, m)-convex function on $[x, my]$, $m \neq 0$, $\alpha \in (0, 1]$, with modulus*

C, then the following fractional integrals inequality holds:

(40)

$$\begin{aligned}&\left|\frac{f(x)+f(y)}{2}-\frac{\Gamma(\lambda+1)}{2(y-x)^{\lambda}}\left[(J_{x^+}^{\lambda}f)(y)+(J_{y^-}^{\lambda}f)(x)\right]\right|\\&\leq\frac{y-x}{2}\Bigg[|f'(x)|\left(\beta\left(\frac{1}{2};\lambda+1,\alpha+1\right)-\beta\left(\frac{1}{2};\alpha+1,\lambda+1\right)+\frac{1-(1/2)^{\lambda+\alpha}}{\lambda+\alpha+1}\right)\\&+m\left|f'\left(\frac{y}{m}\right)\right|\left(\frac{(\frac{1}{2})^{\lambda+\alpha}-1}{\lambda+\alpha+1}+\frac{2}{\lambda+1}\left(1-\frac{1}{2^{\lambda}}\right)-\beta\left(\frac{1}{2};\lambda+1,\alpha+1\right)+\beta\left(\frac{1}{2};\alpha+1,\lambda+1\right)\right)\\&-Cm(\frac{y}{m}-x)^2\Bigg(\beta\left(\frac{1}{2};\lambda+1,\alpha+1\right)-\beta\left(\frac{1}{2};\alpha+1,\lambda+1\right)+\beta\left(\frac{1}{2};2\alpha+1,\lambda+1\right)\\&-\beta\left(\frac{1}{2};\lambda+1,2\alpha+1\right)-\frac{\left(\frac{1}{2}\right)^{\lambda+\alpha}}{\lambda+\alpha+1}+\frac{\left(\frac{1}{2}\right)^{\lambda+2\alpha}}{\lambda+2\alpha+1}+\frac{\alpha}{(\lambda+\alpha+1)(\lambda+2\alpha+1)}\Bigg)\end{aligned}$$

with $\lambda>0$.

Proof. Since $|f'|$ is strongly (α,m)-convex function on $[x,my]$ and $t\in[0,1]$, we have

$$\begin{aligned}|f'(tx+(1-t)y)|\,dt&=\left|f'\left(tx+m(1-t)\frac{y}{m}\right)\right|dt\\&\leq t^{\alpha}|f'(x)|+m(1-t^{\alpha})\left|f'\left(\frac{y}{m}\right)\right|-Cmt^{\alpha}(1-t^{\alpha})\left(\frac{y}{m}-x\right)^2.\end{aligned}\tag{41}$$

By using Lemma 3.1 and (41), we have

(42)

$$\begin{aligned}&\left|\frac{f(x)+f(y)}{2}-\frac{\Gamma(\lambda+1)}{2(y-x)^{\lambda}}\left[(J_{x^+}^{\lambda}f)(y)+(J_{y^-}^{\lambda}f)(x)\right]\right|\\&\leq\frac{y-x}{2}\int_0^1\left|(1-t)^{\lambda}-t^{\lambda}\right|\left|f'\left(tx+m(1-t)\frac{y}{m}\right)\right|dt\\&\leq\frac{y-x}{2}\int_0^1\left|(1-t)^{\lambda}-t^{\lambda}\right|\left(t^{\alpha}|f'(x)|+m(1-t^{\alpha})\left|f'\left(\frac{y}{m}\right)\right|-Cmt^{\alpha}(1-t^{\alpha})\left(\frac{y}{m}-x\right)^2\right)dt\\&\leq\frac{y-x}{2}\Bigg[\int_0^{1/2}\left((1-t)^{\lambda}-t^{\lambda}\right)\left(t^{\alpha}|f'(x)|+m(1-t^{\alpha})\left|f'\left(\frac{y}{m}\right)\right|-Cmt^{\alpha}(1-t^{\alpha})\left(\frac{y}{m}-x\right)^2\right)dt\\&+\int_{1/2}^1\left(t^{\lambda}-(1-t)^{\lambda}\right)\left(t^{\alpha}|f'(x)|+m(1-t^{\alpha})\left|f'\left(\frac{y}{m}\right)\right|-Cmt^{\alpha}(1-t^{\alpha})\left(\frac{y}{m}-x\right)^2\right)dt\Bigg]\\&\leq\frac{y-x}{2}\Bigg[|f'(x)|\left(\beta\left(\frac{1}{2};\lambda+1,\alpha+1\right)-\beta\left(\frac{1}{2};\alpha+1,\lambda+1\right)+\frac{1-(1/2)^{\lambda+\alpha}}{\lambda+\alpha+1}\right)\\&+m\left|f'\left(\frac{y}{m}\right)\right|\left(\frac{(\frac{1}{2})^{\lambda+\alpha}-1}{\lambda+\alpha+1}+\frac{2}{\lambda+1}\left(1-\frac{1}{2^{\lambda}}\right)-\beta\left(\frac{1}{2};\lambda+1,\alpha+1\right)+\beta\left(\frac{1}{2};\alpha+1,\lambda+1\right)\right)\\&-Cm(\frac{y}{m}-x)^2\Bigg(\beta\left(\frac{1}{2};\lambda+1,\alpha+1\right)-\beta\left(\frac{1}{2};\alpha+1,\lambda+1\right)+\beta\left(\frac{1}{2};2\alpha+1,\lambda+1\right)\\&-\beta\left(\frac{1}{2};\lambda+1,2\alpha+1\right)-\frac{\left(\frac{1}{2}\right)^{\lambda+\alpha}}{\lambda+\alpha+1}+\frac{\left(\frac{1}{2}\right)^{\lambda+2\alpha}}{\lambda+2\alpha+1}+\frac{\alpha}{(\lambda+\alpha+1)(\lambda+2\alpha+1)}\Bigg)\Bigg].\end{aligned}$$

After simplifying the last inequality of (42), we get the inequality (40). □

Remark 3.4. (i) If we put $C = 0$ in (40), then we get result for (α, m)-convex function.
(ii) If we put $\alpha = 1$ in (40), then we get [8, Theorem 8].
(iii) If we put $\alpha = 1$, $m = 1 = \lambda = 1$ and $C = 0$, then we get [1, Theorem 2.2].

Corollary 3.5. *For $m = 1$, we get the following inequality for strongly α-convex functions*

(43)
$$\begin{aligned}
&\left|\frac{f(x)+f(y)}{2} - \frac{\Gamma(\lambda+1)}{2(y-x)^\lambda}\left[(J^\lambda_{x+}f)(y) + (J^\lambda_{y-}f)(x)\right]\right| \\
&\leq \frac{y-x}{2}\Bigg[|f'(x)|\left(\beta\left(\frac{1}{2};\lambda+1,\alpha+1\right) - \beta\left(\frac{1}{2};\alpha+1,\lambda+1\right) + \frac{1-(1/2)^{\lambda+\alpha}}{\lambda+\alpha+1}\right) \\
&+ |f'(y)|\left(\frac{(\frac{1}{2})^{\lambda+\alpha}-1}{\lambda+\alpha+1} + \frac{2}{\lambda+1}\left(1-\frac{1}{2^\lambda}\right) - \beta\left(\frac{1}{2};\lambda+1,\alpha+1\right) + \beta\left(\frac{1}{2};\alpha+1,\lambda+1\right)\right) \\
&- C(y-x)^2\Bigg(\beta\left(\frac{1}{2};\lambda+1,\alpha+1\right) - \beta\left(\frac{1}{2};\alpha+1,\lambda+1\right) + \beta\left(\frac{1}{2};2\alpha+1,\lambda+1\right) \\
&- \beta\left(\frac{1}{2};\lambda+1,2\alpha+1\right) - \frac{\left(\frac{1}{2}\right)^{\lambda+\alpha}}{\lambda+\alpha+1} + \frac{\left(\frac{1}{2}\right)^{\lambda+2\alpha}}{\lambda+2\alpha+1} + \frac{\alpha}{(\lambda+\alpha+1)(\lambda+2\alpha+1)}\Bigg)
\end{aligned}$$

with $\lambda > 0$.

Theorem 3.6. *Let $f : [x, y] \to \mathbb{R}$ be a differentiable mapping on (x, y) with $x < y$. If $|f'|^q$ is strongly (α, m)-convex on $[x, my]$ for $q \geq 1$, then the following inequality for fractional integrals holds:*

(44)
$$\begin{aligned}
&\Bigg|\frac{2^{\lambda-1}\Gamma(\lambda+1)}{(my-x)^\lambda}\left[(J^\lambda_{(\frac{x+my}{2})+}f)(my) + m^{\lambda+1}(J^\lambda_{(\frac{x+my}{2m})-}f)\left(\frac{x}{m}\right)\right] \\
&- \frac{1}{2}\left[f\left(\frac{x+my}{2}\right) + mf\left(\frac{x+my}{2m}\right)\right]\Bigg| \leq \frac{my-x}{4(\lambda+1)}\left(\frac{1}{2^\alpha(\lambda+\alpha+1)}\right)^{\frac{1}{q}} \\
&\Bigg[\Bigg((\lambda+1)\left(|f'(x)|^q - m|f'(y)|^q\right) + 2^\alpha m(\lambda+\alpha+1)|f'(y)|^q - \frac{Cm(y-x)^2(\lambda+1)}{2^\alpha} \\
&\left(2^\alpha - \frac{\lambda+\alpha+1}{\lambda+2\alpha+1}\right)\Bigg)^{\frac{1}{q}} + \Bigg((\lambda+1)\left(|f'(y)|^q - m|f'(\frac{x}{m^2})|^q\right) + 2^\alpha m\,(\lambda+\alpha+1)\,|f'(\frac{x}{m^2})|^q \\
&- \frac{Cm(y-\frac{x}{m^2})^2(\lambda+1)}{2^\alpha}\left(2^\alpha - \frac{\lambda+\alpha+1}{\lambda+2\alpha+1}\right)\Bigg)^{\frac{1}{q}}\Bigg],
\end{aligned}$$

with $\lambda > 0$.

Proof. By applying Lemma 3.2 and strongly (α, m)-convexity of $|f'|$, we have

(45)
$$\left|\frac{2^{\lambda-1}\Gamma(\lambda+1)}{(my-x)^{\lambda}}\left[(J^{\lambda}_{(\frac{x+my}{2})+}f)(my)+m^{\lambda+1}(J^{\lambda}_{(\frac{x+my}{2m})-}f)\left(\frac{x}{m}\right)\right]-\frac{1}{2}\left[f\left(\frac{x+my}{2}\right)+mf\left(\frac{x+my}{2m}\right)\right]\right|$$
$$\leq\frac{my-x}{4}\left[\int_0^1\left|t^{\lambda}f'\left(\frac{t}{2}x+m\left(\frac{2-t}{2}\right)y\right)\right|dt+\int_0^1\left|t^{\lambda}f'\left(\left(\frac{2-t}{2}\right)\frac{x}{m}+\frac{t}{2}y\right)\right|dt\right]$$
$$\leq\frac{my-x}{4}\Bigg[\left(\frac{|f'(x)|-m|f'(y)|+|f'(y)|-m|f'(\frac{x}{m^2})|}{2^{\alpha}}\right)\int_0^1 t^{\lambda+\alpha}dt+m\left(|f'(y)|+\left|f'\left(\frac{x}{m^2}\right)\right|\right)\int_0^1 t^{\lambda}dt$$
$$-\frac{Cm}{2^{2\alpha}}\left((y-x)^2+\left(y-\frac{x}{m^2}\right)^2\right)\int_0^1 t^{\lambda+\alpha}(2^{\alpha}-t^{\alpha})dt\Bigg]$$
$$=\frac{my-x}{4}\Bigg[\left(\frac{|f'(x)|-m|f'(y)|+|f'(y)|-m|f'(\frac{x}{m^2})|}{2^{\alpha}(\lambda+\alpha+1)}\right)+\frac{m\left(|f'(y)|+|f'(\frac{x}{m^2})|\right)}{\lambda+1}$$
$$-\frac{Cm}{2^{2\alpha}}\left((y-x)^2+\left(y-\frac{x}{m^2}\right)^2\right)\left(\frac{2^{\alpha}}{\lambda+\alpha+1}-\frac{1}{\lambda+2\alpha+1}\right)\Bigg].$$

Now, for strongly (α, m)-convexity of $|f'|^q$, $q > 1$, using power mean inequality we get

(46)
$$\left|\frac{2^{\lambda-1}\Gamma(\lambda+1)}{(my-x)^{\lambda}}\left[(J^{\lambda}_{(\frac{x+my}{2})+}f)(my)+m^{\lambda+1}(J^{\lambda}_{(\frac{x+my}{2m})-}f)\left(\frac{x}{m}\right)\right]-\frac{1}{2}\left[f\left(\frac{x+my}{2}\right)+mf\left(\frac{x+my}{2m}\right)\right]\right|$$
$$\leq\frac{my-x}{4}\left(\int_0^1 t^{\lambda}dt\right)^{1-\frac{1}{q}}\left[\left(\int_0^1 t^{\lambda}\left|f'\left(\frac{t}{2}x+m\left(\frac{2-t}{2}\right)y\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1 t^{\lambda}\left|f'\left(\left(\frac{2-t}{2}\right)\frac{x}{m}+\frac{t}{2}y\right)\right|^q dt\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{my-x}{4(\lambda+1)^{\frac{1}{p}}}\left(\frac{|f'(x)|^q-m|f'(y)|^q}{2^{\alpha}}\int_0^1 t^{\lambda+\alpha}dt+m|f'(y)|^q\int_0^1 t^{\lambda}dt-\frac{Cm(y-x)^2}{2^{2\alpha}}\int_0^1 t^{\lambda+\alpha}(2^{\alpha}-t^{\alpha})dt\right)^{\frac{1}{q}}$$
$$+\left(\frac{|f'(y)|^q-m|f'(\frac{x}{m^2})|^q}{2^{\alpha}}\int_0^1 t^{\lambda+\alpha}dt+m\left|f'\left(\frac{x}{m^2}\right)\right|^q\int_0^1 t^{\lambda}dt-\frac{Cm(y-\frac{x}{m^2})^2}{2^{2\alpha}}\int_0^1 t^{\lambda+\alpha}(2^{\alpha}-t^{\alpha})dt\right)^{\frac{1}{q}}$$
$$\leq\frac{my-x}{4(\lambda+1)^{\frac{1}{p}}}\Bigg[\left(\frac{|f'(x)|^q-m|f'(y)|^q}{2^{\alpha}(\lambda+\alpha+1)}+\frac{m|f'(y)|^q}{(\lambda+1)}-\frac{Cm(y-x)^2}{2^{2\alpha}}\left(\frac{2^{\alpha}}{\lambda+\alpha+1}-\frac{1}{\lambda+2\alpha+1}\right)\right)^{\frac{1}{q}}$$
$$+\left(\frac{|f'(y)|^q-m|f'\left(\frac{x}{m^2}\right)|^q}{2^{\alpha}(\lambda+\alpha+1)}+\frac{m|f'(\frac{x}{m^2})|^q}{(\lambda+1)}-\frac{Cm(y-\frac{x}{m^2})^2}{2^{2\alpha}}\left(\frac{2^{\alpha}}{\lambda+\alpha+1}-\frac{1}{\lambda+2\alpha+1}\right)\right)^{\frac{1}{q}}\Bigg]$$
$$=\frac{my-x}{4(\lambda+1)}\left(\frac{1}{2^{\alpha}(\lambda+\alpha+1)}\right)^{\frac{1}{q}}\Bigg[\Bigg((\lambda+1)\left(|f'(x)|^q-m|f'(y)|^q\right)+2^{\alpha}m(\lambda+\alpha+1)|f'(y)|^q$$
$$-\frac{Cm(y-x)^2(\lambda+1)}{2^{\alpha}}\left(2^{\alpha}-\frac{\lambda+\alpha+1}{\lambda+2\alpha+1}\right)\Bigg)^{\frac{1}{q}}+\Bigg((\lambda+1)\left(|f'(y)|^q-m|f'(\frac{x}{m^2})|^q\right)$$
$$+2^{\alpha}m(\lambda+\alpha+1)|f'(\frac{x}{m^2})|^q-\frac{Cm(y-\frac{x}{m^2})^2(\lambda+1)}{2^{\alpha}}\left(2^{\alpha}-\frac{\lambda+\alpha+1}{\lambda+2\alpha+1}\right)\Bigg)^{\frac{1}{q}}\Bigg].$$

Hence we get the inequality (44). □

COROLLARY 3.7. *For $\alpha = 1$ in (44), we get the following inequality for strongly m-convex functions*

$$\left|\frac{2^{\lambda-1}\Gamma(\lambda+1)}{(my-x)^{\lambda}}\left[(J^{\lambda}_{(\frac{x+my}{2})^+}f)(my)+m^{\lambda+1}(J^{\lambda}_{(\frac{x+my}{2m})^-}f)\left(\frac{x}{m}\right)\right]\right. \tag{47}$$

$$\left.-\frac{1}{2}\left[f\left(\frac{x+my}{2}\right)+mf\left(\frac{x+my}{2m}\right)\right]\right|\leq\frac{my-x}{4(\lambda+1)}\left(\frac{1}{2(\lambda+2)}\right)^{\frac{1}{q}}$$

$$\left[\left((\lambda+1)|f'(x)|^q+m(\lambda+3)|f'(y)|^q-\frac{Cm(y-x)^2(\lambda+1)(\lambda+4)}{2(\lambda+3)}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left((\lambda+1)|f'(y)|^q+m\,(\lambda+3)\,|f'(\frac{x}{m^2})|^q-\frac{Cm(y-\frac{x}{m^2})^2(\lambda+1)(\lambda+4)}{2(\lambda+3)}\right)^{\frac{1}{q}}\right],$$

with $\lambda > 0$.

COROLLARY 3.8. *For $\alpha = 1$, $m = 1$ in (44), we get the following inequality for strongly convex functions*

(48)

$$\left|\frac{2^{\lambda-1}\Gamma(\lambda+1)}{(y-x)^{\lambda}}\left[(J^{\lambda}_{(\frac{x+y}{2})^+}f)(y)+(J^{\lambda}_{(\frac{x+y}{2})^-}f)(x)\right]-f\left(\frac{x+y}{2}\right)\right|\leq\frac{y-x}{4(\lambda+1)}\left(\frac{1}{2(\lambda+2)}\right)^{\frac{1}{q}}$$

$$\left[\left((\lambda+1)|f'(x)|^q+(\lambda+3)|f'(y)|^q-\frac{C(y-x)^2(\lambda+1)(\lambda+4)}{2(\lambda+3)}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left((\lambda+1)|f'(y)|^q+(\lambda+3)\,|f'(x)|^q-\frac{C(y-x)^2(\lambda+1)(\lambda+4)}{2(\lambda+3)}\right)^{\frac{1}{q}}\right],$$

with $\lambda > 0$.

COROLLARY 3.9. *For $m = 1$ in (44), we get the following inequality for strongly α-convex functions*

(49)

$$\left|\frac{2^{\lambda-1}\Gamma(\lambda+1)}{(y-x)^{\lambda}}\left[(J^{\lambda}_{(\frac{x+y}{2})^+}f)(y)+(J^{\lambda}_{(\frac{x+y}{2})^-}f)(x)\right]-f\left(\frac{x+y}{2}\right)\right|$$

$$\leq\frac{y-x}{4(\lambda+1)}\left(\frac{1}{2^{\alpha}(\lambda+\alpha+1)}\right)^{\frac{1}{q}}\left[\left((\lambda+1)\left(|f'(x)|^q-|f'(y)|^q\right)+2^{\alpha}(\lambda+\alpha+1)|f'(y)|^q\right.\right.$$

$$\left.-\frac{C(y-x)^2(\lambda+1)}{2^{\alpha}}\left(2^{\alpha}-\frac{\lambda+\alpha+1}{\lambda+2\alpha+1}\right)\right)^{\frac{1}{q}}+\left((\lambda+1)\left(|f'(y)|^q-|f'(x)|^q\right)\right.$$

$$\left.\left.+2^{\alpha}\,(\lambda+\alpha+1)\,|f'(x)|^q-\frac{C(y-x)^2(\lambda+1)}{2^{\alpha}}\left(2^{\alpha}-\frac{\lambda+\alpha+1}{\lambda+2\alpha+1}\right)\right)^{\frac{1}{q}}\right],$$

with $\lambda > 0$.

COROLLARY 3.10. *For $\alpha = \lambda = 1$, $m = 1$ in (44), we get following inequality*

$$\left|\frac{1}{y-x}\int_x^y f(u)du-f\left(\frac{x+y}{2}\right)\right|\leq\frac{y-x}{4}\left(\frac{1}{6}\right)^{\frac{1}{q}}\left[\left(|f'(x)|^q+2|f'(y)|^q\right.\right. \tag{50}$$

$$\left.\left.-\frac{5C(y-x)^2}{8}\right)^{\frac{1}{q}}+\left(|f'(y)|^q+2|f'(x)|^q-\frac{5C(y-x)^2}{8}\right)^{\frac{1}{q}}\right].$$

Corollary 3.11. *For $\alpha=\lambda=1$, $m=q=1$ in (44), we get following inequality*

$$\left|\frac{1}{y-x}\int_x^y f(u)du - f\left(\frac{x+y}{2}\right)\right| \leq \frac{y-x}{8}\left(|f'(x)|+|f'(y)|-\frac{10C}{24}(y-x)^2\right), \tag{51}$$

Remark 3.12. (i) If we put $\alpha=1$ and $C=0$ in (44), then we get [4, Theorem 2.4].
(ii) If we put $\alpha=1$, $m=1$ and $C=0$ in (26), then we get [27, Theorem 5].

Theorem 3.13. *Let $f:[x,y]\to\mathbb{R}$ be a differentiable mapping on (x,y) with $x<y$. If $|f'|^q$ is strongly (α,m)-convex function on $[x,my]$, $m\neq 0$ and $\alpha\in(0,1]$, $q>1$, then the following fractional integrals inequality holds:*

(52)

$$\Bigg|\frac{2^{\lambda-1}\Gamma(\lambda+1)}{(my-x)^{\lambda}}\left[(J^{\lambda}_{(\frac{x+my}{2})+}f)(my)+m^{\lambda+1}(J^{\lambda}_{(\frac{x+my}{2m})-}f)\left(\frac{x}{m}\right)\right]$$
$$-\frac{1}{2}\left[f\left(\frac{x+my}{2}\right)+mf\left(\frac{x+my}{2m}\right)\right]\Bigg| \leq \frac{my-x}{4}\left(\frac{1}{\lambda p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2^{\alpha}(\alpha+1)}\right)^{\frac{1}{q}}$$
$$\Bigg[\left((|f'(x)|+[m(2^{\alpha}(\alpha+1)-1)]^{\frac{1}{q}}|f'(y)|)^q - Cm(y-x)^2\left(1-\frac{\alpha+1}{2^{\alpha}(2\alpha+1)}\right)\right)^{\frac{1}{q}}$$
$$+\left((|f'(y)|+[m(2^{\alpha}(\alpha+1)-1)]^{\frac{1}{q}}|f'(\frac{x}{m^2})|)^q - Cm(\frac{x}{m^2}-y)^2\left(1-\frac{\alpha+1}{2^{\alpha}(2\alpha+1)}\right)\right)^{\frac{1}{q}}\Bigg]$$
$$\leq \frac{my-x}{4}\left(\frac{1}{\lambda p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2^{\alpha}(\alpha+1)}\right)^{\frac{1}{q}}\Bigg[|f'(x)|+|f'(y)|+\left(|f'(y)|+|f'\left(\frac{x}{m^2}\right)|\right)$$
$$[m(2^{\alpha}(\alpha+1)-1)]^{\frac{1}{q}} - Cm\left(1-\frac{\alpha+1}{2^{\alpha}(2\alpha+1)}\right)\left((y-x)^2+(\frac{x}{m^2}-y)^2\right)\Bigg],$$

where $\lambda>0$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By applying Lemma 3.2, using Hölder inequality and strong (α, m)-convexity of $|f'|^q$, we get

(53)

$$\left|\frac{2^{\lambda-1}\Gamma(\lambda+1)}{(my-x)^{\lambda}}\left[(J^{\lambda}_{(\frac{x+my}{2})^+}f)(my)+m^{\lambda+1}(J^{\lambda}_{(\frac{x+my}{2m})^-}f)\left(\frac{x}{m}\right)\right]-\frac{1}{2}\left[f\left(\frac{x+my}{2}\right)\right.\right.$$
$$\left.\left.+mf\left(\frac{x+my}{2m}\right)\right]\right|\leq\frac{my-x}{4}\left(\int_0^1 t^{p\lambda}dt\right)^{\frac{1}{p}}\left[\left(\int_0^1\left|f'\left(x\frac{t}{2}+m\left(\frac{(2-t)}{2}\right)y\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left|f'\left(y\frac{t}{2}+\left(\frac{2-t}{2}\right)\frac{x}{m}\right)\right|^q dt\right)^{\frac{1}{q}}\right]\leq\frac{my-x}{4}\left(\frac{1}{\lambda p+1}\right)^{\frac{1}{p}}\left[\left(|f'(x)|^q\int_0^1\left(\frac{t}{2}\right)^{\alpha}dt\right.\right.$$
$$\left.+m|f'(y)|^q\int_0^1\left(1-\left(\frac{t}{2}\right)^{\alpha}\right)dt-\frac{Cm(y-x)^2}{2^{2\alpha}}\int_0^1 t^{\alpha}(2^{\alpha}-t^{\alpha})dt\right)^{\frac{1}{q}}+\left(|f'(y)|^q\int_0^1\left(\frac{t}{2}\right)^{\alpha}dt\right.$$
$$\left.\left.+m\left|f'\left(\frac{x}{m^2}\right)\right|^q\int_0^1\left(1-\left(\frac{t}{2}\right)^{\alpha}\right)dt-\frac{Cm(\frac{x}{m^2}-y)^2}{2^{2\alpha}}\int_0^1 t^{\alpha}(2^{\alpha}-t^{\alpha})dt\right)^{\frac{1}{q}}\right]$$
$$=\frac{my-x}{4}\left(\frac{1}{\lambda p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2^{\alpha}(\alpha+1)}\right)^{\frac{1}{q}}\left[\left(|f'(x)|^q-m|f'(y)|^q+m2^{\alpha}(\alpha+1)|f'(y)|^q\right.\right.$$
$$\left.-Cm(y-x)^2\left(1-\frac{\alpha+1}{2^{\alpha}(2\alpha+1)}\right)\right)^{\frac{1}{q}}+\left(|f'(y)|^q-m|f'\left(\frac{x}{m^2}\right)|^q+m2^{\alpha}(\alpha+1)|f'\left(\frac{x}{m^2}\right)|^q\right.$$
$$\left.\left.-Cm(\frac{x}{m^2}-y)^2\left(1-\frac{\alpha+1}{2^{\alpha}(2\alpha+1)}\right)\right)^{\frac{1}{q}}\right]\leq\frac{my-x}{4}\left(\frac{1}{\lambda p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2^{\alpha}(\alpha+1)}\right)^{\frac{1}{q}}$$
$$\left[\left((|f'(x)|+[m(2^{\alpha}(\alpha+1)-1)]^{\frac{1}{q}}|f'(y)|)^q-Cm(y-x)^2\left(1-\frac{\alpha+1}{2^{\alpha}(2\alpha+1)}\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left((|f'(y)|+[m(2^{\alpha}(\alpha+1)-1)]^{\frac{1}{q}}|f'(\frac{x}{m^2})|)^q-Cm(\frac{x}{m^2}-y)^2\left(1-\frac{\alpha+1}{2^{\alpha}(2\alpha+1)}\right)\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{my-x}{4}\left(\frac{1}{\lambda p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2^{\alpha}(\alpha+1)}\right)^{\frac{1}{q}}\left[|f'(x)|+|f'(y)|+\left(|f'(y)|+|f'\left(\frac{x}{m^2}\right)|\right)\right.$$
$$\left.[m(2^{\alpha}(\alpha+1)-1)]^{\frac{1}{q}}-Cm\left(1-\frac{\alpha+1}{2^{\alpha}(2\alpha+1)}\right)\left((y-x)^2+(\frac{x}{m^2}-y)^2\right)\right].$$

Here we have used $A^q+B^q\leq(A+B)^q$, for $A\geq 0, B\geq 0$ and $q>1$. This completes the proof. □

Corollary 3.14. *For $\alpha = 1$, we get the following inequality for strongly m-convex functions*

(54)
$$\left|\frac{2^{\lambda-1}\Gamma(\lambda+1)}{(my-x)^{\lambda}}\left[(J^{\lambda}_{(\frac{x+my}{2})+}f)(my)+m^{\lambda+1}(J^{\lambda}_{(\frac{x+my}{2m})-}f)\left(\frac{x}{m}\right)\right]-\frac{1}{2}\Big[f\left(\frac{x+my}{2}\right)\right.$$
$$\left.+mf\left(\frac{x+my}{2m}\right)\Big]\right|\leq\frac{my-x}{16}\left(\frac{4}{\lambda p+1}\right)^{\frac{1}{p}}\left[\left((|f'(x)|+(3m)^{\frac{1}{q}}|f'(y)|)^{q}-\frac{2Cm(y-x)^{2}}{3}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left((|f'(y)|+(3m)^{\frac{1}{q}}|f'(\frac{x}{m^{2}})|)^{q}-\frac{2}{3}Cm(\frac{x}{m^{2}}-y)^{2}\right)^{\frac{1}{q}}\right]\leq\frac{my-x}{16}\left(\frac{4}{\lambda p+1}\right)^{\frac{1}{p}}$$
$$\left[|f'(x)|+|f'(y)|+(3m)^{\frac{1}{q}}\left(|f'(y)|+|f'\left(\frac{x}{m^{2}}\right)|\right)-\frac{2}{3}Cm\left((y-x)^{2}+(\frac{x}{m^{2}}-y)^{2}\right)\right],$$
where $\lambda > 0$ and $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 3.15. *For $\alpha = m = 1$, we get the following inequality for strongly convex functions*

(55)
$$\left|\frac{2^{\lambda-1}\Gamma(\lambda+1)}{(y-x)^{\lambda}}\left[(J^{\lambda}_{(\frac{x+y}{2})+}f)(y)+(J^{\lambda}_{(\frac{x+y}{2})-}f)(x)\right]-f\left(\frac{x+y}{2}\right)\right|\leq\frac{y-x}{16}\left(\frac{4}{\lambda p+1}\right)^{\frac{1}{p}}$$
$$\left[\left((|f'(x)|+(3)^{\frac{1}{q}}|f'(y)|)^{q}-\frac{2}{3}C(y-x)^{2}\right)^{\frac{1}{q}}+\left((|f'(y)|+(3)^{\frac{1}{q}}|f'(x)|)^{q}-\frac{2}{3}C(y-x)^{2}\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{y-x}{16}\left(\frac{4}{\lambda p+1}\right)^{\frac{1}{p}}\left[\left((3)^{\frac{1}{q}}+1\right)\left(|f'(x)|+|f'(y)|\right)-\frac{4}{3}C(y-x)^{2}\right],$$
where $\lambda > 0$ and $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 3.16. *For $m = 1$, we get the following inequality for strongly α-convex functions*

(56)
$$\left|\frac{2^{\lambda-1}\Gamma(\lambda+1)}{(y-x)^{\lambda}}\left[(J^{\lambda}_{(\frac{x+y}{2})+}f)(y)+(J^{\lambda}_{(\frac{x+y}{2})-}f)(x)\right]-f\left(\frac{x+y}{2}\right)\right|\leq\frac{y-x}{4}\left(\frac{1}{\lambda p+1}\right)^{\frac{1}{p}}$$
$$\left(\frac{1}{2^{\alpha}(\alpha+1)}\right)^{\frac{1}{q}}\left[\left((|f'(x)|+[(2^{\alpha}(\alpha+1)-1)]^{\frac{1}{q}}|f'(y)|)^{q}-C(y-x)^{2}\left(1-\frac{\alpha+1}{2^{\alpha}(2\alpha+1)}\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left((|f'(y)|+[(2^{\alpha}(\alpha+1)-1)]^{\frac{1}{q}}|f'(x)|)^{q}-C(y-x)^{2}\left(1-\frac{\alpha+1}{2^{\alpha}(2\alpha+1)}\right)\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{y-x}{4}\left(\frac{1}{\lambda p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2^{\alpha}(\alpha+1)}\right)^{\frac{1}{q}}\left[\left(|f'(x)|+|f'(y)|\right)\left(1+[(2^{\alpha}(\alpha+1)-1)]^{\frac{1}{q}}\right)\right.$$
$$\left.-2C(y-x)^{2}\left(1-\frac{\alpha+1}{2^{\alpha}(2\alpha+1)}\right)\right],$$
where $\lambda > 0$ and $\frac{1}{p}+\frac{1}{q}=1$.

Remark 3.17. (i) If we put $\alpha = 1$ and $C = 0$ in (52), then we get [4, Theorem 2.7].
(ii) If we put $\alpha = 1$, $m = 1$ and $C = 0$ in (52), then we get [27, Theorem 6].
(iii) If we put $\alpha = \lambda = 1$, $m = q = 1$, and $C = 0$ in (52), then we get [27, Theorem 2].

4. conclusion

In this article, we have proved two generalized fractional versions of the Hadamard inequality. For proving these inequalities we have utilized Riemann-Liouville fractional integrals and strongly (α, m)-convex functions. The presented results hold for several types of convexity. Also, the presented results have some connection with already published results.

References

[1] S.S.Dragomir and R.P.Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett. **11** (5) (1998), 91–95.

[2] S.S.Dragomir and C.E.M.Pearce, *Selected topics on Hermite-Hadamard inequalities and applications*, RGMIA Monographs. Victoria University, 2000.

[3] G.Farid, A.U.Rehman, B.Tariq and A.Waheed, *On Hadamard type inequalities for m-convex functions via fractional integrals*, J. Inequal. Spec. Funct. **7** (4) (2016), 150–167.

[4] G.Farid, A.U.Rehman and B.Tariq, *On Hadamard-type inequalities for m-convex functions via Riemann-Liouville fractional integrals*, Studia Univ. Babes-Bolyai, Math. **62** (2) (2017), 141–150.

[5] Y.Dong, M.Saddiqa, S.Ullah and G.Farid, *Study of fractional integral operators containing Mittag-Leffler functions via strongly (α, m)-convex functions*, Math. Prob. Eng. **2021** (2021), 13 pages.

[6] G.Farid, A.U.Rehman and S.Mehmood, *Hadamard and Fejér-Hadamard type integral inequalities for harmonically convex functions via an extended generalized Mittag-Leffler function*, J. Math. Comput. Sci. **8** (5) (2018), 630–643.

[7] İ.İşcan, *Hermite-Hadamard-Fejér type inequalities for convex functions via fractional integrals*, Stud. Univ. Babeş-Bolyai Math. **60** (3) (2015), 355–366.

[8] Y.C.Kwun, G.Farid, S.B.Akbar and S.M.Kang, *Rieman-Liouville fractional version of Hadamard inequality for strongly m-convex functions*, Submitted.

[9] S.M.Kang, G.Farid, W.Nazeer and S.Mehmood, *(h, m)-convex functions and associated fractional Hadamard and Fejér-Hadamard inequalities via an extended generalized Mittag-Leffler function*, J. Inequal. Appl. **2019** (2019), 78 pages.

[10] A.A.Kilbas, H.M.Srivastava, J.J.Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, 204, Elsevier, New York-London, 2006.

[11] U.S.Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula*, Appl. Math. Comput. **147** (5) (2004), 137–146.

[12] T.Lara, N.Merentes, R.Quintero and E.Rosales, *On strongly m-convex functions*, Mathematica Aeterna. **5** (3) (2015), 521–535.

[13] S.Mehmood and G.Farid *Fractional Hadamard and Fejér-Hadamard inequalities for exponentially m-convex function*, Stud. Univ. Babeş-Bolyai Math., to appear.

[14] S.Mehmood, G.Farid, K.A.Khan and M.Yussouf, *New Hadamard and Fejér-Hadamard fractional inequalities for exponentially m-convex function*, Eng. Appl. Sci. Lett. **3** (1) (2020), 45–55.

[15] S.Mehmood, G.Farid, K.A.Khan and M.Yussouf, *New fractional Hadamard and Fejér-Hadamard inequalities associated with exponentially $(h - m)$-convex function*, Eng. Appl. Sci. Lett. **3** (2) (2020), 9–18.

[16] E.Mittal, S.Joshi, R.M.Pandey and V.N.Mishra, *Fractional integral and integral transform formulae using generalized Appell hypergeometric function*, Nonlinear Science Letters A. **8**, (2) (2017), 221–227.

[17] V.N.Mishra, D.L.Suthar and S.D.Purohit, *Marichev-Saigo-Maeda Fractional Calculus Operators, Srivastava Polynomials and Generalized Mittag-Leffler Function*, Cogent Math. **4** (2017), 1–11.

[18] V.G.Mihesan, *A generalization of the convexity*, Seminar on Functional Equations, Approx. and Convex, Cluj-Napoca, Romania, 1993.

[19] J.Mako and A.Hazy, *On strongly convex functions*, Carpathian J. Math. **32** (1) (2016), 87–95.
[20] A.F.Nikiforov and V.B.Uvarov, *Special Functions of Mathematical Physics*, Springer Basel AG, 1988.
[21] K.Nikodem and Z.Páles, *Characterizations of inner product spaces by strongly convex functions*, Banach J. Math. Anal. **5** (1) (2011), 83–87.
[22] C.P.Niculescu and L.E.Persson, *Old and new on the Hermite-Hadamard inequality*, Real Anal. Exch. **29** (2) (2003/2004), 663–685.
[23] B.T.Polyak, *Existence theorems and convergence of minimizing sequences in extremum problems with restrictions*, Soviet Mathematics Doklady. **7** (1966), 72–75.
[24] A.W.Roberts and D.E.Varberg, *Convex Functions*, Academic Press, New York, 1973.
[25] Z.Retkes, *An extension of the Hermite-Hadamard Inequality*, Acta Sci. Math. (Szeged), **74** (1) (2008), 95–106.
[26] M.Z.Sarikaya, E.Set, H.Yaldiz and N.Başak, *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. comput. model. **57** (9) (2013), 2403–2407.
[27] M.Z.Sarikaya, and H.Yildirim, *On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals*, Miskolc Math. Notes. **17** (2) (2017), 1049–1059.
[28] G.Toader, *Some generalizations of the convexity, Proceedings of the Colloquium on Approximation and Optimization*, Univ. Cluj-Napoca, Cluj-Napoca, (1985), 329–338.
[29] J.P.Vial, *Strong convexity of sets and functions*, J. Math. Econ. **9** (1-2) (1982), 187–205.

Ghulam Farid
Department of Mathematics, COMSATS University Islamabad, Attock Campus, Pakistan
E-mail: faridphdsms@hotmail.com

Saira Bano Akbar
Department of Mathematics, COMSATS University Islamabad, Attock Campus, Pakistan
E-mail: sairabano644@yahoo.com

Laxmi Rathour
Ward number – 16, Bhagatbandh, Anuppur 484 224, Madhya Pradesh, India
E-mail: laxmirathour817@gmail.com, rathourlaxmi562@gmail.com

Lakshmi Narayan Mishra
Department of Mathematics, School of Advanced Sciences,
Vellore Institute of Technology (VIT) University,
Vellore 632 014, Tamil Nadu, India
E-mail: lakshminarayanmishra04@gmail.com,
lakshminarayan.mishra@vit.ac.in